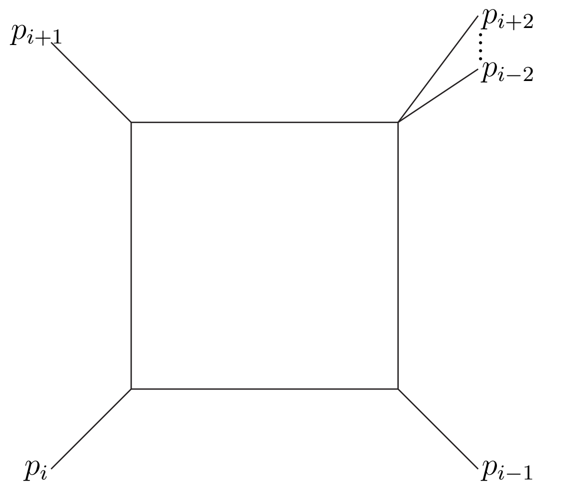
10 Cut Construction

We shall outline the calculation of the colour ordered 1-loop MHV amplitude in $\mathcal{N} = 4$ SUSY using the method of cut construction.

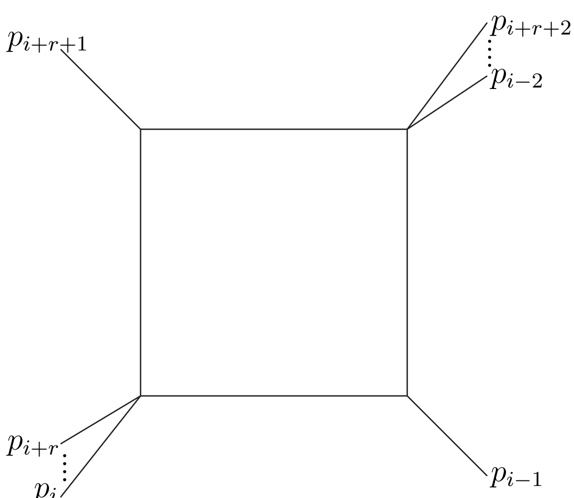
All 1-loop $\mathcal{N} = 4$ SUSY amplitudes can be expressed in terms of scalar box integrals $(I_4(s, t, p_1^2, p_2^2, p_3^2, p_4^2))$, so that in general we have a sum of terms



$\sum_{i=i}^{n} C_i^{1m}(I_4(q_{i,i+1}^2, q_{i-1,i}^2, 0, 0, q_{i+2,i-2}^2, 0)$

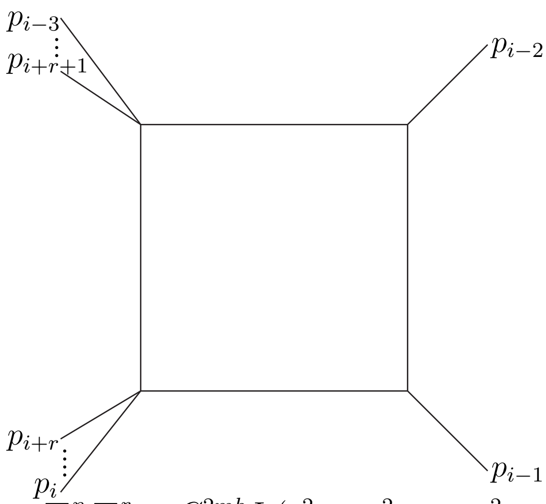
(recall that the indices are all MOD(n) so

$$q_{i+2,i-2} = p_{i+2} + \cdots p_n + p_1 + \cdots p_{i-2}$$

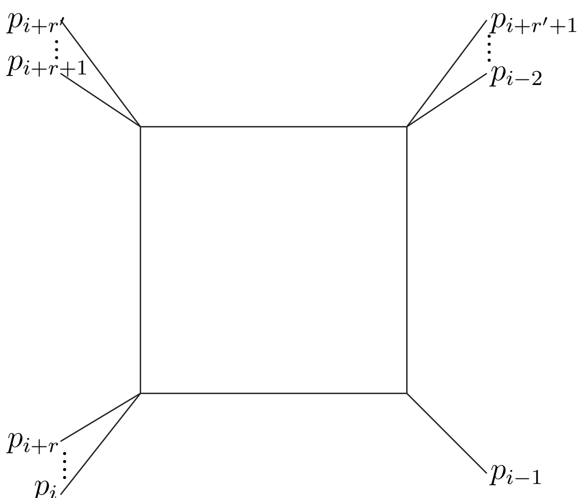


$\sum_i^n \sum_{r=1}^{r_{max}} C_{i,r}^{2me} I_4(q_{i,r+1+1}^2, q_{i-1,r+i}^2, q_{i,i+r}^2, 0, q_{i+r+2,i-2}^2, 0)$

where $r_{max} = (n-4)/2$ for even n and $r_{max} = (n-5)/2$ for odd n.

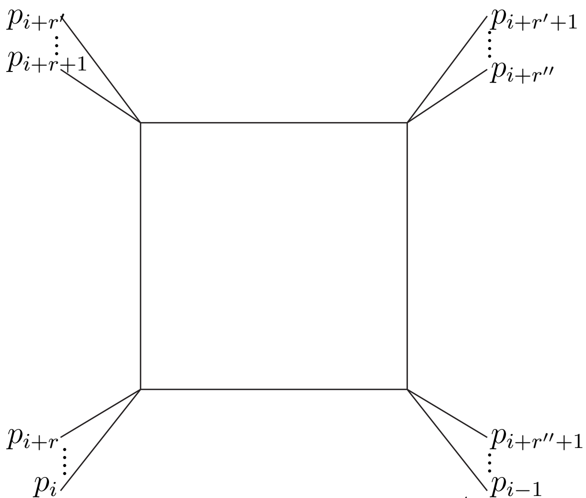


$\sum_i^n \sum_{r=1}^{r_{max}} C_{i,r}^{2mh} I_4(q^2_{i,i-3}, q^2_{i-1,r+i}, q^2_{i,r+i}, q^2_{i+r+1,i-3}, 0, 0)$



$\sum_{i=1}^n \sum_{r'=r+1}^{r_{max}} \sum_{r=1}^{r_{max}} C_{i,r,r'}^{3m} I_4(q^2_{i,i+r'}, q^2_{i-1,r+i}, q^2_{i,i+r}, q^2_{i+r+1,i+r'}, q^2_{i+r'+1,i-2}, 0)$

where $r_{max} = (n-6)/2$ for n even and $(n-5)/2$ for n odd.

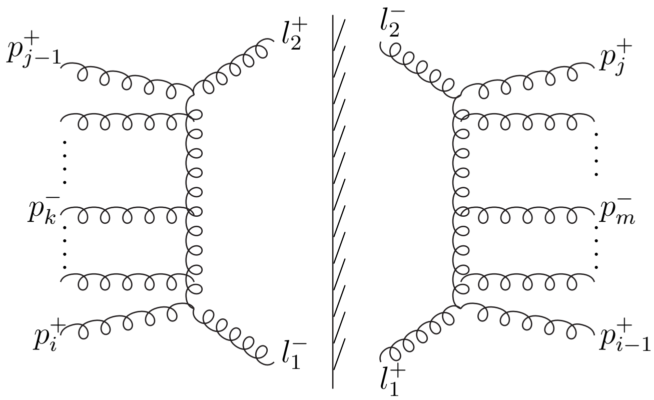


$$\sum_{i=1}^{n}\sum_{r''=r'+1}^{r_{max}}\sum_{r'=r+1}^{r_{max}}\sum_{r=1}^{r_{max}} C^{4m}_{i,r,r',r''} I_4(q^2_{i,i+r'}, q^2_{r''+i+1,r+i}, q^2_{i,i+r}, q^2_{i+r+1,i+r'}, q^2_{i+r'+1,i+r''} q^2_{i+r''+1,i-1})$$

where $r_{max} = (n-6)/2$ for n even and $(n-7)/2$ for n odd.

We need to examine the cuts in order to determine the coefficients C^{1m}_i, $C^{2me}_{i,r}$, $C^{2mh}_{i,r}$, $C^{3m}_{i,r,r'}$, $C^{4m}_{i,r,r',r''}$.

Consider the one-loop MHV amplitude from a gluon loop a cut between external legs $i-1$ and i and between legs $j-1$ and j and suppose that the cut negative helicity external states k and m are on either side of the cut.



The product of the MHV's on either side of the cut is

$$(-ig)^n \frac{\langle p_k|l_2\rangle^4}{\langle l_1|p_i\rangle\cdots\langle p_{j-1}|l_2\rangle\langle l_2|l_1\rangle}\,\frac{\langle l_1|p_m\rangle^4}{\langle l_2|p_j\rangle\cdots\langle p_{i-1}|l_1\rangle\langle l_1|l_2\rangle}$$

which we can write as

$$-i\frac{\mathcal{A}_{tree}(p_1^+\cdots p_k^-\cdots p_m^-\cdots p_n^+)}{\langle p_k\ p_m\rangle^4}\,\frac{\langle p_{j-1}|p_j\rangle\langle p_{i-1}|p_i\rangle\langle p_k|l_2\rangle^4\langle l_1|p_m\rangle^4}{\langle l_1|l_2\rangle^2\langle p_{j-1}|l_2\rangle\langle l_1|p_i\rangle\langle p_{i-1}|l_1\rangle\langle l_2|p_j\rangle}$$

Now add the contribution from internal gluons with the helicities of the internal gluons reversed ($l_1 \leftrightarrow l_2$) and the contributions from the 4 Majorana fermions and the three complex scalars, obtained using the supersymmetry Ward identities.

After some *considerable* algebra we end up with

$$-i\mathcal{A}_{tree}(p_1^+ \cdots p_k^- \cdots p_m^- \cdots p_n^+)\frac{\langle l_1|l_2\rangle^2 \langle p_{j-1}|p_j\rangle\langle p_{i-1}|p_i\rangle}{\langle p_{i-1}|l_1\rangle\langle p_i|l_1\rangle\langle p_j|l_2\rangle\langle p_{j-1}|l_2\rangle}$$

Exercise: Show that if both the negative helicities are on one side of the cut then the only contributing graphs are the ones with internal gluons and that this leads to a cut graph with the same form as the above expression.

Multiplying top and bottom by

$$[p_i|l_1][p_{i-1}|l_1][p_j|l_2][p_{j-1}|l_2]$$

then the denominators become propagators

$$(l_1+p_i)^2(l_1-p_{i-1})^2(l_2+p_j)^2(l_2-p_{j-1})^2$$

so that we have a cut hexagon integral proprtional to

$$\int \frac{d^4l}{(2\pi)^4}\delta(l_1^2)\delta(l_2^2)\frac{\langle l_1|l_2\rangle^2\langle p_{j-1}|p_j\rangle\langle p_{i-1}|p_i\rangle[p_i|l_1][p_{i-1}|l_1][p_j|l_2][p_{j-1}|l_2]}{(l_1+p_i)^2(l_1-p_{i-1})^2(l_2+p_j)^2(l_2-p_{j-1})^2}$$

with

$$l_2 = l_1 - p_1 \cdots - p_{i-1}.$$

If we now use the Schouten identity

$$\langle l_1|l_2\rangle\langle p_{j-1}|p_j\rangle \;=\; \langle p_{j-1}|l_2\rangle\langle l_1|p_j\rangle \;-\; \langle p_j|l_2\rangle\langle l_1|p_{j-1}\rangle$$

and

$$\langle l_1|l_2\rangle\langle p_{i-1}|p_i\rangle \;=\; \langle p_{i-1}|l_2\rangle\langle l_1|p_i\rangle \;-\; \langle p_i|l_2\rangle\langle l_1|p_{i-1}\rangle$$

we see that we can always cancel two of the denominator factors in the hexagon and end up with four cut box integrals and we end up with four terms of the form

$$\int \frac{d^4l}{(2\pi)^4}\delta(l_1^2)\delta(l_2)^2\frac{[p_i|l_1]\langle l_1|p_j\rangle[p_j|l_2]\langle l_2|p_i\rangle}{(l_1+p_i)^2(l_2+p_j)^2}$$

which is a contribution to the imaginary part of the loop-integral is a contribution to the imaginary part of the integral (noting that the numerator can be written as a trace)

$$(-ig)^n\mathcal{A}_{tree}\int \frac{d^{4-2\epsilon}}{(2\pi)^{4-2\epsilon}}\frac{\mathrm{Tr}\,(\gamma\cdot p_i\gamma\cdot l_1\gamma\cdot p_j\gamma\cdot l_2)}{l_1^2(l_1+p_1)^2)l_2^2(l_2+p_j)^2}$$

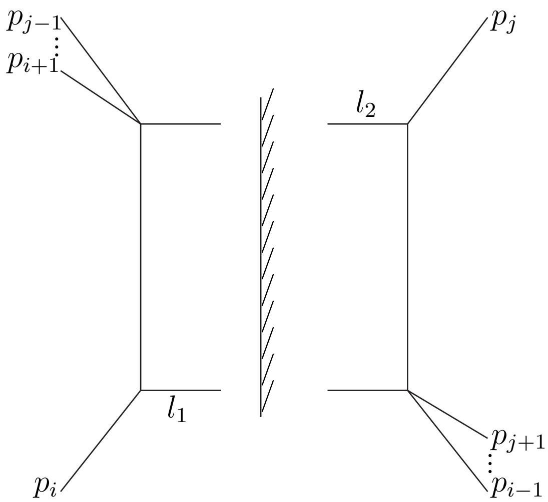
where

$$l_2 = l_1 - p_1 \cdots - p_{i-1}.$$

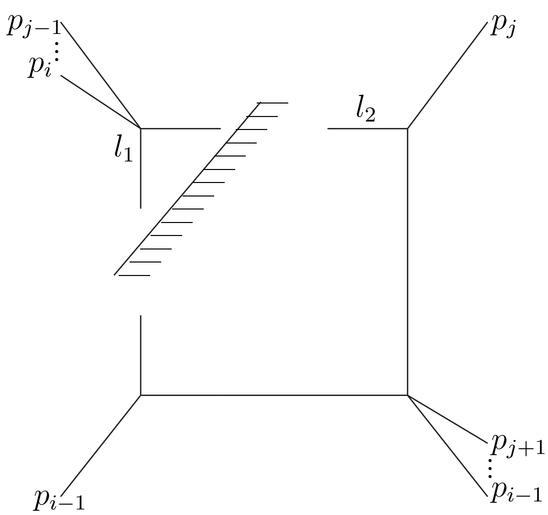
But we see that since p_i^2 and p_j^2 are both zero this is a 2-mass-easy integral.

There are still factors of l in the numerator and when these are reduced using VP reduction they give rise to triangle and bubble graphs, but when all the terms are added together these triangles and bubble integrals cancel as expected for $\mathcal{N} = 4$ SUSY.

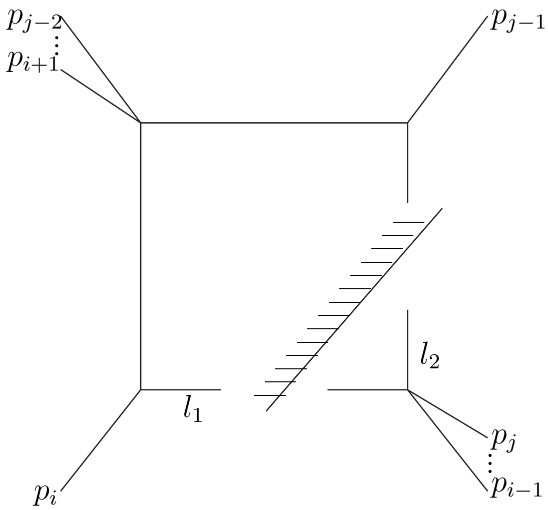
The four different terms obtained by cancelling either $\langle p_{i-1}|l_1\rangle$ or $\langle p_i|l_1\rangle$ and either $\langle p_{j-1}|l_2\rangle$ or $\langle p_j|l_2\rangle$ in the denominator give the four different cuts of the I_4^{2me} integral:



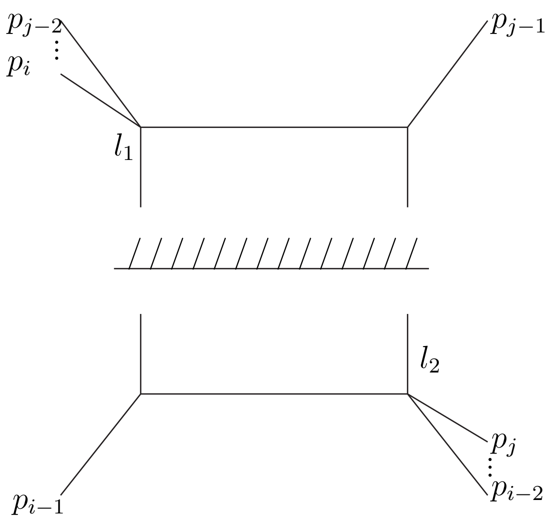
$$\int d^4l \frac{\delta(l_1^2)\delta(l_2^2)}{(l_1 + p_i)^2(l_2 + p_j)^2}$$



$$\int d^4l \frac{\delta(l_1^2)\delta(l_2^2)}{(l_1 - p_{i-1})^2(l_2 + p_j)^2}$$



$$\int d^4 l \frac{\delta(l_1^2)\delta(l_2^2)}{(l_1+p_i)^2(l_2-p_{j-1})^2}$$



$$\int d^4 l \frac{\delta(l_1^2)\delta(l_2^2)}{(l_1-p_{i-1})^2(l_2-p_{j-1})^2}$$

The momentum labels of the external lines have been shuffled a little, but we need to sum over all possible (distinct) ways of cutting, (i.e. over i and j).

It has been shown rigorously by Brandhuber, Spence and Travaglini, that the two-mass-easy integral $I_4(s,t,p_1^2,0,p_3^2,0)$ can be constructed from these four cuts using the sum of dispersion integrals in the cut kinematic variable

$$I_4^{2me} = \frac{i}{2\pi}\left\{\int ds' \frac{\Delta_{s'} I_4^{2me}}{(s-s\prime)} + \frac{\Delta_{t'} I_4^{2me}}{(t-t\prime)} + \frac{\Delta_{p_1^2{}'} I_4^{2me}}{(p_1^2-p_1^2{}')} + \frac{\Delta_{p_3^2{}'} I_4^{2me}}{(p_3^2-p_3^2{}')}\right\}$$

The coefficient of this I_4^{2me} integral is obtained from the part of the trace

$$\mathrm{Tr}\left(\gamma\cdot p_i \gamma\cdot l_1 \gamma\cdot p_j \gamma\cdot l_2\right) = -2\left(l_1\cdot l_2 p_i\cdot p_j - p_i\cdot l_1 p_j\cdot l_2 - -p_i\cdot l_2 p_j\cdot l_1\right)$$

Using

$$l_2 = l_1 + p_i + q_{i+i,j-1}$$

so that

$$p_i \cdot l_2 \;=\; p_i \cdot l_1 + \frac{1}{2}\left(q_{i,j-1}^2 - q_{i+1,j-1}^2\right)$$

similarly

$$\begin{aligned} p_j \cdot l_1 &= p_j \cdot l_2 + \frac{1}{2}\left(q_{j,i-1}^2 - q_{j+1,i-1}^2\right) \\ l_1 \cdot l_2 &= \frac{1}{2} q_{i,j-1}^2 \end{aligned}$$

(using conservation of momentum $l_1 + l_2 = -q_{i,j-1}$) and the relations

$$l_1 \cdot p_i \;=\; \frac{1}{2}(l_1 + p_i)^2$$

$$l_2 \cdot p_j \;=\; \frac{1}{2}(l_2 + p_i)^2$$

The trace becomes

$$\frac{1}{2}\left(q_{i,j-1}^2 q_{i-1,j}^2 - q_{i+1,j-1}^2 q_{j+1,i-1}^2\right),$$

plus terms proportional to $(l_1 + p_i)^2$ or $(l_2 + p_j)^2$ (or both), which give the triangle or bubble graphs that cancel between the four cuts.

Summing over all possible cuts and recalling that for some of these cuts we will generate one-mass integrals, we end up with

$$\begin{aligned} \mathcal{A}_{1-loop}^{\mathcal{N}=4}(p_1^+ \cdots p_k^- \cdots p_m^- \cdots p_n^+) \;=\; & \frac{g^2}{2} \mathcal{A}_{tree}^{\mathcal{N}=4}(p_1^+ \cdots p_k^- \cdots p_m^- \cdots p_n^+) \times \\ & \left\{ \sum_{i=1}^{n} \sum_{r=1}^{r_{max}} \left(q_{i,i+r+1}^2 q_{i-1,i+r}^2 - q_{i,i+r}^2 q_{i+r+2,i-2}^2\right) I_4(q_{i,i+r+1}^2, q_{i-1,i+r}^2, q_{i,i+r}^2, 0, q_{i+r+2,i-2}^2, 0) \right. \\ & \left. + \sum_{i=1}^{n} \left(q_{i,i+1}^2 q_{i-1,i}^2\right) I_4(q_{i,i+1}^2, q_{i-1,i}^2, 0, 0, q_{i+2,i-2}^2, 0) \right\} \end{aligned}$$

For non-MHV amplitudes, we would expect terms which involve the other box integrals, and for $\mathcal{N} = 1$ SUSY we would also expect to pick up triangle and bubble integrals.

10.1 Use of Quadruple Cuts

It would be convenient if we could identify the coefficients of the required box integrals in $\mathcal{N} = 4$ SUSY without having to go through all the manipulations involving the spinors constructed from the loop momentum. Indeed, for non-MHV amplitudes the identification of a particular cut does not unambiguously identify one of the box integrals because a particular cut can be shared by more than one box integral.

Britto, Cachazo, and Feng noticed, however, that there is a "leading singularity" associated with each box integral and that these leading singularities *are* unique to the integral.

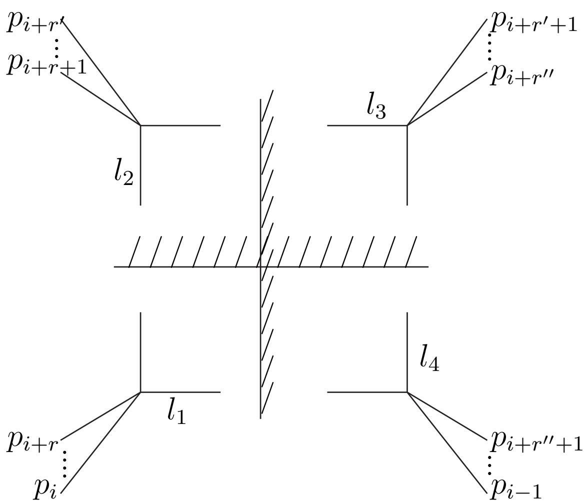
For example, if we look at the four-mass box integral,

$$I_4^{4m} \equiv I_4(q^2_{i,r'+i}, q^2_{i+r''+1,i+r}, q^2_{i,i+r}, q^2_{i+r+1,r'+i}, q^2_{i+r'+1,r''+i}, q^2_{i+r''+1,i-1})$$

and let the variable $q^2_{i,i+r}$ become much larger than any of the other kinematic variables then the integral has a double logarithm term

$$\ln\left(-q^2_{i,i+r}\right)\ln\left(q^2_{i+r'+1,i+r''}\right),$$

which does NOT occur in any other integral with the same number of external particles. All other integral have a similar unique leading logarithm in the limit where one of the kinematic variables becomes large.



$$l_1 = l$$
$$l_2 = l + q_{i,i+r}$$
$$l_3 = l + q_{i,i+r'}$$
$$l_4 = l + q_{i,i+r''}$$

The coefficient of this leading singularity is obtained from the "quadruple cuts" that is the graph cut in the s-channel and $t-$channel in such a way that all the internal propagators are on shell. The corners of the graph are nothing other than on-shell tree-level amplitudes (not necessarily MHV). The leading cut is therefore given by

$$\int \frac{d^4 l}{(2\pi)^4}\delta(l^2)\delta((l+q_{i,i+r})^2)\delta((l+q_{i,i+r'})^2)\delta((l+q_{i,i+r''})^2)$$

$$\times\mathcal{A}_{tree}(l, p_i \cdots l + q_{i,i+r})\mathcal{A}_{tree}(l + q_{i,i+r}, p_{i+r+1} \cdots l + q_{i,i+r'})$$
$$\times\mathcal{A}_{tree}(l + q_{i,i+r'}, p_{i+r'+1} \cdots l + q_{i,i+r''})\mathcal{A}_{tree}(l + q_{i,i+r''}, p_{i+r''+1} \cdots l)$$

The four delta functions determine the loop momentum l so there is no further integration. There may be a discrete set of solutions $l_m, \quad m = 1 \cdots s$ and the cut is obtained from averaging over these. The integral merely gives the Jacobian of the arguments of the delta-function w.r.t. the components of l. Writing the integral as $I_4(s, t, m_1^2, m_2^2, m_3^2, m_4^2)$, This Jacobian is

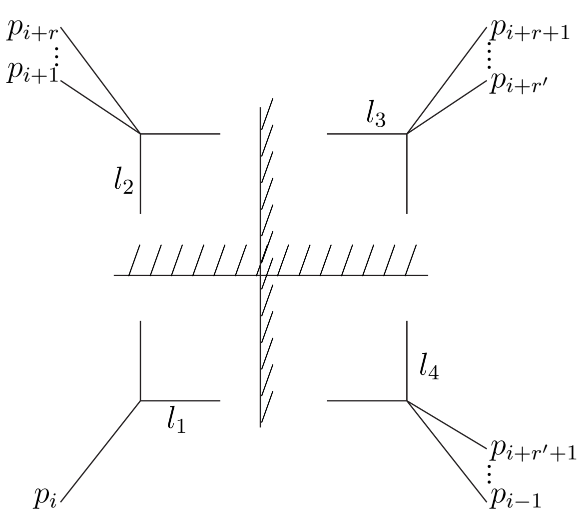
$$J = 4\lambda^{1/2}(st, m_1^2 m_3^2, m_2^2 m_4^2),$$
$$\lambda(x, y, z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz$$

The leading singularity of the four-mass box integral is then

$$\frac{1}{J}\frac{1}{s}\sum_{m=1}^{s}\mathcal{A}_{tree}(l_m, p_i \cdots l_m + q_{i,i+r})\mathcal{A}_{tree}(l_m + q_{i,i+r}, p_{i+r+1} \cdots l_m + q_{i,i+r'})$$
$$\times\mathcal{A}_{tree}(l_m + q_{i,i+r'}, p_{i+r'+1} \cdots l_m + q_{i,i+r''})\mathcal{A}_{tree}(l_m + q_{i,i+r''}, p_{i+r''+1} \cdots l_m).$$

By comparing this with the coefficient of the double logarithm of the four-mass box integral we can then obtain the coefficient of the four-mass box integral for that particular (pinched) diagram.

We can do exactly the same for the other box integrals, but care must be taken in the case of boxes with massless legs.



The factor

$$\mathcal{A}_{tree}(l_1, p_i, l_1 + p_i)$$

only exists if we make the transformation to complex momentum and then use the 3-point MHV vertex or its conjugate, as appropriate. (Britto, Cachazo, Feng solved this problem by working with a metric whose signature was (-1,-1,1,1), which yields the same results).