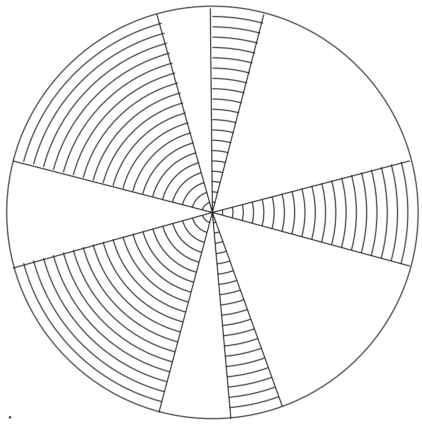

Figure 43: A representation of the spin part of the electron wavefunction as postulated by the theory of hidden variables. A measurement of the spin in a direction within a shaded area yields the result $+\frac{1}{2}\hbar$, whereas a measurement of the spin in a direction within a unshaded area yields the result $-\frac{1}{2}\hbar$.

15 Hidden Variables

If we return to the EPR experiment described above, the hidden variable hypothesis postulates that the electron is in a state such that if one were to choose to measure the spin of an electron in any given direction, there is a pre-determined result which is either $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$. This means that if we are going to measure the spin transverse to the electron's direction of motion and we imagine a circle in the plane transverse to the electron's direction of motion, then there are sectors of that circle in which the spin component is $+\frac{1}{2}\hbar$ and others in which it is $-\frac{1}{2}\hbar$. An example is shown in Fig. 43, which shows a circle which is a representation of the state of an electron whose spin in any given direction is determined. The shaded sectors are the directions in which a measurement of the spin will return the result $+\frac{1}{2}\hbar$ and the unshaded sectors are those in which a measurement of the spin will return the result $-\frac{1}{2}\hbar$. Note that each shaded sector has an unshaded sector exactly opposite reflecting the fact that if a measurement of spin in a particular direction yields the result $+\frac{1}{2}\hbar$ then a measurement in exactly the opposite direction will give the opposite result.

In 1964, John Bell [35] proved that Quantum Mechanics was not compatible with the theory of hidden variables, in which all classical properties of a system are determined but not measured. He also proposed a clean experiment which was able to distinguish between the predictions of Quantum Mechanics and the predictions of the theory of hidden variables. The experiment consists of sending one electron from a decaying particle to Bob and the other to Alice. The two electron spins are correlated in the sense that if both Alice and Bob measure the spins of their electrons in the same direction, conservation of spin leads to the fact that one will be spin up and yield the result that the component of spin in that direction is $+\frac{1}{2}\hbar$, whereas the other will be spin down and yield the result that the component of spin in that direction is $-\frac{1}{2}\hbar$. The electrons are in an entangled state, being a superposition of [Alice's electron spin-up; Bob's electron spin-down] and [Alice's electron spin-down; Bob's electron spin-up):

$$\frac{1}{\sqrt{2}}\left(\Psi^A_\uparrow\Psi^B_\downarrow - \Psi^A_\downarrow\Psi^B_\uparrow\right)$$

so before the measurement, we do not know which of the two combinations we will find. Now suppose that Alice and Bob do not measure the spins of their electrons in the same direction, but in two different directions making an angle θ to each other. In this case there is a possibility that we will find both spins up or both spins down (i.e. both electrons having spin component $+\frac{1}{2}\hbar$ or both having spin component $-\frac{1}{2}\hbar$. Note that the total number of measurements N is equal to the sum of these:

$$N \;=\; N_{\uparrow\downarrow} + N_{\downarrow\uparrow} + N_{\uparrow\uparrow} + N_{\downarrow\downarrow},$$

where $N_{ab}(\theta)$, with a, b running over $\uparrow, \downarrow$, is the number of measurements for which Alice's electron turns out to have spin component a in a given direction, and Bob's electron turns out to have spin component b in a direction at an angle θ to the direction in which Alice made her measurement.

We define the correlation factor for an angular separation of the 2 directions of measurement, θ, as

$$C(\theta) = \frac{(N_{\uparrow\downarrow} + N_{\downarrow\uparrow} - N_{\uparrow\uparrow} - N_{\downarrow\downarrow})}{N}, \tag{15.1}$$

Note that at $\theta = 0,\ C = 1$, the spins of the two electrons are totally correlated.

The hypothesis of hidden variables tells us that if we consider, for example, the $x - z$ plane then at any direction in that plane, the component of spin in that direction is either $+\frac{1}{2}\hbar$ for Alice's electron and $-\frac{1}{2}\hbar$ for Bob's or the other way around - so that a circle in the $x-z$ plane is partitioned into sectors with one or the other allowed combination, as shown in Fig. 44. When Alice's and Bob's chosen directions for the measurement of the spin both fall in the same sector as shown in Fig. 44(a) the spins are correlated (one is up and the other is down), but if they fall in adjacent sectors as shown in Fig. 44(b) then they are uncorrelated – they both have spin up or both spin down.

The maximum correlation within the hidden variable hypothesis occurs when the circle in the $x - z$ plane is divided into two semi-circles, as shown in Fig. 45 . The two semi-circles define two regions - one for which Alice's electron is spin up and Bob's is spin down and the other which Alice's electron is spin down and and Bob's is spin down and the other which Alice's electron is spin down and Bob's is spin up. In Fig. 45 the boundary between the two is shown at 45^0 to the axes, from top-left to bottom right - but it could have been at any random angle.

If the hidden variable distribution being as shown in Fig. 45, and the spins of the two electrons are measured in two directions with angular separation θ, then the probability that

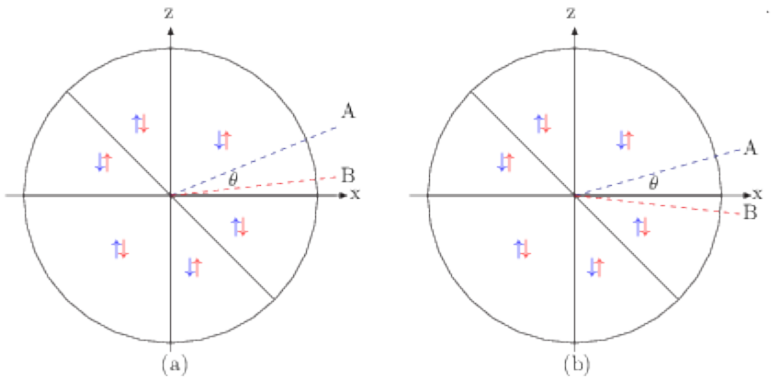

Figure 44: Assuming that in any direction the spin of the two electrons is determined, this shows a typical distribution of directions in which Alice's electron (blue) is up and Bob's (red) is down, or vice versa. In (a) the two directions of measurement by Alice and Bob fall in the same sector, so that one electron is measured to be up and the other is found to be down. In (b) the two directions of measurement crosses a boundary between sectors. The direction of Alice's measurement in in the sector in which her electron (blue) is spin down, whereas the direction of Bob's measurement is in an adjacent sector for which his electron (red) is also spin down.

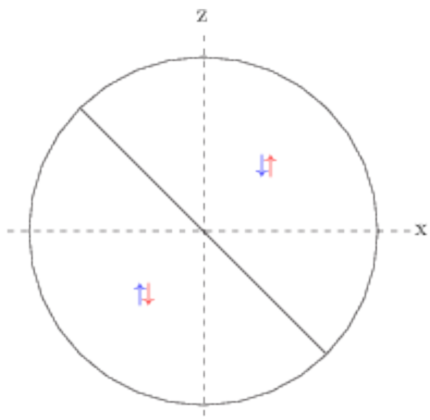

Figure 45: The maximum correlation allowed by the hypothesis of hidden variables is the case where the directions in which Alice's electron is spin up and Bob's is spin down all lie in one semi-circle – with the directions in which Alice's electron is spin down and Bob's electron is spin up lie in the other semi-circle.

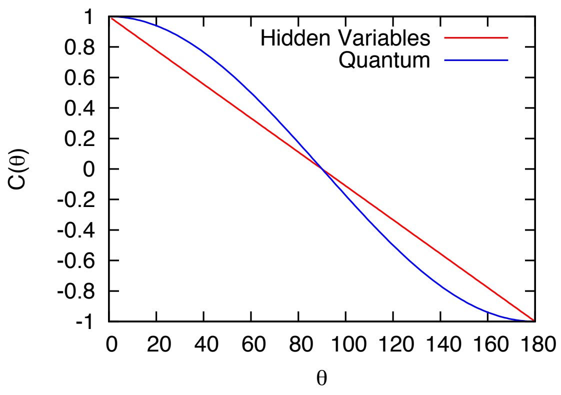

Figure 46: The red line shows the maximum correlation allowed within the hypothesis of hidden variables, whereas the blue curve is the result obtained from Quantum Mechanics. We see that for angles less than 90^0 the prediction of Quantum Physics exceeds the maximum allowed by the theory of hidden variables.

these two directions straddle both semi-circles, so that the results of the measurement of the two spins are the same (as in diagram (b) of Fig. 44), is $\theta/180$ and the probability that they are both in the same semi-circle, so that the results of the measurement of the two spins are opposite (as in diagram (a) of Fig. 44), is $1 - \theta/180$. From this we conclude that if the experiment is repeated over a very large number, N, of identical systems then

$$\frac{(N_{\uparrow\uparrow} + N_{\downarrow\downarrow})}{N} = \frac{\theta}{180} \text{ and}$$

and

$$\frac{(N_{\uparrow\downarrow} + N_{\downarrow\uparrow})}{N} = 1 - \frac{\theta}{180},$$

so that the correlation function $C(\theta)$ is given by:

$$C(\theta) = \frac{(N_{\uparrow\uparrow} + N_{\downarrow\downarrow} - N_{\uparrow\downarrow} - N_{\downarrow\uparrow})}{N} = 1 - \frac{\theta}{90}. \quad (15.2)$$

This then is the maximum value that this quantity can have within the hypothesis of hidden variables. It is shown in the red line of Fig. 46. Since this is the *maximum* value that the correlation $C(\theta)$ can have within the hypothesis of hidden variables, i.e. the correlation is smaller than or equal to that shown in the red line, this is known as **"Bell's inequality"**.

However, the result of a calculation using the rules of Quantum Mechanics gives

$$C(\theta) = \cos\theta. \quad (15.3)$$

This is shown in the blue curve in Fig. 46. We see that if the angle between the directions is less than 90^0 the blue curve is above the red line, meaning that the prediction of Quantum

Mechanics for the probability that the spins are opposite, exceeds the maximum probability allowed within the hypothesis of hidden variables.

The first test of Bell's inequality was conducted by Stuart Freedman and John Clauser [36] in 1972, but the most decisive experiment was carried out in 1982 by Alain Aspect [37]. It was found that the inequality was violated for angles between the two spin measurement of less than 90^0 and that the measured correlation agreed to a very high degree of accuracy with the prediction of Quantum Mechanics. We therefore have experimental evidence that an electron does not possess a well-defined component of spin in more than one direction. *There are no hidden variables.*