4 The Renormalization Group

4.1 The β-function

The value of renormalized coupling constants are renormalization prescription dependent. In particular, they depend on the value of the momentum μ, at which the infinite Green functions are subtracted - in dimensional regularization this is the mass scale, μ, which controls the dimensionful bare coupling outside four dimensions.

In $d = 4 - 2\varepsilon$ dimensions in the MS scheme, the renormalized dimensionless coupling, $g_R(\mu)$, is related to the dimensionful bare coupling g_0 by

$$g_R(\mu) \;=\; \mu^{-\varepsilon} Z_g(g_R) g_0, \tag{4.1}$$

with

$$Z_g \;=\; \frac{Z_2 Z_3^{1/2}}{Z_1}$$

We define the beta function in d dimensions, $\tilde{\beta}(g_R)$ to be a function of the renormalized coupling, such that

$$\tilde{\beta}(g_R) \;=\; \frac{\partial}{\partial(\ln(\mu))} g_R(\mu)). \tag{4.2}$$

From eq.(4.1) this gives

$$\tilde{\beta}(g_R) \;=\; -\varepsilon g_R + \frac{g_R}{Z_g(g_R)} \frac{\partial Z_g(g_R)}{\partial g_R} \tilde{\beta}(g_R) \tag{4.3}$$

$\tilde{\beta}$ is finite as $\varepsilon \to 0$, so by comparing powers of ε (the coefficient of ε) we must have

$$\tilde{\beta} \;=\; -\varepsilon g_R + \beta, \tag{4.4}$$

where β is the value of $\tilde{\beta}$ in four dimensions.

Using the fact that in the MS scheme, Z_g contains only poles at $\varepsilon = 0$, (at n^{th} order it will contain poles up to order n) so it can be written as

$$Z_g \;=\; 1 + \sum_n \sum_{k=1}^{n} a_k^n \frac{g_R^{2n}}{\varepsilon^k}$$

and substituting eq.(4.4) into the RHS of eq.(4.3)

$$Z_g(g_R)\beta \;=\; -\varepsilon g_R^2 \frac{\partial Z_g(g_R)}{\partial g_R} + \beta \frac{\partial Z_g(g_R)}{\partial g_R} g_R$$

Comparing coefficients of ε^0 we get

$$\beta \;=\; -2g_R^2 \sum_n n a_1^n g_R^{2n-1}, \tag{4.5}$$

i.e. in any order of perturbation theory the β-function may be obtained from (minus $2\times$) the simple pole part of the coupling constant renormalization factor, Z_g.

β has a perturbative expansion

$$\beta(g_R) \;=\; g_R \left[\beta_0 \frac{g_R^2}{16\pi^2} + \beta_1 \left(\frac{g_R^2}{16\pi^2}\right)^2 + \cdots\right]$$

Suppose that in a different renormalization scheme

$$g'_R \;=\; g_R + A\frac{g_R^2}{16\pi^2}$$

$$\begin{aligned}
\beta'(g'_R) \;&=\; g'_R \left[\beta'_0 \frac{g_R'^2}{16\pi^2} + \beta'_1 \left(\frac{g_R'^2}{16\pi^2}\right)^2 + \cdots\right] \\
&=\; g_R \left[\beta_0 \frac{g_R^2}{16\pi^2} + (\beta_1 + 3A\beta_0) \left(\frac{g_R^2}{16\pi^2}\right)^2 + \cdots\right] \\
&=\; g'_R \left[\beta_0 \frac{g_R'^2}{16\pi^2} + \beta_1 \left(\frac{g_R'^2}{16\pi^2}\right)^2 + \cdots\right]
\end{aligned} \tag{4.6}$$

We see here that the first two terms in the β-function are renormalization prescription independent - so we may use the *MS* scheme for convenience (this statement is *not* true beyond the first two terms in the expansion.)

In the case of QED ($g \equiv e$), we have $Z_1 = Z_2$ so that

$$Z_g \;=\; \sqrt{Z_3}.$$

From eq.(3.19) we see that the simple pole part of Z_3 is given at the one loop level by

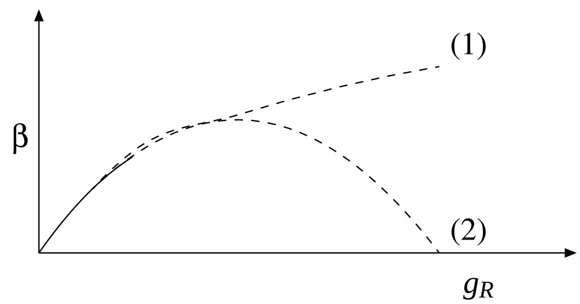
$$-\frac{4}{3}\frac{e^2}{16\pi^2}$$

So we see that in this case β_0 is given by

$$\beta_0^{QED} \;=\; \sum_i \frac{4}{3} Q_i^2, \tag{4.7}$$

the sum and factor Q_i^2 arising from the fact that for the QED calculation we considered the electron as the only charged particle. In reality all charged particles, i, contribute to the higher order corrections to the photon propagator with a coupling proportional to the square of their electric charge, Q_i.

The fact that β_0 is positive means that at least for small values of the coupling, β itself is positive and this means that as the renormalization scale increases, the renormalized coupling constant also rises.



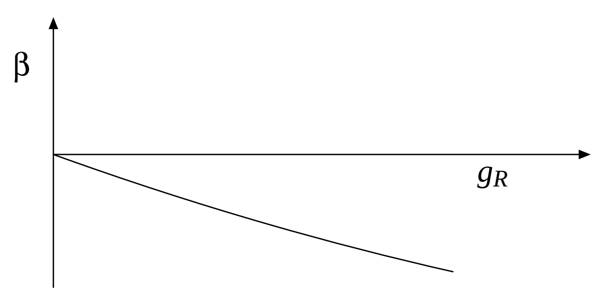
Whether the renormalized coupling continues to increase indefinitely or stops at some "ultraviolet fixed point" depends on whether the higher order contributions to β are such that it remains positive for all values of the coupling (curve(1)) , or whether it acquires negative contributions in higher orders such that it decreases again and crosses the axis at a value of g_R equal to that fixed point (curve(2)). In the second case, once the fixed point has been reached, $\beta = 0$, so that the renormalized coupling ceases to change as μ is further increased.

For non-Abelian gauge theories, the calculation of β_0 is more involved. Here we do *not* have $Z_1 = Z_2$. Furthermore there are extra Feynman graphs for the vertex correction and the gauge boson propagator correction which account for the self-interactions of the gauge bosons (and where appropriate the Faddeev-Popov ghosts). The couplings carry colour factors, so the contributions from each graph will be proportional to Casimir operators of the gauge group in various different representations.

For an $SU(N)$ gauge theory with n_f copies of fermions transforming as the defining representation of $SU(N)$, β_0 takes the value

$$\beta_0 = -\frac{11N - 2n_f}{3} \tag{4.8}$$

Provided the number of fermions is not too large this is negative. Theories with negative values of β_0 are called "asymptotically free" theories.



As $\mu \to \infty$, the renormalized coupling goes to zero. Non-Abelian gauge theories are the only known examples of such asymptotically free gauge theories.

For such theories we may solve the differential equation

$$\beta(g_R) \;=\; \frac{\partial g_R(\mu)}{\partial \ln(\mu)} \;=\; \beta_0 \frac{g_R^3}{16\pi^2} \;+\; \beta_1 \frac{g_R^5}{(16\pi^2)^2} \;+\; \cdots$$

It is usually more convenient to work in terms of

$$\alpha_R \;\equiv \frac{g_R^2}{4\pi},$$

for which the differential equation becomes

$$\frac{\partial \alpha_R(\mu^2)}{\partial \ln(\mu^2)} \;=\; \beta_0 \frac{\alpha_R^2}{4\pi} \;+\; \beta_1 \frac{\alpha_R^3}{(4\pi)^2}.$$

We have truncated the series at the two-loop level. This is a first order differential equation with a constant of integration, which is usually expressed in terms of the value of the renormalized coupling, α_0, at some reference scale μ_0 - the renormalization prescription must be specified. † The solution of the differential equation, accurate to the order of the truncation is

$$\alpha_R(\mu^2) \;=\; \frac{\alpha_0}{\left[1+\left(\frac{\alpha_0}{4\pi}|\beta_0| - \beta_1 \frac{\alpha_0^2}{(4\pi)^2}\right) \ln\left(\frac{\mu^2}{\mu_0^2}\right)\right]} \tag{4.9}$$

This expression is valid provided μ_0 is sufficiently large that $\alpha_0 \ll 1$ and $\mu \geq \mu_0$.

An older way of writing the solution is to introduce a scale Λ_{QCD} (which is again renormalization prescription invariant) and expressing the renormalized coupling as

$$\alpha_R(\mu^2) \;=\; \frac{4\pi}{\left[|\beta_0| \ln\left(\frac{\mu^2}{\Lambda_{QCD}^2}\right) + \frac{\beta_1}{\beta_0} \ln\left(\ln\left(\frac{\mu^2}{\Lambda_{QCD}^2}\right)\right)\right]}$$

These two expressions are equivalent (up to corrections of order α_0^3) provided we identify

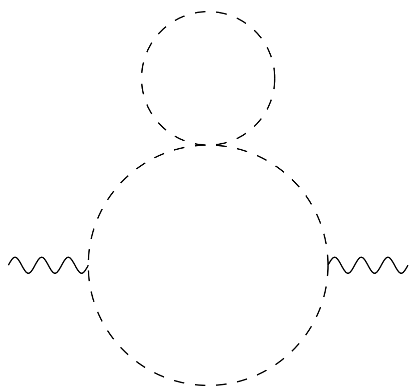
$$\alpha_0 \;=\; \frac{4\pi}{\left[|\beta_0| \ln\left(\frac{\mu_0^2}{\Lambda_{QCD}^2}\right) + \frac{\beta_1}{\beta_0} \ln\left(\ln\left(\frac{\mu_0^2}{\Lambda_{QCD}^2}\right)\right)\right]}$$

If we consider a Green function whose invariant square momenta are of order q^2 and we subtract the ultraviolet infinities at μ^2, we are always left with corrections which are proportional to $\ln\left(q^2/\mu^2\right)$ in the renormalized Green functions. If $q^2 \ll \mu^2$ or $q^2 \gg \mu^2$ these logarithms can become

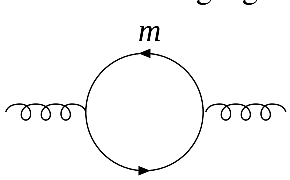
† Nowadays the most popular reference scale is the mass of the $Z-$boson and the $\overline{MS}$ prescription is used - i.e. the renormalized coupling is expressed in terms of $\alpha_{\overline{MS}}(M_Z^2)$.

large and give large coefficients of the renormalized coupling, $\alpha_R(\mu^2)$ so that $|\alpha_R(\mu^2)\ln\left(q^2/\mu^2\right)| \sim 1$ and this may spoil the convergence of the perturbation expansion. It is therefore convenient to choose $\mu^2 \approx q^2$, i.e. we choose a subtraction scale to be of the order of the typical energy of the process under consideration. This way we obtain a perturbative expansion in the "effective coupling", $\alpha_R(q^2)$, with no large logarithms. For asymptotically free theories the effective coupling decrease as the energy scale increases. Strong interactions are believed to be described by such a theory (QCD), so that the interactions actually become sufficiently small at large energies for a perturbation expansion to be valid.

In general, spontaneously broken gauge theories are *not* asymptotically free. This is not because the β-function for the gauge coupling is positive at small couplings (the β-function acquires a small positive contribution from the interaction of the gauge-bosons with the Higgs scalar fields, but unless there are a very large number of fermions or scalar fields β_0 remains negative), but rather because the β-function for the self-interaction of the Higgs scalar field is positive, so the ϕ^4 coupling grows as μ increases and this increasing scalar coupling feeds into the other couplings at higher orders.



Thresholds
When calculating the effective coupling, $\alpha_R(q^2)$, we include in the number of fermion multiplets, n_f, only the fermions whose masses are less than $\sqrt{q^2}$. This is because the derivative w.r.t. $\ln(q^2)$ of the contribution to the gauge-boson propagator from a graph with a massive fermion loop

$\frac{\partial}{\partial \ln(q^2)}$ 

$\sim q^2/m^2, \quad (q^2 \ll m^2)$

On the other hand the contribution it clearly reaches its full value, if $q^2 \gg m^2$. Thus a fermion only contributes to the q^2 development of the effective coupling for $q^2 \geq m^2$. The exact threshold can be calculated from the finite part of the Feynman graph, but a good approximation for the threshold is $q^2 = 4m^2$, which is the energy threshold at which a gauge-boson can produce a quark-antiquark pair of mass m.

4.2 Callan-Symanzik equation

Consider an n-point 1-particle-irreducible Green function, which depends on external momenta p_i and masses m_j, the coupling g, and, in general, the gauge parameter ξ, and possibly an ultraviolet cut-off Λ. The renormalized green function is independent of the ultraviolet cutoff Λ, but depends explicitly on the subtraction scale μ.

$$\Gamma_0(p_1\cdots p_{n-1},g_0,m_0,\xi_0,\Lambda) \; = \; \prod_{i=1}^{n} Z_i^{-1/2}(\mu)\Gamma_R(p_1\cdots p_{n-1},g_R(\mu),m_R(\mu),\xi_R(\mu),\mu) \qquad (4.10)$$

The renormalized quantities, g_R, m_R, ξ_R depend on the subtraction scale μ as does the wavefunction renormalization constant Z_i, for each external particle.

However, the LHS of eq.(4.10) is independent of μ, and this leads to a first-order (partial) differential equation for the renormalized Green function, Γ_R (the "Callan-Symanzik equation"),

$$\frac{\partial}{\partial \ln(\mu)}\Gamma_R + \beta\frac{\partial \Gamma_R}{\partial g_R} + m_{Rj}\gamma_{m_j}\frac{\partial \Gamma_R}{\partial m_{Rj}} + \xi_R\gamma_\xi\frac{\partial \Gamma_R}{\partial \xi_R} - \sum_i \gamma_i \Gamma_R \; = \; 0, \qquad (4.11)$$

where

$$\gamma_{m_j} \; = \; \frac{1}{m_{Rj}}\frac{\partial m_{Rj}}{\partial \ln(\mu)}$$

$$\gamma_\xi \; = \; \frac{1}{\xi_R}\frac{\partial \xi_R}{\partial \ln(\mu)}$$

and

$$\gamma_i \; = \; \frac{1}{2Z_i}\frac{\partial Z_i}{\partial \ln(\mu)}$$

γ_i are called the "anomalous dimensions" of the field corresponding to particle i.

If we are in a high-energy region of momentum-space in which the masses may be neglected we may neglect the terms involving

$$\frac{\partial \Gamma_R}{\partial m_{Rj}}.$$

Such a region is the "deep-Euclidean region" in which all the external square momenta and scalar products between momenta are negative (space-like) and large compared with all the masses, ($p_i^2 < 0$ and $|p_i^2|, |p_i \cdot p_k \gg m_j^2$).

In this region we may write $p_i = p y_i$, where y_i are dimensionless vectors and p sets the scale for the momenta. We can write the renormalized Green function as

$$\Gamma_R(p_1 \cdots p_{n-1}, g_R, \xi_R, \mu) \;=\; p^d \tilde{\Gamma}_R\left(y_1 \cdots y_{n-1}, g_R, \xi_R, \frac{\mu^2}{p^2}\right),$$

where d is the naive ("engineering") dimension of the Green function (obtained from simple power counting). $\tilde{G}_R$ depends explicitly on μ^2, through the dimensionless ratio μ^2/p^2. This gives us the Callan-Symanzik equation for the p dependence of the renormalized Green-function

$$\frac{\partial}{\partial \ln p}\Gamma_R \;=\; \left(d - \frac{\partial}{\partial \ln(\mu)}\right)\Gamma_R. \tag{4.12}$$

We restrict ourselves to gauge invariant quantities, so that we may now discard the terms involving the derivative w.r.t. ξ. G_R now obeys the equation

$$\left(\frac{\partial}{\partial \ln(\mu)} + \beta(g_R(\mu))\frac{\partial}{\partial g_R(\mu)} + \sum_i \gamma_i(g_R(\mu))\right)\Gamma_R \;=\; 0. \tag{4.13}$$

β and γ_i depend on μ through their dependence on $g_R(\mu)$. Using the relation between the explicit μ dependence and the p dependence (eq.(4.12)), we get

$$\left(\frac{\partial}{\partial \ln p} - \beta(g_R(p))\frac{\partial}{\partial g_R(p)} - d + \sum_i \gamma_i(g_R(p))\right)\Gamma_R \;= 0, \tag{4.14}$$

where $g_R(p)$ is the value for the renormalized coupling at $p^2 = \mu^2$ (the effective coupling).

The solution to eq.(4.14) is

$$\Gamma_R(p_1 \cdots p_{n-1}, g_R(\mu), \mu) \;=\; p^d \tilde{\Gamma}_R(y_1 \cdots y_{n-1}, g_R(p), 1) \exp\left\{\int_{g_R(\mu)}^{g_R(p)} \frac{-\sum_i \gamma_i(g')}{\beta(g')} dg'\right\}. \tag{4.15}$$

$\tilde{\Gamma}_R(y_1 \cdots y_{n-1}, g_R(p), 1)$ is $\tilde{\Gamma}_R$ at $p^2 = \mu^2$, calculated in an ordinary perturbation expansion in $g_R(p)$. For asymptotically free theories $g_R(p) \to 0$ as $p \to \infty$. This then tells us something about the high energy behaviour of Euclidean Green functions.

Using perturbative expansions

$$\beta(g') \;=\; \frac{g'^3}{16\pi^2}\beta_0 + \frac{g'^5}{(16\pi^2)^2}\beta_1 + \cdots$$

and

$$\gamma_i(g') \;=\; \frac{g'^2}{16\pi^2}\gamma_{i0} + \frac{g'^4}{(16\pi^2)^2}\gamma_{i1} + \cdots$$

The power series for $\tilde{\Gamma}_R(y_1 \cdots y_{n-1}, g_R(\mu), 1)$ may be written as

$$\tilde{\Gamma}^0_R(y_1, \cdots y_{n-1}) \left[1 + c(y_1, \cdots y_{n-1}) \frac{g_R^2}{16\pi^2} + \cdots \right].$$

We can use this to determine the behaviour of the renormalized Green function under a change of the momentum scale from p to p'. To leading order (in terms of $\alpha_R \equiv g_R^2/(4\pi)$) we have

$$\begin{aligned} \Gamma_R(p'_1, \cdots p'_{n-1}, g_R(\mu), \mu) &= \Gamma_R(p_1, \cdots p_{n-1}, g_R(\mu), \mu) \left(\frac{p'}{p}\right)^d \left(\frac{\alpha_R(p'^2)}{\alpha_R(p^2)}\right)^{-\sum_i \gamma_{i0}/(2\beta_0)} \\ &\times \left[1 + O\left(\alpha_R(p^2) - \alpha_R(p'^2)\right)\right] \end{aligned} \tag{4.16}$$

Calculation of the Anomalous Dimensions

In the MS scheme

$$Z_i = 1 + \sum_n \sum_{k=1}^{n} a_k^n \frac{g_R^{2n}}{\varepsilon^k}$$

In $4 - 2\varepsilon$ dimensions

$$\frac{\partial g_R}{\partial \ln(\mu)} = \beta(g_R) - \varepsilon g_R,$$

so that (differentiating w.r.t $\ln(\mu)$)

$$Z_i \gamma_i = \frac{1}{2} \sum_n \sum_{k=1}^{n} (2n) a_k^n \frac{g_R^{2n-1}}{\varepsilon_k} \left(\beta(g_R) - \varepsilon g_R\right)$$

Comparing coefficients of ε^0, we have

$$\gamma_i = -\sum_n n a_1^n g_R^{2n}$$

We see that to any order in perturbation theory, the anomalous dimension is simply related to the coefficient of the simple pole.

For a fermion interacting with a non-Abelian gauge-boson, we have to one loop order (in Feynman gauge)

$$Z_2 = 1 - \frac{g_R^2}{16\pi^2} \frac{C_F}{\varepsilon} + \cdots$$

($C_F = (N^2 - 1)/(2N)$ for an $SU(N)$ theory). This gives

$$\gamma_{i0} = +\frac{\alpha_R}{4\pi} C_F.$$

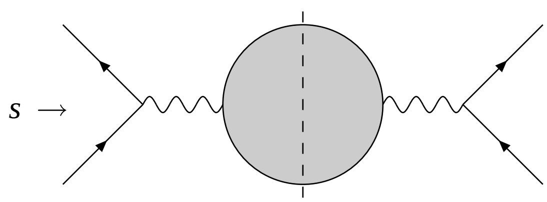
Likewise the next coefficient γ_{i1} may be determined from the two-loop calculation of the wavefunction renormalization Z_2. Only the leading order term γ_{i0} is renormalization scheme independent - the higher order terms are different in different renormaliztion schemes.

Deep Euclidean region Green functions are not physical, but there are a number of techniques available for using the scale dependence of such Green functions to calculate the energy dependence of physical processes.

The simplest example of this is the total cross-section in electron-positron annihilation. This is usually expressed in terms of a ratio of the cross-section into hadrons to the pure QED process in which the electron-positron pair annihilates into a muon pair

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)}.$$

Unitarity and the Optical Theorem relate the total cross-section to the imaginary part of the off-shell photon propagator

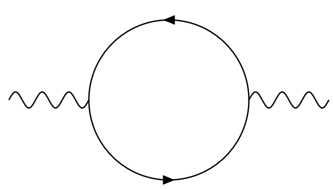


Here the shaded blob refers to the sum of all graphs with a quark loop plus gluon corrections. A graph with a quark loop with quarks of mass m_i has an imaginary part provided $s > 4m_i^2$. We can make an analytic continuation into negative (space-like) s and for such values of s we are indeed in the Deep Euclidean region where the Callan-Symanzik equation is valid. However, in this case the external line is the off-shell photon with square momentum s. Since this is not a strongly interacting particle there is no QCD contribution to the anomalous dimension. The solution to the Callan-Symanzik equation then greatly simplifies and we have

$$\Pi(s, \alpha_S(\mu^2), \mu) \ = \ \Pi(\alpha_S(s), s = \mu^2),$$

(α_S means the renormalized strong coupling)

To leading order, Π is just given by the one-loop graph

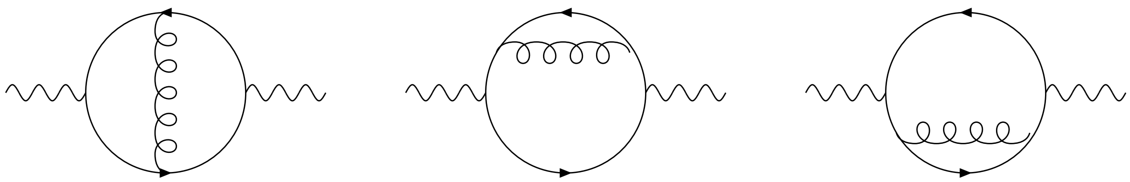


The contribution to the ratio R is just the sum of the squares of the electric charges Q_i^2, of all the quarks with masses below s, multiplied by a phase-space factor which accounts for the non-zero quark masses.

$$R = \sum_{i:\ 4m_i^2<s} Q_i^2 \times \left[\sqrt{1-4\frac{m_i^2}{s}}\left(1+2\frac{m_i^2}{s}\right)\right],$$

where the factor inside the square brackets is the phase-space factor.

In next order in perturbation ($O(\alpha_S)$ corrections) we have the graphs



These are calculated at $s=\mu^2$ with α_S renormalized at $\mu^2=s$. This gives

$$R(s) = \sum_{i:\ 4m_i^2<s} Q_i^2\left[1+3C_F\frac{\alpha_S(s)}{4\pi}+\cdots\right] \times \left(\sqrt{1-4\frac{m_i^2}{s}}\left(1+2\frac{m_i^2}{s}\right)\right),$$

Deep inelastic electron proton scattering is another example of a process whose energy dependence can be determined using the Callan-Symanzik equation.