6 Colour Factors

It is now time to restore the colour factors.

The generators T^a in the defining representation are normalised such that

$$\mathrm{Tr}(T^a T^b) \;=\; \delta^{ab},$$

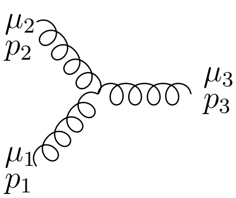
(without the usual factor of $\frac{1}{2}$), so that the commutation relations are

$$\left[T^a, T^b\right] \;=\; i\sqrt{2} f^{abc} T^c$$

Now for a pure gluon amplitude the total amplitude, $\mathcal{A}(1,2,\cdots n)$, is given in terms of the coloured ordered amplitudes $\tilde{\mathcal{A}}(1,2,\cdots n)$ by

$$\mathcal{A}(1,2,\cdots n) \;=\; \sum_{non-cyclic\ perms.} \tilde{\mathcal{A}}(1,\{2,\cdots n\})\mathrm{Tr}\left(T^{a_1}T^{\{a_2}\cdots T^{a_n\}}\right)$$

Let us see how this works for the fundamental vertices.



Define

$$\tilde{V}(1,2,3) \;\equiv\; \frac{1}{\sqrt{2}} ig\left(g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3} + g_{\mu_2\mu_3}(p_2-p_3)_{\mu_1} + g_{\mu_3\mu 1}(p_3-p_1)_{\mu_2}\right)$$

The only non-cyclic permutation is $\tilde{V}(3,2,1)$, and we can see that

$$\tilde{V}(3,2,1) \;=\; -\tilde{V}(3,2,1)$$

so the complete Feynman rule is

$$V(1,2,3) \;=\; \tilde{V}(1,2,3)\left(\mathrm{Tr}(T^{a_1}T^{a_2}T^{a_3}) - \mathrm{Tr}(T^{a_3}T^{a_2}T^{a_1})\right)$$

From the commutation rules this is the usual vertex

$$V(1,2,3) \;=\; -gf^{a_1a_2a_3}\left(g_{\mu_1\mu_2}(p_1-p_2)_{\mu_3} + g_{\mu_2\mu_3}(p_2-p_3)_{\mu_1} + g_{\mu_3\mu 1}(p_3-p_1)_{\mu_2}\right)$$

For the four-point vertex we have

$$\tilde{V}(1,2,3,4) \;=\; ig^2 g_{\mu_1\mu_3} g_{\mu_2\mu_4} - \frac{1}{2} ig^2 \left(g_{\mu_1\mu_2} g_{\mu_3\mu_4} + g_{\mu_1\mu_4} g_{\mu_2\mu_3}\right)$$

so that the complete vertex is given by the sum of six terms

$$\begin{aligned} V(1,2,3,4) \;=\;& \tilde{V}(1,2,3,4)\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_3}T^{a_4}\right) + \tilde{V}(1,3,4,2)\mathrm{Tr}\left(T^{a_1}T^{a_3}T^{a_4}T^{a_2}\right) \\ +\;& \tilde{V}(1,4,2,3)\mathrm{Tr}\left(T^{a_1}T^{a_4}T^{a_2}T^{a_3}\right) + \tilde{V}(1,4,3,2)\mathrm{Tr}\left(T^{a_1}T^{a_4}T^{a_3}T^{a_2}\right) \\ +\;& \tilde{V}(1,2,4,3)\mathrm{Tr}\left(T^{a_1}T^{a_2}T^{a_4}T^{a_3}\right) + \tilde{V}(1,3,2,4)\mathrm{Tr}\left(T^{a_1}T^{a_3}T^{a_2}T^{a_4}\right) \end{aligned}$$

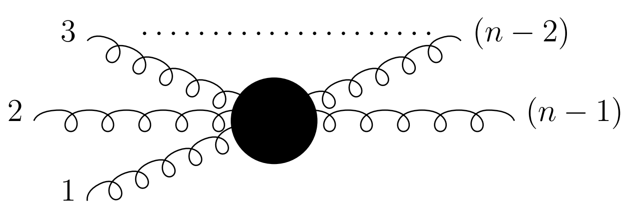
After some manipulations we can rewrite this as

$$\begin{aligned} V(1,2,3,4) \;=\;& ig^2 \left\{\mathrm{Tr}\left(\left[T^{a_1},T^{a_2}\right]\left[T^{a_3},T^{a_4}\right]\right)\left(g_{\mu_1\mu_3}g_{\mu_2\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}\right)\right. \\ +\;& \mathrm{Tr}\left(\left[T^{a_1},T^{a_3}\right]\left[T^{a_2},T^{a_4}\right]\right)\left(g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_1\mu_4}g_{\mu_2\mu_3}\right) \\ +\;& \left.\mathrm{Tr}\left(\left[T^{a_1},T^{a_4}\right]\left[T^{a_2},T^{a_3}\right]\right)\left(g_{\mu_1\mu_2}g_{\mu_3\mu_4} - g_{\mu_2\mu_4}g_{\mu_1\mu_3}\right)\right\}, \end{aligned}$$

which is the usual four-point vertex Feynman rule.

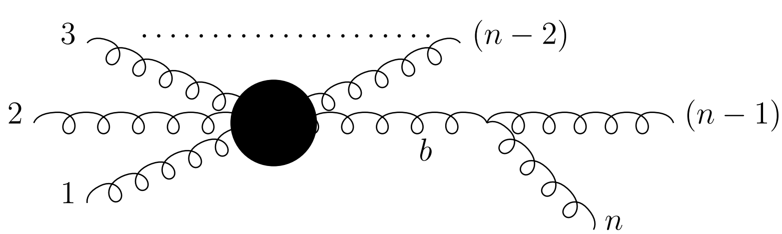
In order to demonstrate that this works for any amplitude we will assume that it works for amplitudes up to $(n-2)$-point and show that it then works for the n-point amplitude. We have shown explicitly that this works for $n = 3$ (it also works trivially for $n = 2$ - the gluon propagator whose colour factor is $\mathrm{Tr}(T^aT^b) = \delta^{ab}$).

Consider an $(n-1)$-point colour-ordered amplitude



to which we wish to add an n^{th} gluon whilst maintaining the colour ordering. Using the cyclic symmetry, this means that the n^{th} gluon is either attached to gluon 1 or to gluon $n-1$.

In the latter case we have a graph of the form:



The colour factor for the factor on the left is

$$\mathrm{Tr}\left(T^{a_1}T^{a_2}\cdots T^{a_{n-2}}T^{b}\right)$$

whereas the colour factor for the (colour ordered) triple-gluon coupling involving gluon n is

$$\mathrm{Tr}\left(T^{b}T^{a_{n-1}}T^{a_n}\right),$$

where b is the colour of the internal gluon that attached the two parts of the graph.

Now performing the sum over index b gives two terms, using

$$\left(T^b\right)^i_j\left(T^b\right)^k_l \;=\; \delta^i_l\delta^k_j - \frac{1}{N}\delta^i_j\delta^k_l$$

The first is

$$\mathrm{Tr}\left(T^{a_1}T^{a_2}\cdots T^{a_{n-1}}T^{a_n}\right),$$

which is precisely the colour factor that we want.

The second is the term

$$\frac{1}{N}\mathrm{Tr}\left(T^{a_1}T^{a_2}\cdots T^{a_{n_2}}\right)\delta\left(a_{n-1}a_n\right)$$

This second term vanishes when summed over all possible colour orderings. This is an example of the “Photon Decoupling Theorem”.

The “extra” terms proportional to $1/N$ arises from the fact that the generators of $SU(N)$ are traceless. They would *not* be present if the gauge group were $U(N)$. On the other hand, we know that since the $(N^2)^{th}$ generator of $U(N)$ commutes with the other N^2-1 generators the $(N^2)^{th}$ gluon will decouple from all the rest. The extra term that we have above can be identified with (minus) the contribution that we would have found if the $(N^2)^{th}$ gluon exited. The fact that it decouples tells us that the extra term cancels when summed over all diagrams.

6.1 Colour factors for squared amplitudes

In order to calculate differential cross-sections the amplitude must be squared and each interference between colour ordered amplitudes will have its own colour factor associated with it. The colour factor will involve products traces of products of generators - each colour is summed over and the colour factor is easily determined using the following:

The relation

$$\left(T^a\right)^i_j\left(T^a\right)^k_l \;=\; \delta^i_l\delta^k_j - \frac{1}{N}\delta^i_j\delta^k_l$$

leads to the following useful formulae

$$P\,T^aQ\,T^aR \;=\; P\,R\mathrm{Tr}(Q) - \frac{1}{N}P\,Q\,R$$

where P, Q, R are any colour matrices.

In the special case where $Q = T^b$, which is traceless, we have simply

$$T^aT^bT^a \;=\; -\frac{1}{N}T^b$$

and when Q is the unit matrix, I, we have

$$T^aT^a \;=\; \frac{(N^2-1)}{N}I.$$

Also

$$\mathrm{Tr}\,(T^aP)\,\mathrm{Tr}\,(T^aQ) \;=\; \mathrm{Tr}\,(P\,Q) - \frac{1}{N}\mathrm{Tr}(P)\mathrm{Tr}(Q)$$

Example: As an example we will calculate the colour factor for the interference between the four-point colour ordered amplitudes $\tilde{\mathcal{A}}(1,2,3,4)$ and $\tilde{\mathcal{A}}(1,3,2,4)$. The colour factor is

$$\mathrm{Tr}\,(T^{a_1}T^{a_2}T^{a_3}T^{a_4})\,\mathrm{Tr}\,(T^{a_1}T^{a_3}T^{a_2}T^{a_4})$$

Contracting the index a_1 we get

$$\mathrm{Tr}\,(T^{a_2}T^{a_3}T^{a_4}T^{a_3}T^{a_2}T^{a_4}) - \frac{1}{N}\mathrm{Tr}\,(T^{a_2}T^{a_3}T^{a_4})\,\mathrm{Tr}\,(T^{a_3}T^{a_2}T^{a_4})$$

Contracting the index a_3 we get

$$-\frac{1}{N}\mathrm{Tr}\,(T^{a_2}T^{a_4}T^{a_2}T^{a_4}) - \frac{1}{N}\mathrm{Tr}\,(T^{a_4}T^{a_2}T^{a_2}T^{a_4}) + \frac{1}{N^2}\mathrm{Tr}\,(T^{a_4}T^{a_2})\,\mathrm{Tr}\,(T^{a_4}T^{a_2})$$

Contracting the index a_2 we get

$$\left(\frac{1}{N^2} - \frac{1}{N}\left(1 - \frac{1}{N}\right) + \frac{1}{N^2}\right)\mathrm{Tr}\,(T^{a_4}T^{a_4})$$

and finally

$$\mathrm{Tr}\,(T^{a_4}T^{a_4}) \;=\; N^2 - 1$$

gives the colour factor as

$$\frac{(3-N)(N^2-1)}{N^2}$$

For amplitudes with more than four gluons the algebra becomes more difficult but the method is still straightforward. It is a simple matter to use an algebraic package to carry out these colour sums.

6.2 Inclusion of Fermions

If we have two (incoming) fermions (with opposite helicity) as well as gluons, then we can without loss of generality choose the fermions to be particle 1 and n of the n-point amplitude. If particle n has positive helicity then the colour factor is

$$T^{a_2}T^{a_3}\cdots T^{a_{n-1}}$$

whereas if the particle n has negative helicity the colour factor is

$$T^{a_{n-1}}\cdots T^{a_3}T^{a_2}$$

Note that there is no traced here. When an amplitude is squared the colour factor for each interference term is a single trace in which all of the indices appear twice and can be determined using the above reduction relations for the colour generators.

Example:
We return to the process $q\,\overline{q} \;\rightarrow\; gg$, discussed in section 1.

We considered the graphs

(a) (b)

and found the amplitude

$$ig^2\frac{\langle p_1|p_3\rangle^3}{\langle p_4|p_1\rangle\langle p_1|p_2\rangle\langle p_2|p_3\rangle}$$

The amplitude is accompanied is the colour factor

$$T^{a_3}T^{a_2}$$

The other ordering

(a) (b)

gives a colour ordered amplitude

$$-ig^2 \frac{\langle p_1|p_3\rangle^3}{\langle p_4|p_1\rangle\langle p_1|p_3\rangle\langle p_2|p_3\rangle}$$

This is accompanied by the colour factor

$$T^{a_2}T^{a_3}.$$

When the total,amplitude is squared, the colour factors the colour factor from the squares of the two colour ordered amplitude is

$$\mathrm{Tr}\left(T^{a_3}T^{a_2}T^{a_2}T^{a_3}\right) \;=\; \frac{(N^2-1)^2}{N}$$

whereas the colour factor from the interference between the two is

$$\mathrm{Tr}\left(T^{a_3}T^{a_2}T^{a_3}T^{a_2}\right) \;=\; -\frac{N^2-1}{N}$$