3 Energy-Momentum Tensor

The energy-momentum tensor, $T_{\mu\nu}$ is defined by

$$T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} \partial_\nu \phi - g_{\mu\nu} \mathcal{L}.$$

We see immediately, using the definition of the canonical momentum, $\pi(x)$, that T_{00} is the Hamiltonian density.

3.1 The Momentum Operator

The momentum operator for a system described by a Lagrangian density $\mathcal{L}$ is given by the $\mu = 0$ components of this tensor, integrated over space (and normal ordered so that the momentum of the vacuum is zero)

$$P_\nu = \int d^3\mathbf{x} : T_{0\nu} :$$

Now in terms of the expansion in creation and annihilation operators, we have

$$\frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)} = \partial_\mu \phi = -i \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} p^\mu \left(a(\mathbf{p}) e^{-ip\cdot x} - a^\dagger(\mathbf{p}) e^{+ip\cdot x} \right),$$

so that the operator P_i, $(i = 1 \cdots 3)$ is

$$\begin{aligned} P_i &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \frac{d^3\mathbf{p}'}{(2\pi)^3 2E_p'} d^3\mathbf{x} \left(E_p p_i' + E_{p'} p_i \right) a^\dagger(\mathbf{p}') a(\mathbf{p}) e^{i(p-p').\cdot x} \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} \frac{d^3\mathbf{p}'}{2E_p'} \delta^3(\mathbf{p} - \mathbf{p}') \left(E_p p_i' + E_{p'} p_i \right) a^\dagger(\mathbf{p}') a(\mathbf{p}) e^{i(E_p - E_{p'}).\cdot t} \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_p} p_i a^\dagger(\mathbf{p}) a(\mathbf{p}). \end{aligned}$$

This is the number of particles with momentum $\mathbf{p}$, multiplied by p_i and integrated over all possible momenta (using the Lorenz invariant integration measure) †

For space-like components, the momentum operator may be written as

$$P_i = \int d^3\mathbf{x} \pi(x) \partial_i \phi(x)$$

which can be seen to obey the commutation relation

$$\left[P_\mu, \phi(x) \right] = -i \partial_\mu \phi(x).$$

† We have dropped terms quadratic in the creation or annihilation operator, which can be shown to vanish.

The momentum operator generates infinitesimal translations, and for finite transformations with parameter a^μ we have

$$e^{iP_\mu a^\mu}\phi(x)e^{-iP_\mu a^\mu} = \phi(x+a).$$

Using the Euler-Lagrange equations it can be shown that the energy-momentum tensor is "conserved", i.e. its divergence cancels:

$$\begin{aligned}
\partial^\mu T_{\mu\nu} &= \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi + \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)}\partial^\mu\partial_\nu\phi - \partial_\nu\mathcal{L} \\
&= \partial^\mu \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\phi + \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)}\partial^\mu\partial_\nu\phi - \frac{\partial \mathcal{L}}{\partial\phi}\partial_\nu\phi - \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\partial_\mu\phi \\
&= \frac{\partial \mathcal{L}}{\partial\phi}\partial_\nu\phi + \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)}\partial^\mu\partial_\nu\phi - \frac{\partial \mathcal{L}}{\partial\phi}\partial_\nu\phi - \frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)}\partial_\nu\partial_\mu\phi \\
&= 0,
\end{aligned}$$

where the Euler-Lagrange equations have been used in the first term in the last step. In components we may write the zero component of this conservation law as

$$\frac{d}{dt}T_{0\nu} - \frac{\partial}{\partial x_i}T_{i\nu} = 0$$

Integrating over all space, the second term vanishes as it is the integral of a derivative (assumed to vanish at spatial infinity) and we are left with

$$\frac{d}{dt}\int d^3\mathbf{x}T_{0\nu} = \frac{d}{dt}P_\nu = 0,$$

i.e, the total energy and momentum of the system are conserved (as expected)

3.2 The Angular Momentum Operator

In 3 dimensions

$$\mathbf{L} = \mathbf{r}\times\mathbf{p}$$

where the component L_i is given in terms of components of the energy-momentum tensor by

$$L_i = \varepsilon_{ijk}\int d^3\mathbf{x} : x^j T^{0k}$$

We can generalise this to four dimensions (thereby including the generators of Lorentz boosts as well as rotations) by defining a 3-rank tensor

$$\mathcal{M}^{\mu\nu\rho} = (x^\mu T^{\nu\rho} - x^\nu T^{\mu\rho})$$

and hence an antisymmetric 2-rank tensor

$$M^{\mu\nu} = \int d^3\mathbf{x}\mathcal{M}^{\mu\nu 0}.$$

We can see that $M^{ij} = \varepsilon^{ijk} L_k,\ (i, j = 1\cdots 3)$ are the usual angular momentum operators, whereas $M^0 i$ generate Lorentz boosts.

By expanding the fields in terms of creation and annihilation operators (as we did for the momentum operators) and performing some algebra we can show that

$$M^{0i}|p\rangle \ = \ -p_i t|p\rangle - E_p \frac{\partial}{\partial p_i}|p\rangle$$

Now consider the infinitesimal operator

$$\left(1 - i\delta v_i M^{0i}\right)$$

acting on a one-particle momentum state $|p\rangle$, where δv_i is an infinitesimal boost velocity in the direction i. $\delta v_i E_p = \delta p_i$, the change in momentum in the i-direction and $p_i \delta v_i = \delta E$, the change in energy. Thus we have

$$\left(1 - i\delta v_i M^{0i}\right)|p\rangle \ = \ e^{-i\delta E t}|p\rangle + \delta p_i \frac{\partial}{\partial p_i}|p\rangle \ = \ e^{-i\delta E t}|p + \delta p\rangle,$$

showing that the above operator is an infinitesimal boost in the direction i.

Likewise it may be shown that the operator

$$\left(1 - i\varepsilon^{ijk}\delta\theta_i M_{jk}\right)$$

acting on $|p\rangle$ gives a state in which the momentum is rotated by a small angle $\delta\theta$ about the i axis.

As before, the divergencelessness of the energy-momentum tensor leads to the conservation law

$$\partial_\rho \mathcal{M}^{\mu\nu\rho},$$

which in turn (after integrating over all space) gives the conservation of angular momentum

$$\frac{d}{dt} M^{\mu\nu} \ = 0.$$