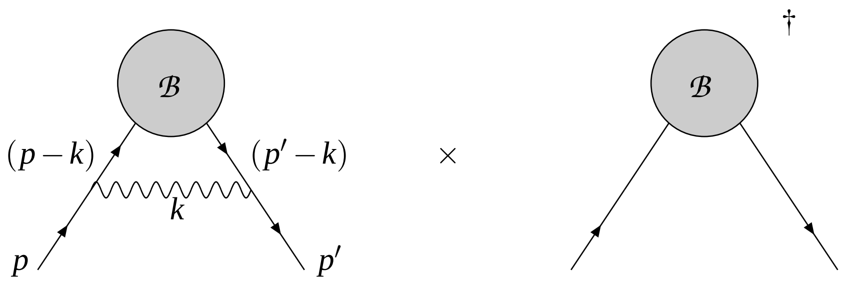
5 Infrared Divergences

We have already seen that some QED graphs have a divergence associated with the masslessness of the photon. The divergence occurs at small values of the photon momentum k. In a general graph there are infrared divergences when *both* ends of a photon are attached to an external charged line.

The contribution to a transition probability or cross-section from such a correction is the interference between the correction graph and the graph without the photon attached to the external lines



The shaded blob stands for any other part of the graph, which could be simply a tree-level or process or it may contain any number of loops of internal photons and fermions.

For small k we neglect any powers of k in the numerator (this is the "eikonal approximation") and similarly in the denominator of the fermion propagator we neglect k^2 and write

$$\frac{i}{((p-k)^2-m^2)} \rightarrow \frac{-i}{2p\cdot k}.$$

In Feynman gauge the numerator may be written (in the eikonal approximation)

$$-(-ie\gamma^\mu)\left(\gamma\cdot p'+m\right)\mathcal{B}\left(\gamma\cdot p+m\right)\left(-ie\gamma_\mu\right)$$

This is sandwiched between on-shell spinors, so that we can anti-commute $\gamma\cdot p'$ or $\gamma\cdot p$ through γ_μ and use the Dirac equation to reduce this to

$$4e^2 p\cdot p'\mathcal{B},$$

where $\mathcal{B}$ represents the contribution from the shaded blob.

The infrared divergent part of this interference may therefore be written

$$-ie^2|\mathcal{B}|^2 8p\cdot p' \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+i\varepsilon)(2p\cdot k-i\varepsilon)(2p'\cdot k-i\varepsilon)} \tag{5.1}$$

We have reinstated the $i\varepsilon$ in the propagators from the time ordering operator. We do this because we choose to perform the above integral by integrating first over the time component k_0 of the loop momentum.

We therefore rewrite eq.(5.1) as

$$-ie^2 2|\mathcal{B}|^2 p\cdot p' \int \frac{dk_0}{(2\pi)} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{(k_0^2-\mathbf{k}^2+i\varepsilon)((p_0k_0-\mathbf{p}\cdot\mathbf{k}-i\varepsilon)(p_0'k_0-\mathbf{p}'\cdot\mathbf{k}-i\varepsilon)} \tag{5.2}$$

and integrate over k_0 by closing the contour in such a way as to pick up the pole at $k_0 = |\mathbf{k}|$. This gives

$$-e^2 2|\mathcal{B}|^2 \int \frac{|\mathbf{k}|^2 d\mathbf{k} d\Omega}{(2\pi)^3 2|\mathbf{k}|} \frac{1}{((p_0|\mathbf{k}|-\mathbf{p}\cdot\mathbf{k})(p_0'|\mathbf{k}|-\mathbf{p}'\cdot\mathbf{k})}$$

This integral diverges at $|\mathbf{k}| \to 0$, so we cut off this lower limit at $|\mathbf{k}| = \lambda$. We also impose an upper limit E above which the infrared approximation is no longer valid (this upper limit is rather arbitrary but we are only interested here in the infrared divergences. We therefore get for the infrared divergent part

$$-\frac{\alpha}{2\pi} p\cdot p' |\mathcal{B}|^2 \ln\left(\frac{E}{\lambda}\right) \int \frac{d\cos\theta\, d\phi}{2\pi} \frac{1}{(E-p\cos\theta)(E'-p'\cos\theta')}, \tag{5.3}$$

E, p and E', p' are the energies and magnitudes of 3-momenta of the external fermion lines and

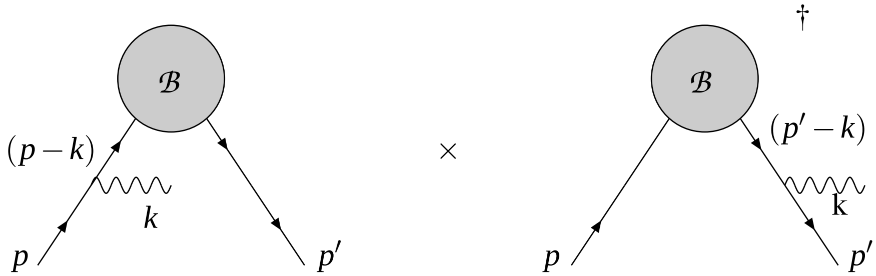
$$\cos\theta' \;=\; \cos\alpha\cos\theta - \sin\alpha\sin\theta\cos\phi,$$

where α is the angle between $\mathbf{p}$ and $\mathbf{p}'$ (θ is the angle between $\mathbf{k}$ and $\mathbf{p}$, whereas θ' is the angle between $\mathbf{k}$ and $\mathbf{p}'$).

These infrared divergences do not cancel within the process considered above. Whenever a process occurs with electromagnetic corrections there is an experimental limit to the accuracy with which the initial and final state energies can be measured. This means that there will *always* be some energy loss in emitted photons (Bremsstrahlung). What is actually observed is the *sum* of the elastic process (no emitted photons) and the process in which a small quantity of energy up to the energy resolution, ΔE, is lost in photon emission. Keeping track of orders of the electromagnetic coupling, we see that a one-loop correction to a tree-level process with no emitted photons is of the same order as the tree-level process involving a single emitted photon. This generalizes to the statement that the α^n correction to a tree-level process consists of the sum of all the process with $n-r$ virtual loops and r emitted photons.

The emission of a (real) photon from an external charged line also introduces an infrared divergences as the energy of the emitted photon goes to zero. It is *this* infrared divergence that cancels the infrared divergence associated with the virtual correction.

Consider the interference between the graphs for the above process in which the photon is emitted from different charge lines



Again, using the eikonal approximation for the numerator (in this case we have exactly $k^2 = 0$ since the emitted photon is on-shell - this means that the denominators of the internal fermion lines are $2p \cdot k$ and $2p' \cdot k$ respectively), the infrared part of this process is

$$e^2 |\mathcal{B}|^2 4p \cdot p' \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \frac{1}{(2p \cdot k)(2p' \cdot k)}, \tag{5.4}$$

We have performed a summation over the polarization of the emitted fermion (which gives a factor $-g_{\mu\nu}$, being the counterpart of the Feynman gauge propagator in the virtual correction). The integral over $\mathbf{k}$ is the integral over the phase-space of the emitted photon. We note that this integral also has a divergence as $|\mathbf{k}| \to 0$ with the opposite sign from that of the virtual correction. Again we cut this lower limit off at $|\mathbf{k}| = \lambda$. We take the upper limit of the integration over $\mathbf{k}$ to be the energy resolution, ΔE. This interference then contributes an infrared divergent part

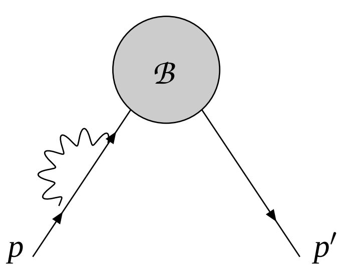
$$\frac{\alpha}{2\pi} p \cdot p' \ln\left(\frac{\Delta E}{\lambda}\right) \int \frac{d\cos\theta \, d\phi}{2\pi} \frac{1}{(E - p\cos\theta)(E' - p'\cos\theta')}. \tag{5.5}$$

If we sum the contributions from eqs.(5.3) and (5.5) we see that the dependence on the infrared cut-off λ cancels and we are left with

$$-\frac{\alpha}{2\pi} p \cdot p' \ln\left(\frac{E}{\Delta E}\right) \int \frac{d\cos\theta \, d\phi}{2\pi} \frac{1}{(E - p\cos\theta)(E' - p'\cos\theta')}, \tag{5.6}$$

to which we must add the contributions from the hard (virtual and real) photons that we have neglected in the eikonal approximation.

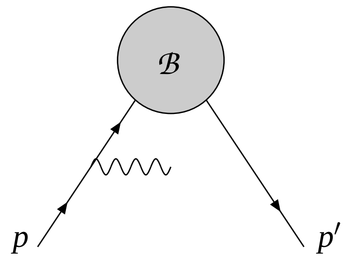
There are other infrared divergent graphs, such as the fermion self-energy insertions, which give rise to an infrared divergence when on-shell renormalization is performed (Z_2 is infrared divergent).



This gives an infrared divergent contribution

$$\frac{\alpha}{\pi} \ln\left(\frac{E}{\lambda}\right)$$

which cancels the infrared divergent part of the square of the graph



This graph squared gives a contribution

$$e^2|\mathcal{B}|^2 m^2 \int \frac{d^3\mathbf{k}}{2\pi^3 2|\mathbf{k}|} \frac{1}{(2k\cdot p)^2}.$$

This also has an infrared divergence as $|\mathbf{k}| \to 0$.

The complete soft photon (i.e. small $\mathbf{k}$) contribution to the sum of the two processes is

$$-|\mathcal{B}|^2 \frac{\alpha}{\pi} \mathcal{K} \ln\left(\frac{E}{\Delta E}\right), \tag{5.7}$$

where

$$\mathcal{K} = \int \frac{d\Omega}{4\pi} \frac{p\cdot p'}{(E - p\cos\theta)(E' - p'\cos\theta')} - 1 \tag{5.8}$$

It can be shown that this cancellation of infrared divergences between the elastic (only virtual photon corrections) part and the inelastic part (one or more real photon emissions with total energy less than ΔE) persists to all orders in perturbation theory. In fact, the infrared divergences can be shown to exponentiate so that for the elastic cross section the sum to all orders of the infrared divergent part is

$$|\mathcal{B}|^2 \exp\left\{-\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{E}{\lambda}\right)\right\}$$

and for the elastic amplitude plus any number of real photons with total energy up to ΔE we have

$$|\mathcal{B}|^2 \exp\left\{-\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{E}{\lambda}\right)\right\} \exp\left\{+\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{\Delta E}{\lambda}\right)\right\}$$

So that the soft photon parts of the complete inelastic cross-section is proportional to

$$\left(\frac{\Delta E}{E}\right)^{\alpha\mathcal{K}/\pi}$$

Since $\mathcal{K}$ is positive, we see that this vanishes as $\Delta E \to 0$, meaning that the probability of a purely elastic process with no energy loss into emitted photons is zero.

5.1 Dimensional Regularization of Infrared Divergences

The method of dimensional regularization can also be used to regularize infrared divergences. After the integration over the energy component k_0 in the case of virtual corrections we have an integral over the $d-1$ space-like components of the photon momentum $d^{d-1}\mathbf{k}$. Likewise the integral over the phase space of the emitted photon for the bremsstrahlung process is carried out in $d-1 = 3-2\varepsilon$ dimensions.

$$\int \frac{d^{3-2\varepsilon}\mathbf{k}}{(2\pi)^{3-2\varepsilon}} = \frac{1}{8\pi^2}\Gamma(1-2\varepsilon)(4\pi)^{\varepsilon}4^{\varepsilon}\int (\sin\theta)^{1-2\varepsilon}d\theta|\mathbf{k}|^{2-2\varepsilon}d|\mathbf{k}|, \tag{5.9}$$

where we have integrated over all but one of the polar angles.

For real photon emission with photon energy up to ΔE we have the phase-space integral

$$-e^2|\mathcal{B}|^2\int_0^{\Delta E}\frac{d^{3-2\varepsilon}\mathbf{k}}{(2\pi)^{3-2\varepsilon}2|\mathbf{k}|}\left(\frac{m^2}{(k\cdot p)^2}+\frac{m^2}{(k\cdot p')^2}-2\frac{p\cdot p'}{(k\cdot p)(k\cdot p')}\right) \tag{5.10}$$

The last term is handled using the Feynman parametrization trick, so we integrate over the Feynman parameter α and define the momentum

$$p_\alpha^\mu = p^\mu\alpha + p'^\mu(1-\alpha).$$

The expression (5.10) becomes

$$-e^2|\mathcal{B}|^2\int_0^{\Delta E}\frac{d^{3-2\varepsilon}\mathbf{k}}{(2\pi)^{3-2\varepsilon}2|\mathbf{k}|}\left(\frac{m^2}{(k\cdot p)^2}+\frac{m^2}{(k\cdot p')^2}-2\int_0^1 d\alpha\frac{p\cdot p'}{(k\cdot p_\alpha)^2}\right) \tag{5.11}$$

Using eq.(5.9) this is

$$-\frac{\alpha}{2\pi}(4\pi)^{\varepsilon}\Gamma(1-2\varepsilon)|\mathcal{B}|^2 4^{\varepsilon}\int_0^{\Delta E}|\mathbf{k}|^{-1-2\varepsilon}(\sin\theta)^{1-2\varepsilon}d|\mathbf{k}|d\theta\left(\frac{m^2}{(k\cdot p)^2}+\frac{m^2}{(k\cdot p')^2}-2\int_0^1 d\alpha\frac{p\cdot p'}{(k\cdot p_\alpha)^2}\right) \tag{5.12}$$

The term

$$4^{\varepsilon}\int(\sin\theta)^{1-2\varepsilon}\frac{1}{(E-p\cos\theta)^2}$$

is a hypergeometric function whose expansion about $\varepsilon = 0$ is

$$\frac{2}{(E^2-p^2)}\left[1-\varepsilon\frac{E}{p}\ln\left(\frac{E-p}{E+p}\right)+O(\varepsilon^2)\right].$$

The integral over $|\mathbf{k}|$ gives a pole at $\varepsilon = 0$. This pole signals the infrared divergence. When dimensional regularization is used to regularize infrared divergences we must think of this as performing the integral initially in *more than* four dimensions (negative ε) for which there is no infrared divergence and then performing an analytic continuation to four dimensions.

The expression (5.12) gives a pole term

$$\frac{\alpha}{2\pi}\Gamma(1-2\varepsilon)|\mathcal{B}|^2(4\pi)^{\varepsilon}\frac{(\Delta E)^{-\varepsilon}}{2\varepsilon}\left[4-4p\cdot p'\int_0^1\frac{d\alpha}{(E_\alpha^2-p_\alpha^2)}\right] \tag{5.13}$$

and a finite term

$$-\frac{\alpha}{2\pi}|\mathcal{B}|^2\left[\frac{E}{p}\ln\left(\frac{E+p}{E-p}\right)+\frac{E'}{p'}\ln\left(\frac{E'+p'}{E'-p'}\right)-2p\cdot p'\int_0^1 d\alpha\frac{E_\alpha}{p_\alpha}\frac{1}{(E_\alpha^2-p_\alpha^2)}\ln\left(\frac{E_\alpha+p_\alpha}{E_\alpha-p_\alpha}\right)\right] \tag{5.14}$$

Expanding the term

$$\frac{(\Delta E)^{-\varepsilon}}{2\varepsilon}$$

in the pole part gives the $\ln(\Delta E)$ dependence found previously.

Now compare this with the virtual correction term. The soft photon contribution is

$$-ie^2|\mathcal{B}|^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\left[\frac{4m^2}{(k^2-2p\cdot k)^2}+\frac{4m^2}{(k^2-2p'\cdot k)^2}-\frac{8p\cdot p'}{(k^2-2p\cdot k)(k^2-2p'\cdot k)}\right] \tag{5.15}$$

The first two terms in square parenthesis coming from the self-energy insertions after performing on-shell wavefunction renormalization and the third term from the correction term in which the virtual photon connects the two external fermions. We have used the eikonal approximation in the numerator, but kept the denominators exact.

Using Feynman parametrization this becomes

$$\begin{aligned}-ie^2|\mathcal{B}|^2\int\frac{d^dk}{(2\pi)^d}&\left[\int_0^1 2\alpha\, d\alpha\frac{4m^2}{(k^2-2p\cdot k\alpha)^3}+\int_0^1 2\alpha\, d\alpha\frac{4m^2}{(k^2-2p'\cdot k\alpha)^3}\right.\\&\left.-\int d\alpha d\beta\theta(1-\alpha-\beta)\frac{16p\cdot p'}{(k^2-2k\cdot(p\alpha+p'\beta))^3}\right]\end{aligned} \tag{5.16}$$

Shifting the momentum k as appropriate this gives

$$-ie^2|\mathcal{B}|^2\int\frac{d^dk}{(2\pi)^d}\left[\int_0^1 16\alpha\, d\alpha\frac{m^2}{(k^2-m^2\alpha^2)^3}-\int d\alpha d\beta\theta(1-\alpha-\beta)\frac{16p\cdot p'}{(k^2-(p\alpha+p'\beta)^2)^3}\right] \tag{5.17}$$

Performing the integration over k gives

$$\frac{\alpha}{4\pi}\Gamma(1+\varepsilon)(4\pi)^\varepsilon|\mathcal{B}|^2\left[(m^2)^{-\varepsilon}\int_0^1 8\alpha^{-1-2\varepsilon}d\alpha-8p\cdot p'\int_0^1 d\alpha d\beta\frac{\theta(1-\alpha-\beta)}{((p_\alpha+p'\beta)^2)^{1+\varepsilon}}\right]. \tag{5.18}$$

The double nested integral over α and β is performed by making the change of variables

$$\alpha=\rho\omega$$

$$\beta=\rho(1-\omega)$$

In the first term we change variable $\alpha \to \rho$, to get

$$4\frac{\alpha}{4\pi}\Gamma(1+\varepsilon)(4\pi)^{\varepsilon}|\mathcal{B}|^2 \int \rho^{-1-2\varepsilon} d\rho \left[(m^2)^{-\varepsilon} - p \cdot p' \int d\omega \frac{1}{\left((p\omega + p'(1-\omega))^2\right)^{1+\varepsilon}} \right] \tag{5.19}$$

Integrating over ρ we are left with

$$4\frac{\alpha}{4\pi}\Gamma(1+\varepsilon)(4\pi)^{\varepsilon}|\mathcal{B}|^2 \frac{1}{2\varepsilon} \left[(m^2)^{-\varepsilon} - p \cdot p' \int d\omega \frac{1}{\left((p\omega + p'(1-\omega))^2\right)^{1+\varepsilon}} \right] \tag{5.20}$$

We see that the pole term in this expression cancels against the pole term for the real emission. The infrared finite term is obtained by expanding up to order ε^0

5.2 Collinear Divergences

The expressions (5.13) and (5.20) for the pole parts of the real emission and virtual corrections respectively contain a factor

$$\int_0^1 d\omega \frac{1}{(p\omega + p'(1-\omega))^2}$$

For $p \cdot p' \gg m^2$ this integral is approximately

$$\frac{1}{p \cdot p'} \ln\left(\frac{2p \cdot p'}{m^2}\right),$$

and diverges as $m \to 0$.

A study of such divergences gives information about the behaviour of processes as the momentum scale increases (the high energy limit, $p \cdot p' \gg m^2$) of QED. For non-Abelian gauge theories such as QCD we have to deal with interacting particles that are strictly massless. In such cases there is a further "collinear" divergence which occurs even if the emitted photon (or gluon) does not carry small momentum, but when it is emitted parallel to the parent particle.

In the case of real photon emission, the double divergence we get when the electron mass is neglected arises from the term in the phase-space integral (in 3+1 dimensions)

$$\int \frac{d|\mathbf{k}| d\Omega}{(2\pi)^3 2|\mathbf{k}|} \frac{1}{(E - p\cos\theta)(E' - p'\cos\theta')}$$

We see that not only is there a divergence as $|\mathbf{k}| \to 0$ but for massless particles for which $E = p$ and $E' = p'$ there is a divergence at angles $\theta = 0$ and $\theta' = 0$. These are the collinear divergences.

These collinear divergences can also conveniently be treated using dimensional regularization. For, example the term under consideration from the expression (5.18) for the virtual correction is

$$\frac{\alpha}{2\pi}|\mathcal{B}|^2\frac{(4\pi)^{\varepsilon}\Gamma(1+\varepsilon)}{2\varepsilon}2p\cdot p'\int_0^1\frac{d\omega}{(p\omega+p'(1-\omega))^{1+\varepsilon}}$$

If $p^2 = p'^2 = 0$ this is

$$\frac{\alpha}{2\pi}|\mathcal{B}|^2\left(\frac{(4\pi)}{2p\cdot p'}\right)^{\varepsilon}\frac{\Gamma(1+\varepsilon)}{2\varepsilon}\int_0^1\frac{d\omega}{(\omega(1-\omega))^{1+\varepsilon}}$$

The integral over ω may now be performed

$$\int_0^1 d\omega w^{-1-\varepsilon}(1-\omega)^{-1-\varepsilon} = \frac{\Gamma^2(-\varepsilon)}{\Gamma(-2\varepsilon)} = -\frac{2}{\varepsilon}\frac{\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

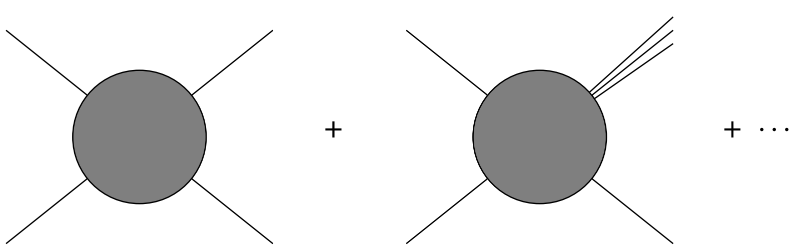
We therefore get

$$-\frac{\alpha}{2\pi}|\mathcal{B}|^2\frac{1}{\varepsilon^2}\left(\frac{4\pi}{p\cdot p'}\right)^{\varepsilon}\frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)}$$

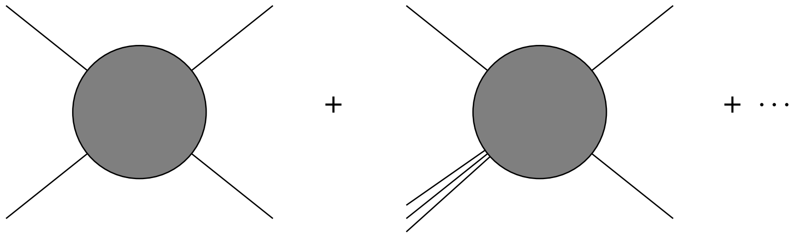
The double pole indicates that we have both a soft photon and a collinear photon divergence.

There is a similar double pole term from the real photon emission, such that the double pole cancels. However, the single pole will *not* cancel in the case of a massless electron. The cancellation between real emission and virtual corrections refers to the soft photon divergences but *not* in general to the collinear divergences. Indeed, for the collinear divergences the eikonal approximation which has been used to extract the numerators of the various graphs is not valid, so we expect more collinear divergent terms than those we have considered here.

The cancellation of both infrared and collinear divergences in massless QED or non-Abelian gauge theories with massless self-interacting particles is far more restrictive. In the case of (massive) QED the cancellation of infrared divergence occurs provided we sum over all processes involving final states that give rise to infrared divergences. For massless interacting particles this means not only summing over processes in which soft massless particles are emitted, but also over states in which hard massless particles are emitted (nearly) parallel to their parent particles. For example we need to sum over processes in which a massless outgoing particle is replaced by a jet of nearly parallel outgoing massless particles.



Unfortunately, this is not sufficient. Kinoshita, Lee & Nauenberg showed that in order to guarantee the cancellation of both soft and collinear divergences we must sum over processes involving all possible *initial* states which can give rise to soft or collinear divergences. For example we need to sum over processes in which an incoming massless particle is replaced by an incoming jet of nearly parallel massless particles.

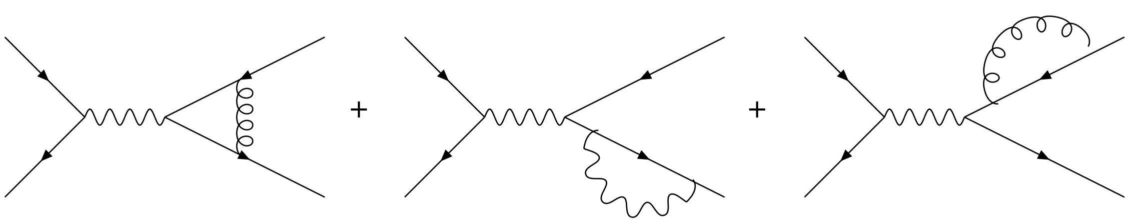


For the summation over final states, this is not really a problem, since one cannot distinguish experimentally between a single particle and a sufficiently narrow jet of particles any more than one can detect soft photon (or gluon) radiation which takes off energy less than the energy resolution of the experiment.

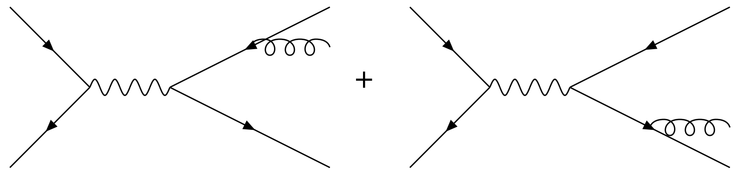
The requirement that one sums over incoming jets in order to cancel the collinear divergences is more problematic. What this means is that if we calculate in perturbation theory the QCD process of quark-quark (or quark-gluon, or gluon-gluon) scattering, we will not get a finite result even when summing over all possible final states. On the other hand, it is important to note that in practice one cannot prepare an initial state which consists of free quarks and/or gluons. The initial states are hadrons which contain quarks and gluons. The remaining divergence arising from the calculation of a process with initial quarks and/or gluons is absorbed into the (momentum scale dependence) of the parton "distribution function", i.e. the probability that a parent hadron contains a parton with a given flavour and momentum fraction.

One case in which we do not need to worry about summing over initial states is the case of electron-positron annihilation. Here the initial state consists of particles which do not have strong interactions and so there are no other initial states that are connected by strong interactions to the initial electron-positron state.

For example, in perturbative QCD to order α_S, the Kinoshita-Lee-Nauenberg theorem tells us that the soft and collinear divergences which arise in the one gluon exchange virtual correction to the cross-section for a quark-antiqark pair



cancels against corresponding divergences in the tree-level process for the production of a quark-antiquark pair *plus* a single gluon



provided we integrate over all phase space for the final state gluon.

At order α_S^2 the cancellation is between the two-loop correction to the quark-antiquark production process, the one-loop correction to the quark-antiquark-gluon production process, and the tree-level quark-antiquark-gluon-gluon production process.

It is not necessary to integrate over the whole of the phase space of the final state particles. Some differential cross-sections are also infrared finite. In such cases we would be able to calculate the differential decay rate into a state in which the final state particles had a particular variable t set equal to a value T. This variable t would be a function of the momenta of the final state particles which would depend on how many particles there were in the final state. For n final state particles we would require

$$t_n(p_1 \cdots p_n) = T$$

If $d\sigma^{(n)}(p_1 \cdots p_n)$ is the differential cross-section for an electron-positron pair to decay into n particles with momenta $(p_1 \cdots p_n)$ (which will in general contain soft and collinear divergences from the virtual corrections), then the total cross-section with respect to the variable T is obtained by inserting a $\delta-$function inside the phase space integral for each of the processes.

$$\frac{d\sigma}{dT} = \sum_n d\sigma^{(n)}(p_1 \cdots p_n)\delta(t(p_1 \cdots p_n) - T)d\{P.S.\}^n, \tag{5.21}$$

where $d\{P.S.\}^n$ means n-particle phase space integration. Each term in the sum of eq.(5.21) contains infrared divergences, but the sum will be finite provided t is what is known as an "infrared-safe" quantity. At order α_S the sum over n with be the two and three particles final states, whereas at order α_S^2 we would also need the four-particle final state.

For t to be an infrared safe quantity we require that that n-particle function becomes equal to the corresponding $n-1$-particle expression if any two final state particles become parallel or if any final stae particle becomes soft, i.e. for any pair of particles i, j we must have

$$t_n(p_1 \cdots p_i, p_j \cdots p_n) \stackrel{(p_i+p_j)^2 \to 0}{\to} t_{n-1}(p_1 \cdots (p_i + p_j) \cdots p_n).$$