11 Beyond Cut Constructibility

A one-loop amplitude with $\mathcal{N} = 4$ SUSY consists of a loop of gluons, four Majorana fermion loops and three complex scalar loops

$$\mathcal{A}^{\mathcal{N}=4} = (\mathcal{A}_g + 4\mathcal{A}_f + 3\mathcal{A}_s)$$

A chiral (matter) loop correction in $\mathcal{N} = 1$ SUSY consists of a loop of Majorana fermions and a loop of complex scalars

$$\mathcal{A}^{\mathcal{N}=1}_{chiral} = (\mathcal{A}_f + \mathcal{A}_s)$$

If we solve for the gluon loop (pure QCD) we obtain

$$\mathcal{A}_g = \mathcal{A}^{\mathcal{N}=4} - 4\mathcal{A}^{\mathcal{N}=1}_{chiral} + \mathcal{A}_s$$

The first two contributions are cut constructible, so we can complete the non-cut-constructible part of QCD by considering the non-cut-constructible part of the contribution from a loop of scalar particles.

The impediment to cut constructibility comes from the tensor bubble integral which obeys the relation

$$I_2(l^\mu l^\nu) - \left(\frac{q^\mu q_1 \nu}{3} - \frac{g^{\mu\nu}}{12}\right) I_2[1] = \frac{1}{18}\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right),$$

However, as shown by Brandhuber, Spence and Traviglini, this is only true in the limit of four dimensions. If we remain in $4 - 2\epsilon$ dimensions we obtain

$$I_2(l^\mu l^\nu) - \left(\frac{q^\mu q_1 \nu}{3} - \frac{g^{\mu\nu}}{12}\right) I_2[1] = \frac{1}{18}\left(g^{\mu\nu}q^2 - q^\mu q^\nu\right)\left(1 - \epsilon\ln(-q^2) + \mathcal{O}(\epsilon^2)\right),$$

which does indeed have an imaginary part, albeit at order ϵ.

Working in $4-2\epsilon$ dimensions, we assume that the external momenta exist in four dimensions only, but the the loop momentum $\tilde{l}$ is in $4 - 2\epsilon$ dimensions so that

$$\tilde{l}^2 = l^2 - \sum_i^{-2\epsilon} l_i^2 = l^2 - \mu^2$$

A loop integral may therefore be written

$$I_n[f(l)] = \pi^{2-\epsilon}\int \frac{d^4 l d^{-2\epsilon}\mu f(l)}{\prod_{i=0}^{n-1}((l + q_{i-1})^2 - \mu^2)},$$

where l is understood to be in 4-dimensions. The integral is first performed over l in four dimensions, treating μ as a mass, common to all internal propagators, followed by an integral over μ, now treated as a vector in $(d-4)$ dimensions and analytically continued to -2ϵ dimensions.

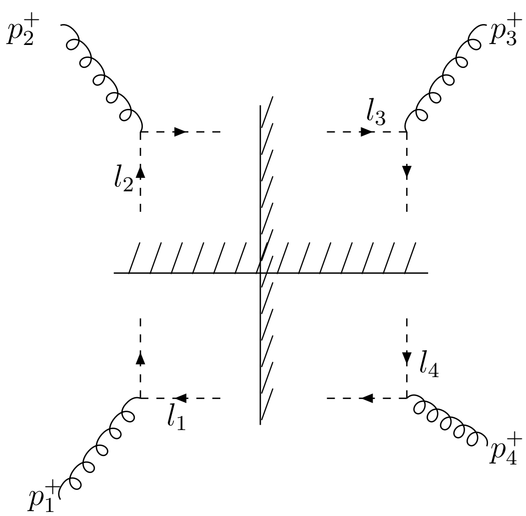
We adopt the approach considered in the last section of looking first at the quadruple cuts in order to obtain the leading singularities and hence determine the coefficients of the box integrals. Unlike the case of $\mathcal{N}=4$ SUSY there will also, in general, be triangle and bubble graphs and we will need to examine triple cuts in order to identify the coefficients of these.

11.1 The amplitude $\mathcal{A}_{1-loop}(p_1^+, p_2^+, p_3^+, p_4^+)$

This is forbidden at tree-level and is therefore not cut constructible in four dimensions.

This tells us immediately that there are no contributions from the cut constructible supersymmetric contributions and so the entire amplitude arises from the contribution from the scalar loop.

We first examine the quadruple cut.



The product of the vertices is

$$2g^4 l_1 \cdot \epsilon_1^+ \, l_2 \cdot \epsilon_2^+ \, l_3 \cdot \epsilon_3^+ \, l_4 \cdot \epsilon_4^+$$

Using the representation

$$\epsilon_{i\,\mu}^+ \; = \; \frac{\langle n_i|\gamma_\mu|p_i]}{\sqrt{2}\langle n_i|p_i\rangle},$$

this becomes

$$g^4 \frac{\langle n_1|\gamma \cdot l_1|p_1]\langle n_2|\gamma \cdot l_2|p_2]\langle n_3|\gamma \cdot l_3|p_3]\langle n_4|\gamma \cdot l_4|p_4]}{\langle n_1|p_1\rangle\langle n_2|p_2\rangle\langle n_3|p_3\rangle\langle n_4|p_4\rangle}$$

Multiplying numerator and denominator by

$$\langle p_1|p_2\rangle\langle p_3|p_4\rangle$$

and using

$$l_i = l_{i-1} + p_i$$

and

$$|p_i]\langle p_i| \;=\; \frac{(1-\gamma^5)}{2}\gamma\cdot p_i,$$

we get

$$l_1^2 l_3^2 \frac{[p_1|p_2][p_3|p_4]}{\langle p_1|p_2\rangle\langle p_3|p_4\rangle}$$

We see immediately that in four dimensions this vanishes because of the on-shell condition for l_1 and l_3 provided by the cut. But this "zero" is multiplied by a box integral which is infrared divergent, so care is needed and we must move to $4-2\epsilon$ dimensions for which the cut condition sets $l_1^2 = \mu^2$ and $l_3^2 = \mu^2$.

The contribution from the scalar box integral is therefore

$$g^4 \frac{[p_1|p_2][p_3|p_4]}{\langle p_1|p_2\rangle\langle p_3|p_4\rangle} \int \frac{d^4 l}{(2\pi)^4} \pi^{-\epsilon} d^{-2\epsilon}\mu \frac{\mu^4}{(l^2-\mu^2)((l-q_1)^2-\mu^2)((l-q_2)^2-\mu^2)((l-q_3)^2-\mu^2)}$$

We can Feynman parametrise and perform the four-dimensional integral over l to obtain

$$\frac{g^4}{16\pi^2} \frac{[p_1|p_2][p_3|p_4]}{\langle p_1|p_2\rangle\langle p_3|p_4\rangle} \int d\alpha_1 \cdots d\alpha_4 \delta\left(1 - \sum_i \alpha_i\right) d^{-2\epsilon}\mu \frac{\mu^4}{(A^2-\mu^2)^2}$$

Now use the standard $d-$dimensional integral

$$\pi^{d/2} \int d^d l \frac{l^4}{(l^2 - A^2)^2} \;=\; -\frac{(2+d)}{2}\Gamma(1-d/2)(A^2)^{d/2}$$

and set $d = -2\epsilon$ to arrive at the result

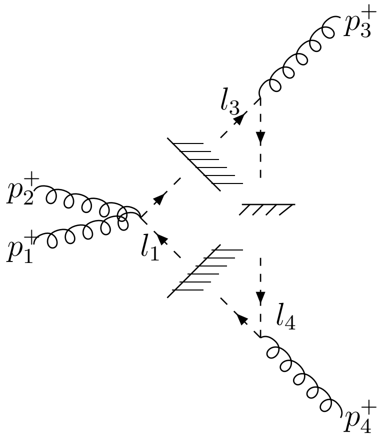
$$-\frac{g^4}{16\pi^2} \frac{[p_1|p_2][p_3|p_4]}{\langle p_1|p_2\rangle\langle p_3|p_4\rangle} \int d\alpha_1 \cdots d\alpha_4 \delta\left(1 - \sum_i \alpha_i\right) (A^2)^{-\epsilon} \;=\; -\frac{g^4}{96\pi^2} \frac{[p_1|p_2][p_3|p_4]}{\langle p_1|p_2\rangle\langle p_3|p_4\rangle} \;+\; \mathcal{O}(\epsilon)$$

11.2 Triple Cuts

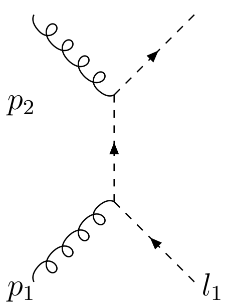
This is not necessarily the entire contribution and we must also examine the triple cuts to see if we can identify any further coefficients of triangle or bubble graphs.

The triple cut graphs are obtained by pinching one of the propagators and then setting all three remaining internal lines on-shell.

For example for the amplitude $\mathcal{A}(p_1^+, p_2^+, p_3^+, p_4^+)$ one of the triple cuts is



The massive vertex with particle 1 and 2 as external legs is actually the subgraph:



(where we have chosen a suitable gauge for the polarisation vectors which suppresses the Feynman graphs involving the four-point coupling and the triple-gluon coupling)

$$\frac{1}{(l_1+p_1)^2-\mu^2)} l_1 \cdot \epsilon_1^+ l_2 \cdot \epsilon_2^+$$

with $l_2 = l_1 + p_1$, and the other vertices give

$$l_3 \cdot \epsilon_3^+ l_4 \cdot \epsilon_4^+$$

so the triple cut is

$$l_1 \cdot \epsilon_1^+ l_2 \cdot \epsilon_2^+ l_3 \cdot \epsilon_3^+ l_4 \cdot \epsilon_4^+ (\pi)^{\epsilon-2} \int \frac{d^4 l d^{-2\epsilon} \mu \delta(l^2-\mu^2)\delta((l+q_2)^2-\mu^2)\delta((l+q_3)^2-\mu^2)}{(l+p_1)^2-\mu^2}.$$

In this case we simply recover a cut of the box integral that was obtained by examining the quadruple cut and we get no new triangle or bubble integrals.

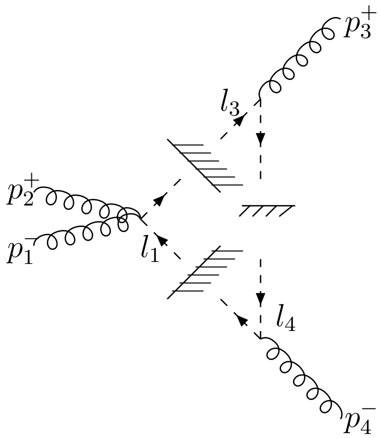
The final result is therefore

$$\mathcal{A}_{1-loop}(p_1^+, p_2^+, p_3^+, p_4^+) \;=\; -\frac{g^4}{96\pi^2} \frac{[p_1|p_2][p_3|p_4]}{\langle p_1|p_2\rangle\langle p_3|p_4\rangle}$$

Note that this is UV and IR convergent and is also a rational function of the kinematic variables - there are no logarithms or dilogarithms which can have cut contributions.

In general, we *do* expect contributions from the triple cut which are not identified from the quadruple cut and which generate triangle and bubble integrals (this is also the case for $\mathcal{N} = 1$ SUSY).

For example, if we consider the one-loop correction to the MHV amplitude $\mathcal{A}(p_1^-, p_2^+, p_3^+, p_4^-)$ one of the triple cut graphs is



The massive vertex contains the propagator

$$\frac{1}{((l_1 + p_1)^2 - \mu^2)}$$

but the numerator will contain up to four powers of the loop momentum.

After performing VP reduction on this numerator we end up with some terms in which the denominator $(l_1 + p_1)^2 - \mu^2)$ has been cancelled, thereby yielding the leading cut of a triangle integral. In general, the coefficient of this triangle will again contain powers of the loop momentum, so that a further VP reduction is needed yielding bubble integrals.