11 Two State Systems

We have mentioned above that a general wavefunction describes a particle which possesses neither a completely well-defined position nor a completely well-defined momentum - until such a time that one or the other of these quantities is measured. It will be slightly more convenient to discuss this strange phenomenon in the case of systems (or properties of systems) which have only two states, i.e. a property which can only have one of two values.

11.1 Photon Polarization

An electromagnetic wave consists of an oscillating electric field and an oscillating magnetic field, which are perpendicular to each other and also perpendicular to the direction of motion of the wave (see Fig. 31). Suppose we have a wave moving along the z-axis. There are two possibilities shown in Fig. 38. The wave is moving out of the plane of the figure, in the z-direction, and in the first diagram the electric field is pointing along the x-axis. We call this horizontal **polarization** and we write the amplitude of the electric field vector, $\mathbf{E}$, as

$$\mathbf{E}_{(H)} \ = \ E\,\mathbf{i}$$

where E is the amplitude of the electric field and $\mathbf{i}$ is a symbol meaning a vector of unit magnitude along the x-axis. In the second diagram the electric field is along the y-axis. We call this vertical polarization and we write the amplitude of the electric field vector, $\mathbf{E}$, as

$$\mathbf{E}_{(V)} \ = \ E\,\mathbf{j}$$

where $\mathbf{j}$ is a vector of unit magnitude along the y-axis.

However, the electric vector need not be exactly along the x- or y-directions but at an angle between them, such as the case shown in the third diagram, in which the electric field is at 45^0 to the x-axis. In this case we write the amplitude of the electric field vector as

$$\mathbf{E} = \cos(45)\mathbf{E}_{(H)} \ + \ \sin(45)\mathbf{E}_{(V)} = \frac{1}{\sqrt{2}}\mathbf{E}_{(H)} \ + \ \frac{1}{\sqrt{2}}\mathbf{E}_{(V)} \qquad (11.1)$$

(we have used $\cos(45) \ = \ \sin(45) \ = \ 1/\sqrt{2}$). This is an example of a superposition state. The photons in this electromagnetic wave are superpositions of horizontally polarized and vertically polarized photon states.

If we now were to put a polaroid filter, which only allows horizontally polarized radiation to pass through, in the path of such a beam of electromagnetic radiation, some of the wave would pass through and some of it would be absorbed by the polaroid filter. In fact the intensity of the radiation is proportional to the square of the amplitude of the electric field, so that in this case half the intensity would be transmitted through the polaroid filter. Furthermore all the radiation which does pass through the polaroid filter has horizontal polarization.

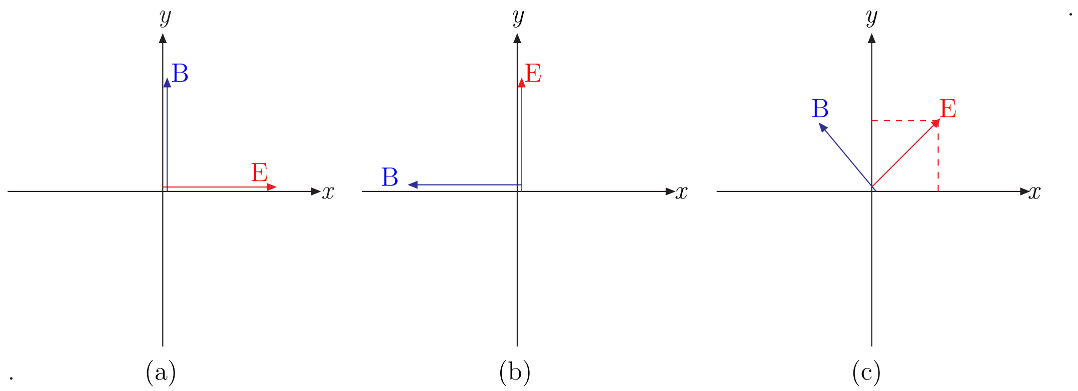

Figure 38: Diagram (a) shows the electric (E) and magnetic (B) field directions for horizontally polarized electromagnetic radiation, diagram (b) shows the electric (E) and magnetic (B) field directions for vertically polarized electromagnetic radiation and diagram (c) shows the electric and magnetic field direction for electromagnetic radiation polarized at 45^0, in which case the electric field can be resolved into a horizontal component and a vertical component, each with an amplitude factor of $1/\sqrt{2}$ leading to an intensity factor of $\frac{1}{2}$. In all three of these diagrams the direction of the wave is in the z-direction, i.e. out of the plane of the diagram.

In terms of a wave, there is no difficulty in conceptualizing this situation. We can do the experiment very easily with two pairs of polaroid sunglasses held with their axes at 45^0 to each other. We will see that the intensity of the light is halved. However, in terms of photons it is a little more difficult to understand. The photon can have one of two states of polarization. We will denote the wavefunction for a photon of horizontally polarized light as Ψ_H and a photon of vertically polarized light as Ψ_V. A photon which passes through a polaroid filter with its axis horizontal is definitely horizontally polarized and has wavefunction Ψ_H. Likewise a photon which passes through a polaroid filter with its axis vertical is definitely vertically polarized and has wavefunction Ψ_V. The wavefunction for a photon of light which passes through a polaroid filter at polarized at 45^0 to the vertical is polarized at 45^0 and its wavefunction is given by

$$\frac{1}{\sqrt{2}}\Psi_H + \frac{1}{\sqrt{2}}\Psi_V,$$

i.e. it is neither horizontally polarized nor vertically polarized but a superposition of the two. If we try to pass this photon through a polaroid filter that will only pass horizontally polarized light then there is a probability of $\frac{1}{2}$ that the photon will pass through and a probability of $\frac{1}{2}$ that it will be absorbed by the polaroid. If the photon does pass through, it will then definitely be horizontally polarized. In this sense the polaroid filter is a measuring device which measures whether the photon is horizontally or vertically polarized. However, the photon is only in such a state of polarization *after* the measurement has been made. Before the photon reaches the polaroid filter it is neither in a horizontal not a vertical polarization state but a superposition of the two. Thus the measurement changes the state of the photon from a superposition of horizontal and vertical polarization, i.e. a photon whose direction of polarization is in a direction at 45^0 to the horizontal axis is transformed by the measurement either into a state in which the direction of polarization is definitely along the horizontal or into a state in which the direction of polarization is definitely along the vertical axis. In the former case (horizontal polarization) the photon passes through the filter, whereas in the latter case (vertical polarization) it is absorbed by the filter.

11.2 Electron Spin

Electrons have a property which is analogous to photon polarization. This is called "spin" and was first postulated in 1925 by Wolfgang Pauli [25]. However there are two features of this spin, which make it impossible to visualize in the same way as we visualize a spinning top:[12]

1. If we measure the spin angular momentum in any direction we can only obtain one of two results: $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$[13] .

2. If the electron is known to have a spin angular momentum $+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$ in a given direction (i.e. the axis of spin is in that direction), then the electron does not have a well-defined component of spin in any other direction. We denote the wavefunction for an electron whose spin component in the z-direction is $+\frac{1}{2}\hbar$ by $\Psi_\uparrow$ (we call this **"spin up"**) and the wavefunction for an electron whose spin component in the z-direction is $-\frac{1}{2}\hbar$ by by $\Psi_\downarrow$ (we call this **"spin down"**). If we have an electron that has a component $+\frac{1}{2}\hbar$ in the x-direction, it would be a superposition of spin up and spin down, with a wavefunction written as

$$\frac{1}{\sqrt{2}}\Psi_\uparrow + \frac{1}{\sqrt{2}}\Psi_\downarrow.$$

The interpretation of this is that this electron has well-defined spin component in the x-direction, but *not* in the z-direction, until such a time that the component in the z-direction is measured. The probability that the result of such a measurement is $+\frac{1}{2}\hbar$ is $\frac{1}{2}$ (the square of the coefficient of $\Psi_\uparrow$) - and similarly the probability that the result of such a measurement is $-\frac{1}{2}\hbar$ is $\frac{1}{2}$ (the square of the coefficient of $\Psi_\downarrow$). The act of a measurement of the z-component of spin changes the spin part of the electron wavefunction either to $\Psi_\uparrow$ or $\Psi_\downarrow$, but before the measurement it is *not* either in one state or the other but in "both". This is analogous to the way that a photon whose electric field is 45^0 to the horizontal axis, is neither horizontally polarized nor vertically polarized, but a superposition of the two - until the photon passes though a polaroid filter with its axis in the horizontal or vertical direction.

[12]An erroneous interpretation of electron spin in terms of a quantized spinning top was proposed at the same time by Kronig, Uhlenbeck and Goudsmit [26].

[13]The symbol $\hbar$ is a short-hand way of writing $h/(2\pi)$), i.e. Planck's constant divided by 2π. It was introduced to Quantum Mechanics because is occurs very often. We shall use this notation from now on.

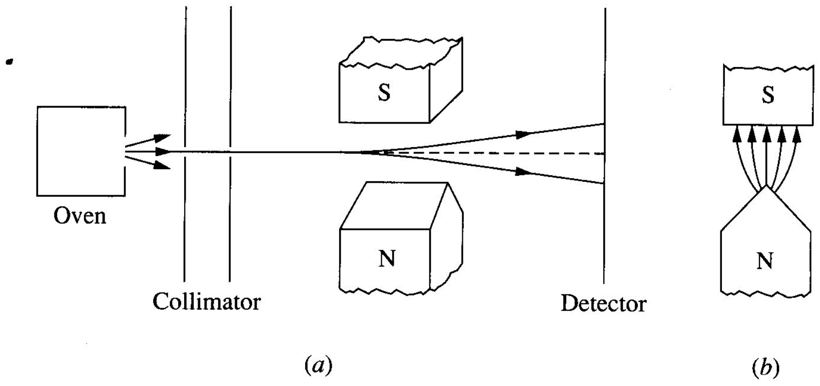

Figure 39: The Stern-Gerlach experiment. Silver atoms from an oven were passed between the poles of a magnet which produced a non-uniform magnetic field in the z-direction. The particles were deflected in the z-direction owing to the fact that they behaved like a small magnet whose strength (i.e. its **magnetic moment**) is proportional to the component of spin in the z-direction. Only two spots were observed, reflecting the fact that the component of spin in the z-direction could only take on one of two possible values.

This is very different from thinking about a spinning top, for which the spin can be of any magnitude, depending on how fast the top is spinning and whichever direction the spin axis is pointing. The classical spin angular momentum has a well-defined component in any direction, but this is not the case for a quantum system, which only possesses a well-defined component of angular momentum in one direction. This is again something that cannot be "understood" or "visualized" but we simply have to accept that since the spin angular momentum of an electron is many orders of magnitude smaller than anything we could ever hope to observe, our usual experience of spin cannot be applied at the sub-microscopic level.

A charged particle with spin behaves like a magnet with a north and south pole in the direction of the spin axis. A magnet is displaced by a non-uniform magnetic field. This is used in the Stern-Gerlach apparatus [27], shown diagrammatically in Fig. 39, in which a beam of electrons[14] passes through a non-uniform magnetic field in the direction of the component that is being measured (e.g the z-axis). An electron with spin component $+\frac{1}{2}\hbar$ (spin up) in the z-direction is displaced in one direction whereas an electron with spin component $-\frac{1}{2}\hbar$ (spin down) is displaced by the same amount but in the opposite direction. After passing through the magnetic field, the path of the charged particle with the same spin property as an electron will be in one of two possible directions depending on two possible values for the z-component of the spin. These two possible directions lead to two spots on the detector screen. The electrons landing on one of these spots have a definite z-component of spin.

[14]In the original experiment of Otto Stern and Walther Gerlach in 1922, silver atoms which emerged from an oven were used, rather than electrons. Silver atoms have the same spin as electrons.

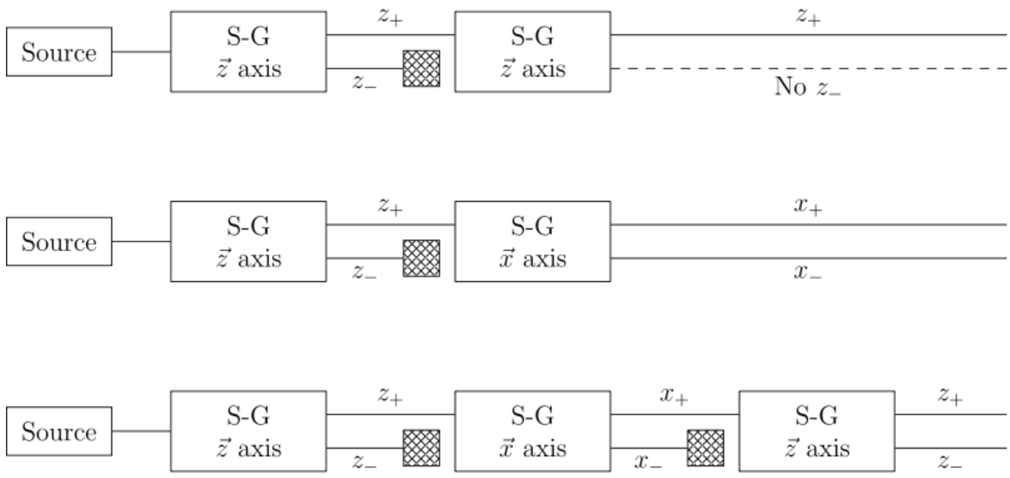

Figure 40: Three different series combinations of Stern-Gerlach apparati. The symbols z^+ and z^- indicate particles with well-defined component of spin ($+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$) in the z-direction, whereas, x^+ and x^- indicate particles with well-defined component of spin ($+\frac{1}{2}\hbar$ or $-\frac{1}{2}\hbar$) in the x-direction.

Note that in the classical picture of spin the component in the z-direction can take any value between these two spots. If the z-component of the spin could take any value between $+\frac{1}{2}\hbar$ and $-\frac{1}{2}\hbar$, as happens in classical physics, then the magnetic field could deflect the path of the charged particle by any amount between two limits and the path which emerges from the magnetic field could be in any direction leading to a continuous line of the screen from the continuum of allowed displacements. The fact that only two spots were seen was one of the surprises foisted upon us by Quantum Physics.

When a particle, whose wavefunction is in general a superposition of spin up and spin down – i.e. a particle that does not possess well-defined component of spin in the z-direction, passes through a Stern-Gerlach apparatus with its magnetic field in the z-direction, its wavefunction is converted either into a spin-up wavefunction (so it emerges in the upper beam) or into a spin-down wavefunction (so it emerges in the lower beam). If we block off the lower beam as shown in Fig. 40 then if the particle passes through, it is definitely in the spin-up state. It will pass through another Stern-Gerlach apparatus with its magnetic field in the z-direction, as shown in the upper diagram of Fig. 40, emerging always in the upper beam.

However, something bizarre happens if between the two Stern-Gerlach apparati which measure the z-component of spin (i.e. the non-uniform magnetic field is in the z-direction), we interpose a Stern-Gerlach apparatus which measures the x-component of spin (i.e. the non-uniform magnetic field is in the x-direction) and block off the beam corresponding to $s_x = -\frac{1}{2}\hbar$. There is then a non-zero probability that the electron will emerge in *either* of the two beams, as seen in the lower diagram of Fig. 40. The point is that the intermediate

Stern-Gerlach apparatus changes the state of the electron into one of well-defined component of spin in the x-direction (either $s_x = +\frac{1}{2}\hbar$ or $s_x = -\frac{1}{2}\hbar$), as indicated in the middle diagram of Fig. 40. Blocking off the beam corresponding to $s_x = -\frac{1}{2}\hbar$ leaves only the electrons with $s_x = +\frac{1}{2}\hbar$, which once again is a superposition of a state in which the z-component of spin is $+\frac{1}{2}\hbar$ and a state in which the z-component of spin is $-\frac{1}{2}\hbar$, so that it could pass into either the upper of lower beam of the third Stern-Gerlach apparatus with the magnetic field in the z-direction.