3 Britto, Cachazo, Feng (BCF) Reduction

This is a reduction formula which enables one to calculate a next-to-MHV (NMHV) amplitude (i.e. an amplitude with *three* negative helicity gluons), in terms of MHV amplitudes (and the extended triple-gluon vertex discussed at the end of the last section).

The reduction can then be iterated so that NNMHV (i.e. four negative helicity states) can be calculated in terms of NMHV and MHV amplitudes, etc.

Useful Notation: Suppose we consider an n-point amplitude with external momenta $p_1 \cdots p_n$.

Define momenta

$$q_{kl} \;=\; -q_{lk} \;=\; p_k + p_{k+1} + \cdots p_l$$

(if $k > l$ then we mean $p_k + \cdots p_n + p_1 + \cdots p_l$).

If $k = 1$, we will just write this as q_l.

Let $\tilde{\mathcal{A}}(1, 2, \cdots n)$ be a coloured ordered n-point amplitude. Using the cyclic symmetry we can always choose particle n to have positive helicity and particle $n-1$ to have negative helicity.

Now define the analytic function of z, $\tilde{\mathcal{A}}(1, 2, \cdots n; z)$, which is the above amplitude but where the "marked" momenta p_{n-1} and p_n have been continued into complex momenta by the transformations on spinors

$$|p_n\rangle \;\rightarrow |\hat{p}_n(z)\rangle \;=\; |p_n\rangle + z|p_{n-1}\rangle$$

$$|p_{n-1}] \;\rightarrow |\hat{p}_{n-1}(z)] \;=\; |p_{n-1}] - z|p_n]$$

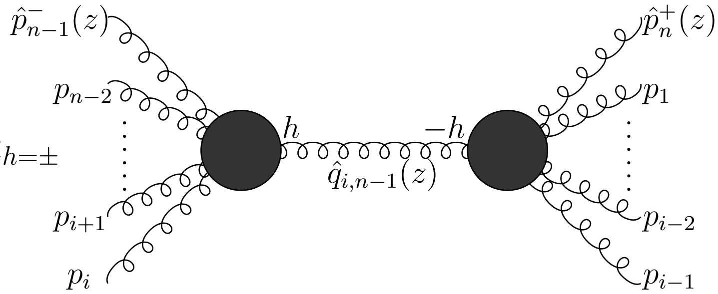
Note that we still have conservation of momentum

$$\sum_i^{n-2} p_i + p_{n-1} + p_n \;=\; \sum_i^{n-2} p_i + \hat{p}_{n-1}(z) + \hat{p}_n(z) \;= 0$$

So that what we want is the value of this analytic function (strictly a meromorphic function) at $z = 0$. On the other hand, provided $\tilde{\mathcal{A}}(1, 2, \cdots n; z) \rightarrow 0$ as $|z| \rightarrow \infty$ (BCF and Witten proved this laboriously - we shall take it as read), then the function is uniquely determined by the residues R_i of the poles at $z = z_i$

$$\tilde{\mathcal{A}}(1, 2, \cdots n; z) = \sum_i \frac{R_i}{(z - z_i)}$$

$\sum_{k=1}^{n-3} \quad \sum_{h=\pm}$



In order to find these poles and their residues we split the amplitude, in all possible ways, into a left-handed amplitude and a right-handed amplitude with a single gluon propagating between them. We only consider terms in which the gluon $(n-1)$ is on the opposite side from gluon n. The poles occur for values of z for which the intermediate gluon propagator diverges - i.e. it carries light-like momentum.

If both the marked gluons are on the same side of the split, then the intermediate gluon must carry momentum (from the side which contains *no* marked momenta) which is independent of z and therefore cannot contribute to the poles.

Thus we have

$$\tilde{\mathcal{A}}(1,2,\cdots n;z) \;=\; \sum_{i=1}^{n-3}\sum_{h=\pm}\mathcal{A}^L(p_i,p_{i+1},\cdots\hat{p}_{n-1}(z),h)\frac{1}{\hat{q}_{i,n-1}(z)^2}\mathcal{A}^R(\hat{p}_n(z),p_1,\cdots p_{i-1},-h),$$

where $\sum_{h=\pm}$ indicates that we need to consider the intermediate gluon propagating for right-helicity to left-helicity and vice-versa.

Now

$$\frac{(1+\gamma^5)}{2}\gamma\cdot q_{i,n-1} \;=\; |p_i\rangle[p_i| + |p_{i+1}\rangle[p_{i+1}| + \cdots |p_{n-1}\rangle[p_{n-1}|$$

so that

$$\frac{(1+\gamma^5)}{2}\gamma\cdot\hat{q}_{i,n-1}(z) \;=\; \frac{1}{2}\left(1+\gamma^5\right)\gamma\cdot\; q_{i,n-1} - z|p_n\rangle[p_{n-1}|$$

and

$$\hat{q}_{i,n-1}(z)^2 \;=\; q_{i,n-1}^2 - z[p_{n-1}|\gamma\cdot q_{i,n-1}|p_n\rangle.$$

This vanishes when

$$z \;\equiv\; z_i \;=\; \frac{q_{i,n-1}^2}{[p_{n-1}|\gamma\cdot q_{i,n-1}|p_n\rangle}$$

Furthermore, at this value the intermediate propagating gluon is *on-shell*, so that the factors on the left and right (whose product gives the residues of the poles) are on-shell amplitudes (albeit with complex momenta) and so we finally arrive at

$$\tilde{\mathcal{A}}(1,2,\cdots n) \;=\; \sum_{i=1}^{n-3}\sum_{h=\pm}\tilde{\mathcal{A}}(p_i,p_{i+1},\cdots\hat{p}_{n-1}(z_i),\hat{q}_{i,n-1}^{h}(z_i))\frac{1}{q_{i,n-1}^2}\tilde{\mathcal{A}}(\hat{p}_n(z_i),p_1,\cdots p_{i-1},-\hat{q}_{i,n-1}^{-h}(z_i)).$$

If $\tilde{\mathcal{A}}(1,2,\cdots n)$ is an NMHV amplitude then the amplitudes on the left and right will either be MHV amplitudes or amplitudes with fewer than two negative helicities, which we discard, with the exception of the triple-gluon vertex with helicities $(+,+,-)$.

For further reductions (e.g.NNMHV) the amplitudes on the left and right of the intermediate gluon will also include NMHV amplitudes, etc.

Thus, the procedure for calculating an NMHV amplitude is as follows:

1. Mark two adjacent gluon lines $(n-1)$, n, where the first has negative helicity and the second has positive helicity (they don't actually need to be adjacent, but it simplifies things if they are.)

2. Draw all diagrams in which one marked gluon is on the left and the other is on the right, with an intermediate gluon propagating from helicity, h to helicity $-h$, and where the amplitudes on either side are either MHV amplitudes or a triple-gluon vertex.

3. Calculate the MHV (or triple-gluon) amplitudes on the left and right with momenta and helicities
$$p_i^{h_i}, p_{i+1}^{h_{i+1}}, \cdots \hat{p}_{n-1}(z_i)^-, \hat{q}_{i,n-1}(z_i)^h$$
and
$$\hat{p}_n(z_i)^+, p_1^{h_1}, \cdots p_{i-1}^{h_{i-1}}, -\hat{q}_{i,n-1}(z_i)^{-h},$$
respectively, where
$$z_i = \frac{q_{i,n-1}^2}{[p_{n-1}|\gamma\cdot q_{i,n-1}|p_n\rangle}$$
and
$$\begin{aligned} |\hat{p}_n(z_i)\rangle &= |p_n\rangle + z_i|p_{n-1}\rangle \\ |\hat{p}_{n-1}(z_i)] &= |p_{n-1}] - z_i|p_n] \\ \hat{q}_{i,n-1}(z_i) &= -\sum_i^{n-2} p_i - \hat{p}(z_i) \end{aligned}$$
Note that
$$\hat{q}_{i,n-1}(z_i)^2 = 0.$$

4. Sum
$$\tilde{\mathcal{A}}(p_i, p_{i+1}, \cdots \hat{p}_{n-1}(z_i), \hat{q}_{i,n-1}(z_i)^h)\frac{1}{q_{i,n-1}^2}\tilde{\mathcal{A}}(\hat{p}_n(z_i), p_1, \cdots p_{i-1}, -\hat{q}_{i,n-1}(z_i)^{-h})$$
over all such diagrams.

The result needs some manipulation in order to simplify it. The following relations are useful in this respect

$$\begin{aligned}\langle k|\hat{q}_{i,n-1}(z_i)\rangle[\hat{q}_{i,n-1}(z_i)|p_n] &= \langle k|\gamma\cdot q_{i,n-1}|p_n]\\ \langle p_{n-1}|\hat{q}_{i,n-1}(z_i)\rangle[\hat{q}_{i,n-1}(z_i)|k] &= \langle p_{n-1}|\gamma\cdot q_{i,n-1}|k].\end{aligned}$$

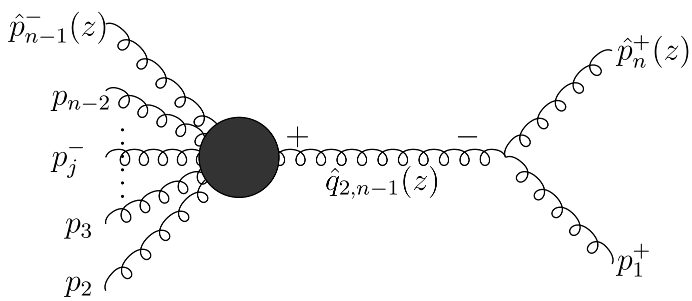
These are seen from the fact that

$$\frac{(1+\gamma^5)}{2}\gamma\cdot\hat{q}_{i,n-1}(z) = \frac{1}{2}(1+\gamma^5)\gamma\cdot q_{i,n-1} + z|p_{n-1}\rangle[p_n|$$

Example 1:

As a first example, we apply this to the n-point MHV vertex in order to prove the Parke-Taylor formula by induction, assuming that it is correct for the $(n-1)$-point function.

There is only one permitted graph, which has an MHV vertex at one end containing both the negative helicity external gluons, and a triple-gluon vertex with two positive helicity external gluons at the other end. The intermediate gluon propagates with the positive helicity end attached to the MHV vertex and the negative end attached to the triple gluon vertex.



The left-handed "vertex" is an $(n-1)$-point MHV vertex

$$i(-g)^{(n-3)}\frac{\langle p_j|\hat{p}_{n-1}(z)\rangle^4}{\langle p_2|p_3\rangle\cdots\langle\hat{p}_{n-1}(z)|\hat{q}_{2,n-1}(z)\rangle\langle\hat{q}_{2,n-1}(z)|p_2\rangle}$$

but $|\hat{p}_{n-1}(z)\rangle = |p_{n-1}\rangle$ so we may write this as

$$i(-g)^{(n-3)}\frac{\langle p_j|p_{n-1}\rangle^4}{\langle p_2|p_3\rangle\cdots\langle p_{n-1}|\hat{q}_{2,n-1}(z)\rangle\langle\hat{q}_{2,n-1}(z)|p_2\rangle}$$

The right-handed vertex is the conjugate of the MHV triple-gluon vertex (as it is a $(+,+,-)$ vertex)

$$-ig\frac{[\hat{p}_n(z)|p_1]^3}{[p_1|\hat{q}_{2,n-1}(z)][\hat{q}_{2,n-1}(z)|[\hat{p}_n(z)]}$$

In this case $|\hat{p}_n(z)] = |p_{n-1}]$, so we may write this as

$$-ig\frac{[p_n|p_1]^3}{[p_1|\hat{q}_{2,n-1}(z)][\hat{q}_{2,n-1}(z)|[p_n]}$$

and the propagator is

$$\frac{-i}{q^2_{2,n-1}} \;=\; \frac{-i}{(p_n+p_1)^2} \;=\; \frac{i}{\langle p_1|p_n\rangle[p_1|p_n]}$$

Combining pairs denominator terms and using

$$\frac{(1+\gamma^5)}{2}\gamma\cdot\hat{q}_{i,n-1}(z) \;=\; \frac{1}{2}(1+\gamma^5)\gamma\cdot q_{i,n-1} + z|p_{n-1}\rangle[p_n|,$$

we have

$$\langle p_2|\hat{q}_{2,n-1}(z)\rangle[\hat{q}_{2,n-1}(z)|p_n] \;=\; \langle p_2|\gamma\cdot q_{2,n-1}|p_n] \;=\; \langle p_2|p_1\rangle[p_1|p_n],$$

where we have used $q_{2,n-1} = p_1 + p_n$ and $\gamma\cdot p_n|p_n] = 0$

Similarly

$$\langle p_{n-1}|\hat{q}_{2,n-1}(z)\rangle[\hat{q}_{2,n-1}(z)|p_1] \;=\; \langle p_{n-1}|\gamma\cdot q_{2,n-1}|p_1] \;=\; \langle p_{n-1}|p_1\rangle[p_{n-1}|p_n].$$

Piecing together we see that there are three factors of $[p_1|p_n]$ in the denominator which cancel $[p_1|p_n]^3$ in the numerator and we are left with

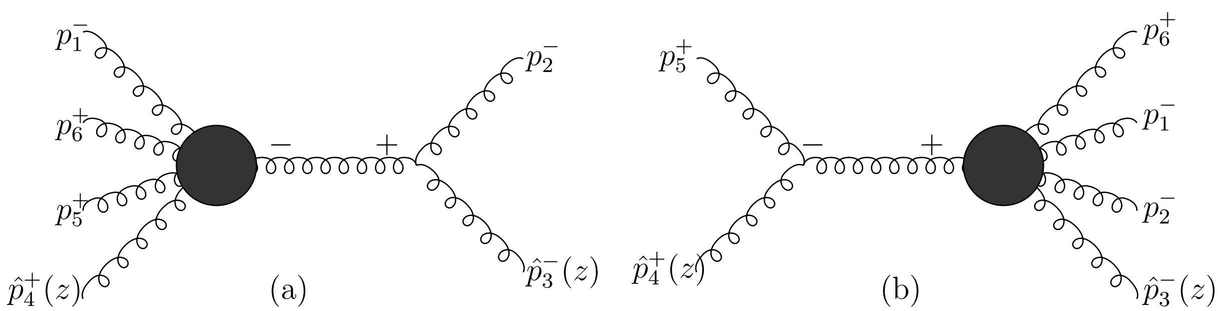
$$i(-g)^{(n-2)}\frac{\langle p_j|p_{n-1}\rangle^4}{\langle p1|p_2\rangle\langle p_2|p_3\rangle\cdots\langle p_{n-1}|p_n\rangle\langle p_n|p_1\rangle},$$

which is the Parke-Taylor result for the n-point amplitude.

Since we have shown explicitly that the Parke-Taylor formula works for $n=4$, this completes the inductive proof that it works for any n-point amplitude.

Example 2:
Now we calculate a 6-point NMHV amplitude, $\tilde{\mathcal{A}}(p_1^-, p_2^-, p_3^-, p_4^+, p_5^+, p_6^+)$.

There are two permitted graphs



Graph (b) can be obtained from graph(a) by complex conjugation and a change of momentum variables, so we will just look at graph (a).

The right vertex is

$$-ig\frac{\langle p_2|\hat{p}_3(z)\rangle^3}{\langle\hat{p}_3(z)|\hat{q}_{2,3}(z)\rangle\langle\hat{q}_{2,3}(z)|p_2\rangle}$$

which we may write as

$$-ig\frac{\langle p_2|p_3\rangle^3}{\langle p|\hat{q}_{2,3}(z)\rangle\langle\hat{q}_{2,3}(z)|p_2\rangle},$$

since $|\hat{p}_3(z)\rangle = |p_3\rangle$.

The left "vertex" is

$$ig^4\frac{\langle p_1|\hat{q}_{2,3}(z)\rangle^3}{\langle\hat{q}_{2,3}(z)|\hat{p}_4(z)\rangle\langle\hat{p}_4(z)|p_5\rangle\langle p_5|p_6\rangle\langle p_6|p_1\rangle}$$

The propagator is

$$\frac{-i}{\langle p_2|p_3\rangle[p_3|p_2]}$$

and

$$z \;=\; \frac{\langle p_2|p_3\rangle[p_3|p_2]}{\langle p_3|\gamma\cdot(p_2+p_3)|p_4]} \;=\; \frac{[p_2|p_3]}{[p_2|p_4]},$$

(using $\langle p_3|p_3| = 0$)

$$\langle p_1|\hat{q}_{2,3}(z)\rangle[\hat{q}_{2,3}(z)|p_4] \;=\; -\langle p_1|\gamma\cdot(p_2+p_3)|p_4]$$

or

$$\langle p_1|\hat{q}_{2,3}(z)\rangle \;=\; -\frac{\langle p_1|\gamma\cdot(p_2+p_3)|p_4]}{[\hat{q}_{2,3}(z)|p_4]}$$

and

$$\langle p_2|\hat{q}_{2,3}(z)\rangle[\hat{q}_{2,3}(z)|p_4] \;=\; -\langle p_2|\gamma\cdot(p_2+p_3)|p_4]$$

or

$$\langle p_2|\hat{q}_{2,3}(z)\rangle \;=\; -\frac{\langle p_2|\cdot p_3|p_4]}{[\hat{q}_{2,3}(z)|p_4]}$$

and

$$\langle p_3|\hat{q}_{2,3}(z)\rangle[\hat{q}_{2,3}(z)|p_4] \;=\; -\langle p_3|\gamma\cdot(p_2+p_3)|p_4]$$

or

$$\langle p_3|\hat{q}_{2,3}(z)\rangle \;=\; -\frac{\langle p_3|\gamma\cdot p_2|p_4]}{[\hat{q}_{2,3}(z)|p_4]}$$

Also,

$$|\hat{p}_4(z)\rangle \;=\; |p_4\rangle + z|p\rangle \;=\; |p_4\rangle + \frac{[p_2|p_3]}{[p_2|p_4]}|p_3\rangle$$

which leads to

$$\langle\hat{p}_4(z)|p_5\rangle \;=\; \langle p_4|p_5\rangle + \frac{[p_2|p_3]\langle p_3|p_5\rangle}{[p_2|p_4]} \;=\; \frac{p_2|\gamma\cdot(p_3+p_4)|p_5\rangle}{[p_2|p_4]}$$

The most difficult one is

$$\begin{aligned}
\langle\hat{p}_4(z)|\hat{q}_{2,3}(z)\rangle[\hat{q}_{2,3}(z)|p_4] \;&=\; \langle\hat{p}_4(z)|\gamma\cdot(p_2+p_3)|p_4] \\
&=\; \langle p_4|\gamma\cdot(p_2+p_3)|p_4] + \frac{[p_2|p_3]}{[p_2|p_4]}\langle p_3|\gamma\cdot(p_2+p_3)|p_4] \\
&=\; \langle p_4|\gamma\cdot(p_2+p_3)|p_4] + [p_2|p_3]\langle p_3|p_2\rangle \\
&=\; 2(p_2\cdot p_4 + p_3\cdot p_4 + p_2\cdot p_3) \;=\; (p_2+p_3+p_4)^2
\end{aligned}$$

Piecing together we find that the $[\hat{q}_{2,3}(z)|p_4]$ factors cancel and we are left with

$$ig^4\frac{1}{(p_2+p_3+p_4)^2}\frac{\langle p_1|\gamma\cdot(p_2+p_3)|p_4]^3}{\langle p_2|p_3\rangle\langle p_3|p_4\rangle\langle p_5|p_6\rangle\langle p_6|p_1\rangle\langle p_5|\gamma\cdot(p_3+p_4)|p_2]}$$

To this we must add the contribution from graph (b) (which can be obtained from the conjugate of the above expression with a rotation of momenta), so the final result becomes quite complicated. In the next session we will discuss an effective field theory from which the result can be obtained far more simply.