7 Perturbation Theory

The fields that appear in the LSZ reduction formula are the full interacting fields, which obey the Heisenberg equations of motion

$$\frac{\partial \phi(x)}{\partial t} = i\left[H[\pi, \phi], \phi(x)\right],$$

where H is the full Hamiltonian including the interaction term

$$H = H_0 + H_I,$$

H_0 being the free part of the Hamiltonian, quadratic in the fields and canonical momenta and H_I is the interacting part which is cubic or higher order in the fields.

$$H_I = -\int d^3\mathbf{x}\mathcal{L}_I(\phi).$$

The (free) "in" fields, $\phi_{in}(x)$, obey the Heisenberg equation of motion

$$\frac{\partial \phi_{in}(x)}{\partial t} = i\left[H_0[\pi_{in}, \phi_{in}], \phi_{in}(x)\right].$$

We seek a unitary operator $U(t)$ such that

$$\phi_{in}(x) = U(t)\phi(x)U^{-1}(t)$$

Differentiating this w.r.t to t and using the Heisenberg equations of motion we have

$$\begin{aligned}
\frac{\partial \phi_{in}(x)}{\partial t} &= i\left[H_0[\pi_{in}, \phi_{in}], \phi_{in}(x)\right] \\
&= \frac{dU(t)}{dt}\phi(x)U^{-1}(t) + U(t)\frac{\partial \phi(x)}{\partial t}U^{-1}(t) + U(t)\phi(x)\frac{U^{-1}(t)}{dt} \\
&= \frac{dU(t)}{dt}U^{-1}(t)\phi_{in}(x) + iU(t)\left[H[\pi, \phi], \phi(x)\right]U^{-1}(t) + \phi_{in}(x)U(t)\frac{U^{-1}(t)}{dt}
\end{aligned}$$

We also have

$$U(t)H[\pi, \phi]U^{-1}(t) = H[\pi_{in}, \phi_{in}],$$

(as can be seen by inserting the unit operator in the form $U^{-1}(t)U(t)$ between the fields or canonical momenta) so that

$$U(t)\left[H[\pi, \phi], \phi(x)\right]U^{-1}(t) = \left[H[\pi_{in}, \phi_{in}], \phi_{in}(x)\right]$$

and we end up with

$$i\left[H_0[\pi_{in}, \phi_{in}], \phi_{in}(x)\right] = \left[\left(\frac{dU(t)}{dt}U^{-1} + i[H[\pi_{in}, \phi_{in}]\right), \phi_{in}(x)\right].$$

From this we conclude that

$$\frac{dU(t)}{dt}U^{-1}(t) = -i\left([H[\pi_{in},(x),\phi_{in}(x)] - [H_0[\pi_{in},\phi_{in}]\right) \;=\; -i[H_I[\pi_{in},\phi_{in}],$$

or

$$\frac{dU(t)}{dt} \;=\; -i[H_I[\pi_{in},\phi_{in}]U(t).$$

This can be solved iteratively as

$$U(t) \;=\; 1 - i\int_{-\infty}^{t} H_I(t_1)dt_1 + (-i)^2\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 H_I(t_1)H_I(t2) + \cdots,$$

Here H_I means the interacting Hamiltonian as a functional of the "in" fields. The second term may be written as

$$\frac{1}{2}\int_{-\infty}^{t} dt_1 dt_2 T\left(H_I(t_1)H_I(t_2)\right),$$

where T is the time ordering operator and likewise the $n^t h$ term is

$$\frac{1}{n!}\int_{-\infty}^{t} dt_1\cdots dt_n T\left(H_I(t_1)\cdots H_I(t_n)\right).$$

From this we may write

$$U(t) \;=\; T\exp\left(-i\int_{-\infty}^{t} dt' H_I(t')\right),$$

where the meaning of the exponential is given in terms of its expansion and for the n^{th} term the n integrals over time are time-ordered.

Now consider a time-ordered product of fields

$$\phi(x_1)\phi(x_2)\cdots\phi(x_n),$$

with $t_1^0 > t_2^0 > \cdots > t_n^0$.

$$U(t_1)\phi_{in}(x_1)U^{-1}(t_1)U(t_2)\phi_{in}(x_2)U^{-1}(t_2)\cdots)U^{-1}(t_{n-1})U(t_{t_n})\phi_{in}(x_n).$$

$$U^{-1}(t_1)U(t_2) \;=\; T\exp\left(-i\int_{t_1}^{t_2} dt' H_I(t')\right),$$

Furthermore, acting on the vacuum

$$U(t)|0\rangle \;=\; |0\rangle$$
$$\langle 0|U(t) \;=\; \langle 0|,$$

so that making use again of the time ordering operator we have

$$\langle 0|T\phi(x1)\cdots\phi(x_n)|0\rangle \;=\; \langle 0|T\exp\left(-i\int_{-\infty}^{\infty} dt' H_I(t')\right)\phi_{in}(x_1)\cdots\phi_{in}(x_n)|0\rangle$$

We may rewrite this in manifestly Lorentz invariant form by noting that

$$\int_{-\infty}^{\infty} dt' H_I(t') \;=\; -\int d^4x \mathcal{L}_I(\phi_{in}),$$

so that we have

$$\langle 0|T\phi(x1)\cdots\phi(x_n)|0\rangle \;=\; \langle 0|T\exp\left(i\int d^4x\mathcal{L}_I(\phi_{in})\right)\phi_{in}(x_1)\cdots\phi_{in}(x_n)|0\rangle.$$

The terms in the interaction Lagrangian density contain a coupling constant (e.g. g in the term $-g\phi^3/3!$), which determines the strength of the interactions. Provided this coupling is sufficiently small we may expand this expression up to a given order in the coupling and calculate the required Green function in the perturbative approximation.

Therefore the calculation is essentially reduced to the determination of the vacuum expectation value of the time ordered product of strings of free fields. For example, at order g^2 in the case of the 2-particle to 2-particle scattering considered previously we would need to calculate

$$\frac{1}{2}\left(\frac{ig}{3!}\right)^2\int d^4xd^4x'\langle 0|T\phi_{in}^3(x)\phi_{in}^3(x')\phi_{in}(x_1)\phi_{in}(x_2)\phi_{in}(y_1)\phi_{in}(y_2)|0\rangle$$

and insert this into the LSZ reduction formula.

7.1 The Wick Contraction

Henceforth we drop the subscript "in" on the ϕ fields, and these are to be understood to be free fields unless stated otherwise.

By expanding in terms of creation and annihilation operators we can show that the time-ordered product of two (free) fields obeys the relation

$$T\phi(x_1)\phi(x_2) \;= :\phi(x_1)\phi(x_2): + \langle 0|T\phi(x_1)\phi(x_2)|0\rangle$$

For four fields we get a more complicated expression

$$\begin{aligned}
T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4) \quad = \quad & :\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4): + \\
&:\phi(x_1)\phi(x_2):\langle 0|T\phi(x_3)\phi(x_4)|0\rangle + :\phi(x_1)\phi(x_3):\langle 0|T\phi(x_2)\phi(x_4)|0\rangle + :\phi(x_1)\phi(x_4):\langle 0|T\phi(x_2)\phi(x_3)|0\rangle + \\
&:\phi(x_2)\phi(x_3):\langle 0|T\phi(x_1)\phi(x_4)|0\rangle + :\phi(x_2)\phi(x_4):\langle 0|T\phi(x_1)\phi(x_3)|0\rangle + :\phi(x_3)\phi(x_4):\langle 0|T\phi(x_1)\phi(x_2)|0\rangle + \\
&\langle 0|T\phi(x_1)\phi(x_2)|0\rangle\langle 0|T\phi(x_3)\phi(x_4)|0\rangle + \langle 0|T\phi(x_1)\phi(x_3)|0\rangle\langle 0|T\phi(x_2)\phi(x_4)|0\rangle + \\
&\langle 0|T\phi(x_1)\phi(x_4)|0\rangle\langle 0|T\phi(x_2)\phi(x_3)|0\rangle + \langle 0|T\phi(x_2)\phi(x_3)|0\rangle\langle 0|T\phi(x_1)\phi(x_4)|0\rangle + \\
&\langle 0|T\phi(x_2)\phi(x_4)|0\rangle\langle 0|T\phi(x_1)\phi(x_3)|0\rangle + \langle 0|T\phi(x_3)\phi(x_4)|0\rangle\langle 0|T\phi(x_1)\phi(x_2)|0\rangle.
\end{aligned}$$

For six fields we have the normal ordered product of all six fields, plus terms with the normal ordered product of four fields multiplied by one Feynman propagtor, plus terms with normal ordered product of two fields multiplied by the product of two Feynman propagators plus terms consisting of the products of three Feynman propagators.

Wick generalised this to

$$\begin{aligned} T\phi(x_1)\cdots\phi(x_{2n}) &= \sum_{pairings} \langle 0|T\phi(x_{i_1}\phi(x_{j_1})|0\rangle \cdots \langle 0|T\phi(x_{i_n}\phi(x_{j_n})|0\rangle \\ &+ \text{ terms involving normal ordered fields} \end{aligned}$$

The vacuum expectation values of a normal ordered product of fields is zero, so the vacuum expectation value of the time-ordered product of $2n$ free fields is given by

$$\begin{aligned} \langle 0|T\phi(x_1)\cdots\phi(x_{2n})|0\rangle &= \sum_{pairings} \langle 0|T\phi(x_{i_1}\phi(x_{j_1})|0\rangle \cdots \langle 0|T\phi(x_{i_n}\phi(x_{j_n})|0\rangle \\ &= i^n \sum_{pairings} \Delta_F(x_{i_1}, x_{j_1}) \cdots \Delta_F(x_{i_n}, x_{j_n}) \end{aligned}$$

For an *odd* number of fields the vacuum expectation value of the time ordered product vanishes