2   The Real Scalar Field

All that is now missing is the fact that momentum $\mathbf{p}$ and position $\mathbf{x}$ should be 3-vectors and so the field ϕ is a function $\phi(\mathbf{x},t)$ which we usually just write as $\phi(x)$, understanding that x refers to a 4-vector $(\mathbf{x},x_0)$, with $x_0 = t$. The Lagrangian is now expressed as an integral over all space

$$L = \int d^3\mathbf{x}\mathcal{L},$$

with

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2.$$

The Lagrangian density, $\mathcal{L}$ is Lorentz invariant, whereas the Lagrangian has dimensions of energy and transforms under Lorentz transformations. We can write the Lagrangian density in manifest Lorentz invariant form

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2,$$

where ∂_μ is the 4-vector operator

$$\partial_\mu \equiv (\partial_t, \partial_x, \partial_y, \partial_z)$$

and

$$\partial^\mu = g^{\mu\nu}\partial_\nu,$$

$g^{\mu\nu}$ being the Minkowski metric (in flat space)

$$g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

The Euler-Lagrange equations are

$$\frac{\partial^2\phi}{\partial t^2} = \nabla^2\phi - m^2\phi,$$

which can be written in manifestly covariant form as

$$\partial^\mu\partial_\mu\phi + m^2\phi \equiv (\Box + m^2)\phi = 0,$$

where

$$\Box \equiv \partial^\mu\partial_\mu = \frac{\partial^2}{\partial t^2} - \nabla.$$

The canonical momentum is

$$\pi(x) = \frac{\partial\mathcal{L}}{\partial\dot{\phi}(x)} = \dot{\phi}(x),$$

and the Hamiltonian is

$$H = \int d^3\mathbf{x}\mathcal{H},$$

where the "Hamiltonian density", $\mathcal{H}$, is given by

$$\mathcal{H} \;=\; \frac{1}{2}\pi(x)^2 + \frac{1}{2}(\nabla\phi(x))^2 + \frac{1}{2}m^2\phi(x)^2.$$

$\pi(x)$ and $\phi(x)$ obey the equal time commutation relations

$$[\pi(x),\phi(y)]_{x_0=y_0} \;=\; -i\delta^3(\mathbf{x}-\mathbf{y}).$$

We can again expand $\phi(\mathbf{x},t)$ in terms of creation and annihilation operators, which are functions of the three-momentum $\mathbf{p}$, as

$$\phi(x) \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}}\left(a(\mathbf{p})e^{-ip\cdot x}+a^\dagger(\mathbf{p})e^{+ip\cdot x},\right)$$

where $p\cdot x$ means $E_p t - \mathbf{p}\cdot\mathbf{x}$, with $p_0=E_p=\sqrt{\mathbf{p}^2+m^2}$. It can easily be seen that this expanjsion for $\phi(x)$ obeys the equation of motion since $e^{\pm ip\cdot x}$ are solutions.

The creation and annihilation operators $a^\dagger(\mathbf{p})$ and $a(\mathbf{p})$ obey the commutation relations,

$$\left[a(\mathbf{p}),a^\dagger(\mathbf{p}')\right] \;=\; (2\pi)^3 2E_p\delta^3(\mathbf{p}-\mathbf{p}')$$

Note that creation operators commute with each other as do annihilation operators.

2.1   Relativistic Normalisation of States

In non-relativistic Quantum Mechanics a one-particle state with momentum $\mathbf{p}$ is normalised by

$$\langle\mathbf{p}|\mathbf{p}'\rangle \;=\; (2\pi)^2\delta^3(\mathbf{p}-\mathbf{p}'),$$

but this is *not* Lorentz invariant since

$$\int d^3\mathbf{p}\,\delta^3(\mathbf{p}-\mathbf{p}') \;=\; 1$$

and $d^3\mathbf{p}$ is a volume element in momentum 3-space, so it is *not* Lorentz invariant.

On the other hand d^4p *is* Lorentz invariant and so is $\delta(p^2-m^2)$. Furthermore we can restrict ourselves to positive energy p_0 without loss of Lorentz invariance. This means that

$$d^4p\,\delta(p^2-m^2)\theta(p_0)$$

is Lorentz invariant.

$$\delta(p^2-m^2)\theta(p_0) \;=\; \delta(2p_0(p_0-E_p))$$

so that

$$d^4p\,\delta(p^2-m^2)\theta(p_0) \;=\; \frac{d^3\mathbf{p}}{2E_p}.$$

The RHS is Lorentz invariant although it does not necessary look like it.

$$\int \frac{d^3\mathbf{p}}{2E_p} 2E_p \delta^3(\mathbf{p}-\mathbf{p}') \;=\; 1,$$

so that $2E_p\delta^3(\mathbf{p}-\mathbf{p}')$ is Lorentz invariant. This gives us the relativistic invariant normalisation of a one-particle momentum eigenstate as

$$\langle \mathbf{p}|\mathbf{p}'\rangle \;=\; (2\pi)^3 2E_p \delta^3(\mathbf{p}-\mathbf{p}').$$

The vacuum is still normalised as

$$\langle 0|0\rangle \;=\; 1,$$

and since

$$|\mathbf{p}\rangle \;=\; a^\dagger(\mathbf{p})|0>$$

and its hermitian conjugate

$$\langle \mathbf{p}'| \;=\; \langle 0|a(\mathbf{p}')$$

we have

$$\langle \mathbf{p}'|\mathbf{p}\rangle \;=\; \langle 0|a(\mathbf{p}')a^\dagger(\mathbf{p})|0\rangle \;=\; \langle 0|\left[a(\mathbf{p}'),a(\mathbf{p})\right]\rangle \;=\; (2\pi)^3 2E_\delta^3(\mathbf{p}-\mathbf{p}').$$

This explains the factors $2E_p$ in the commutation relation between creation and annihilation operators.

Note that

$$a(\mathbf{p})|\mathbf{p}'\rangle \;=\; (2\pi)^3 2E_p \delta(\mathbf{p}-\mathbf{p}')$$

In non-relativistic quantum mechanics the space of states for a fixed number of particles, n, is called a "Hilbert space", and in the representation in which the particles are described by their momenta we would write such a state as $|\mathbf{p_1},\mathbf{p_2},\cdots\mathbf{p_n}\rangle$. The number of particles described by all of these states is always equal to n. In Quantum Field Theory, in which we have creation and annihilation operators which can change the number of particles in a state, we need to consider the union of all such n-particle Hilbert spaces where n ranges from zero (the vacuum) to infinity. This union of Hilbert spaces is known as "Fock space". Later we will need to consider particles with non-zero spin, so that as well as denoting their momenta will will also need to denote the "helicity" of the particles, the component of spin in the direction of their momenta.