5 The Complex Scalar Field

The scalar field expanded as

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(a(\mathbf{p}) e^{-ip\cdot x} + a^\dagger(\mathbf{p}) e^{+ip\cdot x}, \right)$$

is equal to its Hermitian conjugate (under Hermitian conjugation, $a(\mathbf{p}) \leftrightarrow a^\dagger(\mathbf{p})$ and $e^{ip\dot{x}} \leftrightarrow e^{-ip\cdot x}$). This field can only be used to describe particles with the quantum numbers of the vacuum, i.e. no charge or other flavour properties - particles such as the π_0 that are equal to their antiparticle.

In order to describe spin-0 particles with electric charge or other properties (such as strangeness, charm etc.) we need to introduce a new pair of creation and annihilation operators, $b^\dagger(\mathbf{p}),\ b((\mathbf{p})$ which create and annihilate the antiparticles. These also obey the commutation relations

$$\left[b(\mathbf{p}), b^\dagger(\mathbf{p}) \right] = (2\pi)^3 2E_p \delta^3(\mathbf{p} - \mathbf{p}').$$

The complex scalar field is then expanded as

$$\phi(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(a(\mathbf{p}) e^{-ip\cdot x} + b^\dagger(\mathbf{p}) e^{+ip\cdot x}, \right)$$

$$\phi^\dagger(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(b(\mathbf{p}) e^{-ip\cdot x} + a^\dagger(\mathbf{p}) e^{+ip\cdot x}, \right)$$

The field $\phi(x)$ annihilates a particle with positive energy or creates an antiparticle with negative energy, whereas The field $\phi^\dagger(x)$ annihilates an antiparticle with positive energy or creates a particle with negative energy.

The (Feynman) propagator is now

$$\Delta_F(x,y) = -i\langle 0|T\phi(x)\phi^\dagger(y)|0\rangle = \lim_{\varepsilon\to 0} \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\varepsilon},$$

The interpretation is that for $x_0 > y_0$ a particle propagates from y to x, whereas for $x_0 < y_0$ an antiparticle propagates from x to y.

We can define the (electric) charge operator as

$$Q = i \int d^3\mathbf{x} : \left(\phi^\dagger \dot{\phi} - \phi \dot{\phi}^\dagger \right) :$$

Expanding this in terms of the creation and annihilation operators and performing the integral over all space (whcih generates a delta function) and one of the three-momenta we get

$$Q = \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_{\mathbf{p}}} \left(a^\dagger(\mathbf{p}) a(\mathbf{p}) - b^\dagger(\mathbf{p}) b(\mathbf{p}) \right)$$

This counts the number of particles *minus* the number of antiparticles (which have opposite charge).

For a complex scalar field the Lagrangian density is

$$\mathcal{L} \;=\; \left(\partial_\mu\phi\right)^\dagger \partial_\mu\phi - m^2\phi^\dagger\phi$$

The Euler-lagrange equations of motion give

$$\left(\Box + m^2\right)\phi \;=\; 0$$
$$\left(\Box + m^2\right)\phi^\dagger \;=\; 0$$

The canonical momentum is given by

$$\pi(x) \;=\; \frac{\partial\mathcal{L}}{\partial\dot{\phi}} = \dot{\phi}^\dagger$$

(and likewise $\pi^\dagger = \dot{\phi}$).

In terms of the creation and annihilation operators the Hamiltonian density may be written

$$\mathcal{H} \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} E_p \left(a^\dagger(\mathbf{p})a(\mathbf{p}) + b^\dagger(\mathbf{p})b(\mathbf{p})\right)$$

The number operator is

$$N \;=\; \int \frac{d^3\mathbf{p}}{(2\pi)^3 2E_\mathbf{p}} \left(a^\dagger(\mathbf{p})a(\mathbf{p}) + b^\dagger(\mathbf{p})b(\mathbf{p})\right),$$

which counts the number of particles plus the number of antiparticles.