Dynamics of superfluid-superconducting neutron stars

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Introduction

• It is generally accepted that baryons (neutrons and protons) in the internal layers of neutron stars undergo transition into superfluid/superconducting state at $T \lesssim 10^8 \div 10^{10}$ K.

• Thus, to study dynamics of neutron stars at sufficiently low temperatures one has to develop a system of equations describing superfluid-superconducting mixtures.

• Generally, such mixture can be magnetized, relativistic, and can contain both neutron (Feynman-Onsager) and proton (Abrikosov) vortices.

Introduction

- Dynamics of superfluid-superconducting mixtures has been studied, both in the non-relativistic (e.g., *Vardanyan & Sedrakyan'81; Holm & Kupershmidt'87; Mendell & Lindblom'91; Mendell'91; Sedrakyan & Sedrakyan'95; Glampedakis, Andersson & Samuelsson'11*)
and in the relativistic framework (*Lebedev & Khalatnikov'81; Carter & Langlois'95; Carter & Langlois'98; Langlois, Sedrakyan & Carter'98;Kantor & Gusakov'11; Dommes & Gusakov'15; Andersson, Wells & Vickers'16*).

- "State of the art" paper: *Glampedakis, Andersson & Samuelsson'11 (GAS11)*

essentially nonrelativistic formulation
approximation of vanishing temperature
superfluid-superconducting mixture;
type-II proton superconductivity
vortices; mutual friction;
correct treatment of the magnetic field ($B \neq H$);

Introduction

- So, initially, our aim was to extend the results of GAS11 to relativistic framework and to include into consideration the finite-temperature effects.

- Eventually, the equations that we derived turn out to be more general than those of GAS11 (even in the non-relativistic limit)

- We have also found that our equations *differ* from MHD of GAS11

Introduction

All these results will be discussed in my talk,
which is based on the following works

- Gusakov M.E. , **PRD** (2016)
“Relativistic formulation of the Hall-Vinen-Bekarevich-Khalatnikov superfluid hydrodynamics”

- Gusakov M.E., Dommes V.A. , **arXiv: 1607.01629** (submitted to **PRD**)
“Relativistic dynamics of superfluid-superconducting mixtures in the presence of topological defects and the electromagnetic field, with application to neutron stars”

- Dommes V.A., Gusakov M.E. (in preparation)
“Vortex buoyancy in superfluid and superconducting neutron stars”

The result:

Particle and energy-momentum conservation:

$$\partial_\mu j^\mu_{(j)} = 0$$

$$\partial_\mu T^{\mu\nu} = 0$$

$$j^\mu_{(i)} = n_i u^\mu + Y_{ik} w^\mu_{(k)}$$

$$j^\mu_{(e)} = n_e u^\mu$$

$$T^{\mu\nu} = (P + \varepsilon)\, u^\mu u^\nu + P g^{\mu\nu} + Y_{ik}\left(w^\mu_{(i)} w^\nu_{(k)} + \mu_i\, w^\mu_{(k)} u^\nu + \mu_k\, w^\nu_{(i)} u^\mu\right) + \Delta T^{\mu\nu}$$

$$\Delta T^{\mu\nu} = \mathcal{T}^{\mu\nu}_{(\mathrm{E})} + \mathcal{T}^{\mu\nu}_{(\mathrm{M})} + \mathcal{T}^{\mu\nu}_{(\mathrm{VE})} + \mathcal{T}^{\mu\nu}_{(\mathrm{VM})}$$

$$\mathcal{T}^{\mu\nu}_{(\mathrm{E})} = \frac{1}{4\pi}\left(\perp^{\mu\nu} D^\alpha E_\alpha - D^\mu E^\nu\right)$$

$$\mathcal{T}^{\mu\nu}_{(\mathrm{M})} = \frac{1}{4\pi}\left({}^\perp G^{\mu\alpha}\, {}^\perp F^\nu{}_\alpha + u^\nu\, {}^\perp G^{\mu\alpha}\, E_\alpha + u^\mu\, {}^\perp G^{\nu\alpha}\, E_\alpha\right)$$

$$\mathcal{T}^{\mu\nu}_{(\mathrm{VE})} = \perp^{\mu\nu} \mathcal{W}^\alpha_{(\mathrm{E}i)} \mathcal{V}_{(\mathrm{E}i)\alpha} - \mathcal{W}^\mu_{(\mathrm{E}i)} \mathcal{V}^\nu_{(\mathrm{E}i)}$$

$$\mathcal{T}^{\mu\nu}_{(\mathrm{VM})} = {}^\perp\mathcal{W}^{\mu\alpha}_{(i)}\, {}^\perp\mathcal{V}^\nu_{(i)\,\alpha} + u^\nu\, {}^\perp\mathcal{W}^{\mu\alpha}_{(i)}\, \mathcal{V}_{(\mathrm{E}i)\alpha} + u^\mu\, {}^\perp\mathcal{W}^{\nu\alpha}_{(i)}\, \mathcal{V}_{(\mathrm{E}i)\alpha}$$

Second law of thermodynamics:

$$d\varepsilon = T\, dS + \mu_i\, dn_i + \mu_e\, dn_e + \frac{Y_{ik}}{2}\, d\left(w^\alpha_{(i)} w_{(k)\alpha}\right) + d\varepsilon_{\mathrm{add}}$$

$$d\varepsilon_{\mathrm{add}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}^\mu_{(\mathrm{E}i)} d\mathcal{W}_{(\mathrm{E}i)\mu} + \mathcal{W}_{(\mathrm{M}i)\mu} d\mathcal{V}^\mu_{(\mathrm{M}i)}$$

The result:

“Superfluid” equations for neutrons and protons:

$$u^{\nu}\mathcal{V}_{(i)\mu\nu} = \mu_i n_i\, f_{(i)\mu}$$

$$u_{\mu} w^{\mu}_{(i)} = 0$$

$$\mathcal{V}^{\mu\nu}_{(i)} \equiv \partial^{\mu}\left[w^{\nu}_{(i)} + \mu_i u^{\nu}\right] - \partial^{\nu}\left[w^{\mu}_{(i)} + \mu_i u^{\mu}\right] + e_i F^{\mu\nu}$$

vorticity tensor

$$f^{\mu}_{(i)} = \alpha_i \perp^{\mu\nu} \mathcal{V}_{(i)\nu\lambda} W_{(i)\delta} \perp^{\lambda\delta} + \frac{\beta_i - \gamma_i}{\mathcal{V}_{(\mathrm{M}i)}} \perp^{\mu\eta} \perp^{\nu\sigma} \mathcal{V}_{(i)\eta\sigma} \mathcal{V}_{(i)\lambda\nu} W_{(i)\delta} \perp^{\lambda\delta} + \gamma_i \mathcal{V}_{(\mathrm{M}i)} W_{(i)\delta} \perp^{\mu\delta}$$

$$W^{\mu}_{(i)} \equiv \frac{1}{n_i}\left[Y_{ik} w^{\mu}_{(k)} + \partial_{\alpha}\mathcal{W}^{\mu\alpha}_{(i)}\right]$$

Maxwell’s equations in the medium:

$$\begin{cases} \partial_{\alpha}{}^{\star}F^{\alpha\beta} = 0 \\ \partial_{\alpha} G^{\alpha\beta} = -4\pi\, J^{\beta}_{(\mathrm{free})} \end{cases}$$

$$J^{\mu}_{(\mathrm{free})} = e_p(n_p - n_e)u^{\mu} + e_i\, Y_{ik} w^{\mu}_{(k)}$$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$G^{\alpha\beta} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{pmatrix}$$

Idea of derivation

Initial idea *[Bekarevich & Khalatnikov'61]*: consistency between conservation laws and entropy equation.

- Consider a system in the absence of dissipation

- Assume that we know the form of the expressions for particle current densities as well as the form of the second law of thermodynamics

$$d\varepsilon = T dS + \mu_i dn_i + \ldots$$

- Then it is possible to constrain the system energy-momentum tensor from the requirement that the entropy is not produced in the system (which means that the entropy density is subject to continuity equation)

$$\partial_\mu (S u^\mu) = 0$$

S	entropy density
u^μ	four-velocity of normal excitations

That is, by specifying, for example, vortex contribution

$$d\varepsilon = TdS + \mu_i dn_i + \ldots + d\varepsilon_{\mathrm{vortex}}$$

$$T^{\mu\nu} = (P+\varepsilon)u^\mu u^\nu + Pg^{\mu\nu} + \ldots + \Delta T^{\mu\nu}_{\mathrm{vortex}}$$

one finds the correction to the energy-momentum tensor

What physics is included (brief account)?

- **fully relativistic formulation**
- **npe-composition** (additional particle species can be easily included)
- **neutrons are superfluid, protons are superconducting**
- **entrainment and finite temperature effects**
- **both types (I and II) of proton superconductivity**
- **electromagnetic effects**
- **neutron and proton vortices** (or magnetic domains for type-I proton SP)
- **dissipation (e.g., mutual friction)**

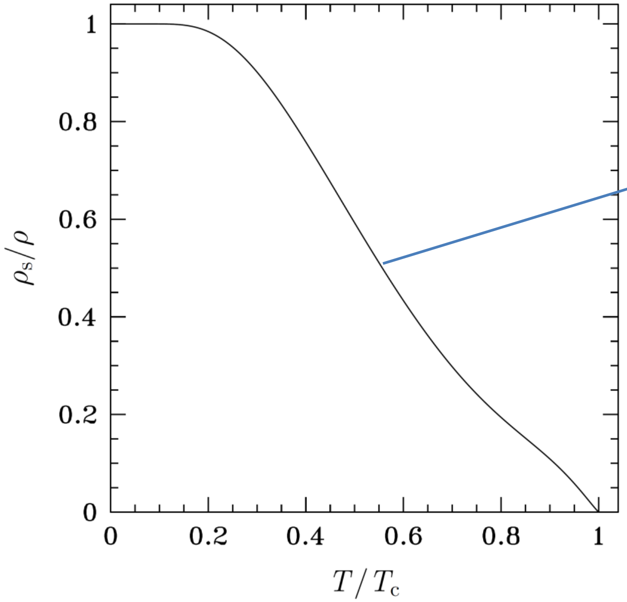
In what follows I will discuss some of these physical “ingredients” in more detail

Importance of finite-temperature effects

- Zero-temperature approximation is justified only if $T \ll T_{cn},\, T_{cp}$ everywhere in the star.

  In many interesting situations (e.g., in magnetars, LMXBs) this is not the case.

- Note that the condition $T \ll \mu_n,\, \mu_p$ does ***not*** justify the use of the zero-temperature hydrodynamics.

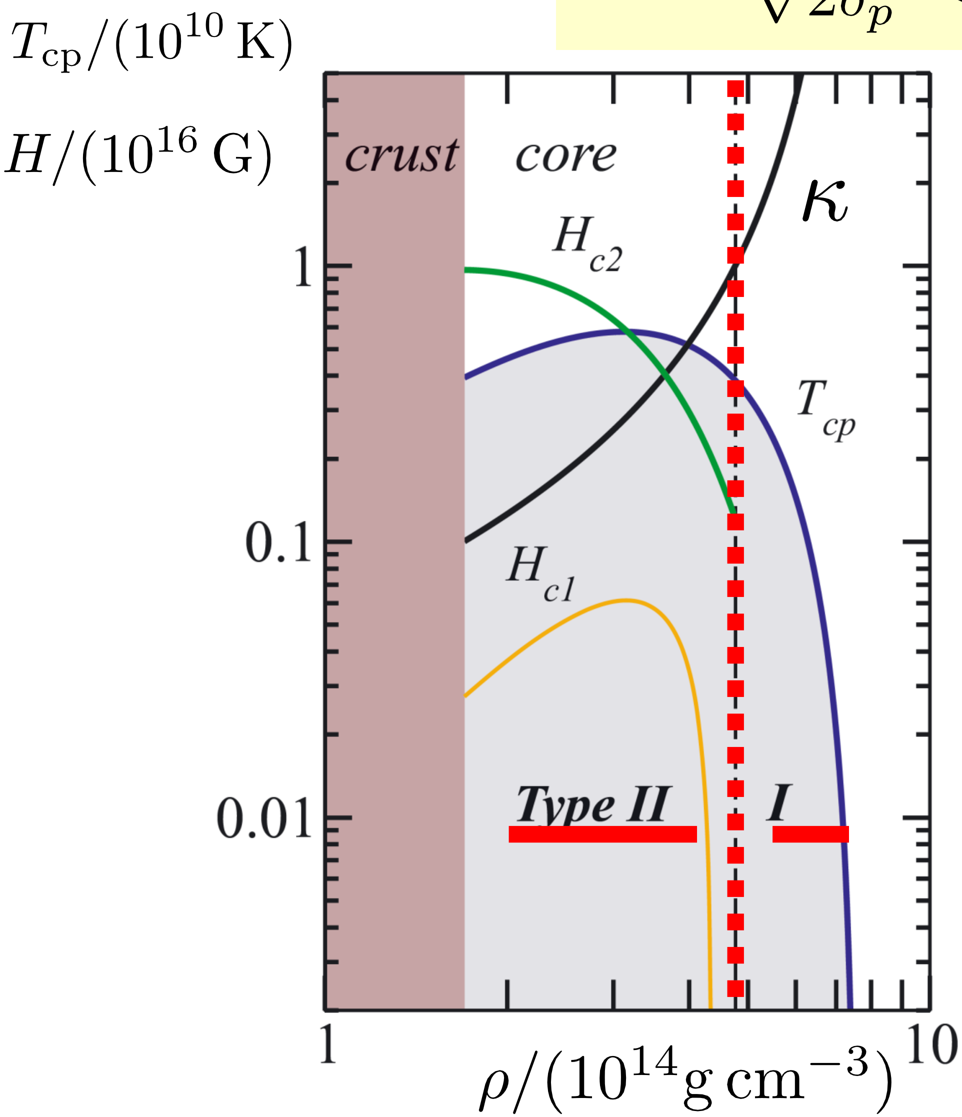


superfluid density is a strong function of temperature!

$$\rho_s = \rho \text{ at } T = 0$$
$$\rho_s = 0 \text{ at } T = T_c$$

What physics is included: Type I/II proton superconductivity

$$\kappa = \frac{\xi_p}{\sqrt{2}\delta_p} \begin{array}{l} > 1 \\ < 1 \end{array}$$
=> I type
=> II type



Coherence length:

$$\xi_\mathrm{p} \approx 5\times10^{-12}\left(\frac{m_\mathrm{p}}{m_\mathrm{p}^*}\right)\rho_{14}^{1/3}\left(\frac{x_\mathrm{p}}{0.1}\right)^{1/3}\left(\frac{10^9\,\mathrm{K}}{T_\mathrm{cp}}\right)\ \mathrm{cm}$$

London penetration depth:

$$\delta_p \approx 9\times10^{-12}\left(\frac{m_\mathrm{p}^*}{m_\mathrm{p}}\right)^{1/2}\rho_{14}^{-1/2}\left(\frac{0.1}{x_\mathrm{p}}\right)^{1/2}\ \mathrm{cm}$$

$$H_c \sim 10^{15}\ \mathrm{G}$$

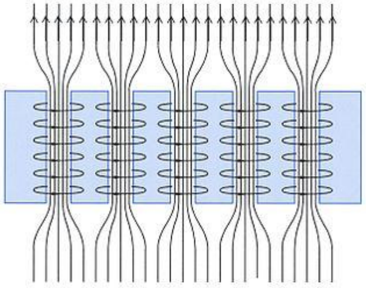
Credit: *Glampedakis et al.'11*

What is the difference between neutron star interiors with type-I and type-II superconductors?

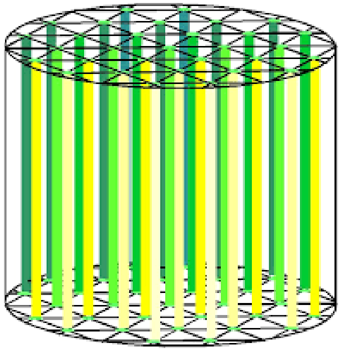
• Transition to superconducting state occurs at constant magnetic flux (*Baym et al. 1969*; typical cooling timescale is much shorter than the magnetic flux expulsion timescale)

• Under these conditions type-I superconductor undergoes transition into an "intermediate" state, while type-II superconductors – into mixed state.

Intermediate state of type-I superconductor:
consists of alternating domains of superconducting (field-free) regions and normal regions hosting magnetic field

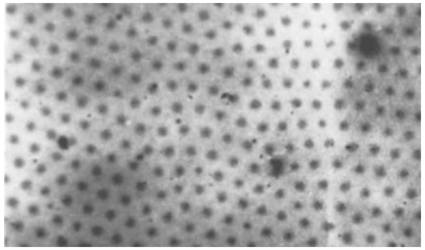


Mixed state of type-II superconductor:
consists of Abrikosov vortices (fluxtubes)



Intermediate vs mixed state

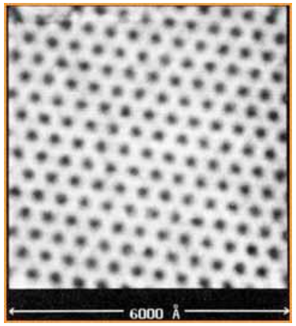
Huebener'00



normal regions are dark

Typical "open topology" intermediate state domain structure

Hess et al'89



Mixed state: Abrikosov vortices

Distance between neighboring flux tubes:

$$b \sim \sqrt{R\delta} \sim 2 \times 10^{-3} \text{ cm}$$

(Huebener'13, Sedrakian'05, DeGennes'66)

Distance between neighboring vortices:

$$b \sim \sqrt{\frac{\hat{\phi}_{p0}}{\pi\, B}} \approx 2.6 \times 10^{-10} \sqrt{\frac{10^{12}\text{ G}}{B}} \text{cm}$$

Flux tube radius:

$$a \approx b(B/H_c)^{1/2} \sim 6 \times 10^{-5} \text{ cm}$$

"Vortex radius":

$$\delta_p \approx 9 \times 10^{-12} \left(\frac{m_{\rm p}^*}{m_{\rm p}}\right)^{1/2} \rho_{14}^{-1/2} \left(\frac{0.1}{x_{\rm p}}\right)^{1/2} \text{ cm}$$

Number of flux quanta in a flux tube:

$$N_\phi \approx \pi a^2 H_{\rm c}/\hat{\phi}_{p0} \approx 6 \times 10^{13}$$

Number of flux quanta in a vortex:

$$1$$

$$B = 10^{12} \text{ G} \qquad H_{\rm c} = 10^{15} \text{ G}$$

What physics is included: vortices

NEUTRON VORTICES

Neutron vortices appear in neutron stars in order to imitate solid-body rotation with a non-superfluid component.

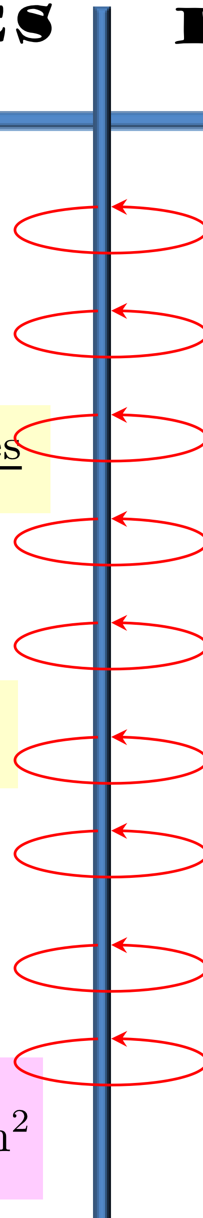
Vortex density = $\dfrac{2m_{\rm n}\Omega}{\pi\hbar} \approx \dfrac{6\times10^3}{P}\,\dfrac{\rm vortices}{\rm cm^2}$

P is the neutron star period in seconds.

Total number of vortices $\sim \dfrac{2\times10^{16}}{P}$

Intervortex spacing $\sim \dfrac{10^{-2}}{\sqrt{P}}$ cm

Magnetic flux $= \dfrac{\pi\hbar c}{e_p}\dfrac{Y_{np}}{Y_{pp}} \sim 4\times10^{-8}\ {\rm G\,cm^2}$



PROTON VORTICES

(assuming proton SP of type-II)

Vortex density=

$$= \frac{B}{\hat{\phi}_{p0}} \approx 4.8\times10^{18}\left(\frac{B}{10^{12}\ {\rm G}}\right)\ {\rm cm}^{-2}$$

Total number of vortices $\sim$

$$\sim \frac{B\,\pi R^2}{\hat{\phi}_{p0}} \approx 1.5\times10^{31}\left(\frac{B}{10^{12}\ {\rm G}}\right)$$

Intervortex spacing $\sim$

$$\sim \sqrt{\frac{\hat{\phi}_{p0}}{\pi\,B}} \approx 2.6\times10^{-10}\sqrt{\frac{10^{12}\ {\rm G}}{B}}\,{\rm cm}$$

Magnetic flux $= \dfrac{\pi\hbar c}{e_p} \sim 2\times10^{-7}\ {\rm G\,cm^2}$

The suggested dynamic equations naturally account for:

✓ Both neutron and proton vortex energies

$$E_{\mathrm{V}} = \frac{\rho_s \kappa^2}{4\pi} \ln\left(\frac{b}{a}\right)$$

✓ Mutual friction (as well as Magnus force etc.)

$$\boldsymbol{F}_{\mathrm{D}} = -\kappa \rho_s \mathcal{R}(\boldsymbol{V}_{\mathrm{L}} - \boldsymbol{V}_{\mathrm{norm}})$$

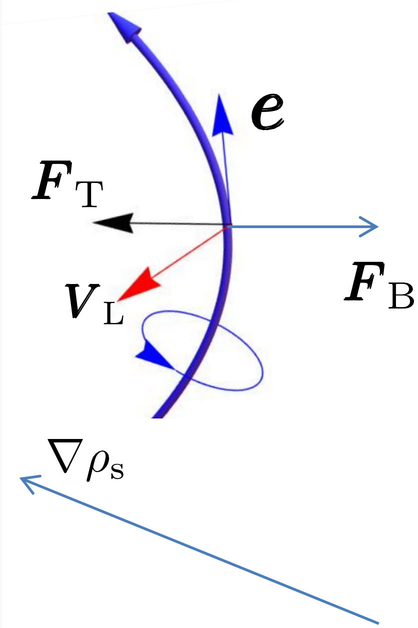
✓ Vortex tension (appears when vortex is bent)

$$\boldsymbol{F}_{\mathrm{T}} = E_{\mathrm{V}} \underbrace{(\mathbf{e}\boldsymbol{\nabla})\mathbf{e}}_{1/R}$$

vortex energy per unit length divided by curvature radius *R*

✓ Vortex buoyancy

$$\boldsymbol{F}_{\mathrm{B}} = -\nabla E_{\mathrm{V}} = -E_{\mathrm{V}} \frac{\nabla_{\perp} \rho_{\mathrm{s}}}{\rho_{\mathrm{s}}}$$



$$\nabla_{\perp} = \nabla - \boldsymbol{e}(\boldsymbol{e}\nabla)$$

Vortex buoyancy in more detail

$$\boldsymbol{F}_{\rm B} = -\nabla E_{\rm V} = -E_{\rm V} \frac{\nabla_\perp \rho_{\rm s}}{\rho_{\rm s}}$$

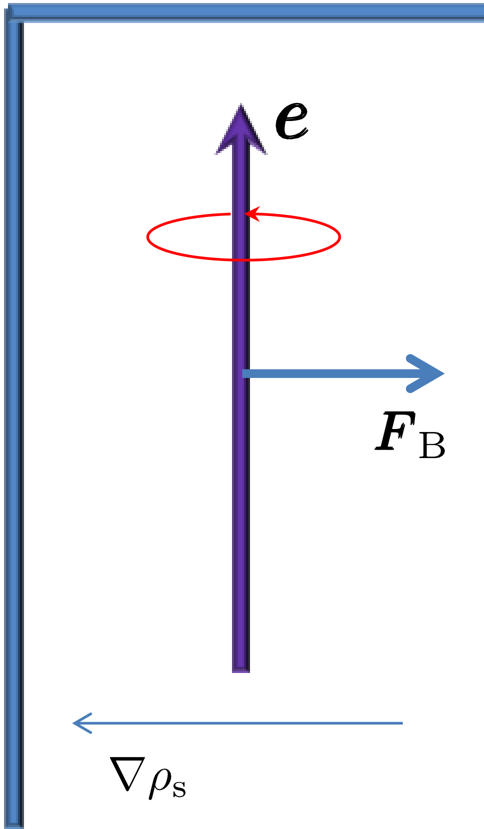
acts to push a vortex out into the region with **smaller** superfluid density

• usually it is either ignored (as in the Hall-Vinen hydrodynamics) or introduced "by hands" in the form (*e.g., Muslimov & Tsygan'85, Elfritz et al.'16, …*)

$$\boldsymbol{F}_{\rm B} = -E_{\rm V} \frac{\boldsymbol{g}}{c_{\rm s}^2}$$

$\boldsymbol{g}$ gravitation acceleration

$c_{\rm s}$ speed of sound



which is popular in studies of the magnetic flux expulsion.

• The latter expression reduces to the correct one **only** for a one-component liquid at zero temperature.

• It should be noted that the correct buoyancy force is contained implicitly in the Bekarevich & Khalatnikov superfluid hydrodynamics and its multifluid extensions.

SP-SFL mixture as a medium with $\boldsymbol{H} \neq \boldsymbol{B}$ and $\boldsymbol{D} \neq \boldsymbol{E}$

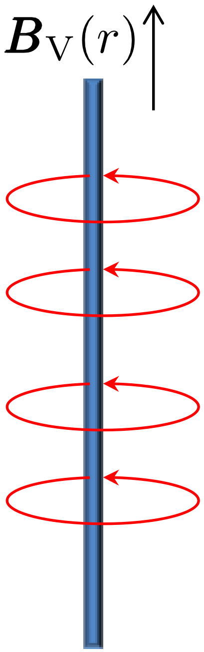
• The next interesting feature of the dynamic equations that we propose is that they consider a superfluid-superconducting mixture as a medium in which $\boldsymbol{H} \neq \boldsymbol{B}$ and $\boldsymbol{D} \neq \boldsymbol{E}$.
Thus they are coupled with the standard Maxwell's equations in the medium.

Maxwell's equations in the medium:

$$
\begin{cases}
\operatorname{div} \boldsymbol{D} = 4\pi \rho_{\text{free}}, \\
\operatorname{rot} \boldsymbol{E} = -\dfrac{1}{c}\dfrac{\partial \boldsymbol{B}}{\partial t}, \\
\operatorname{div} \boldsymbol{B} = 0, \\
\operatorname{rot} \boldsymbol{H} = \dfrac{4\pi}{c}\boldsymbol{J}_{\text{free}} + \dfrac{1}{c}\dfrac{\partial \boldsymbol{D}}{\partial t}
\end{cases}
$$

$\boldsymbol{B}$	magnetic induction
$\boldsymbol{H}$	magnetic field
$\boldsymbol{E}$	electric field
$\boldsymbol{D}$	electric displacement

SP-SFL mixture as a medium with $\boldsymbol{H} \neq \boldsymbol{B}$ and $\boldsymbol{D} \neq \boldsymbol{E}$



- Why $\boldsymbol{H} \neq \boldsymbol{B}$? *Carter, Prix, Langlois'00; Glampedakis et al.'11*

- Short answer: Because $\boldsymbol{H} = \boldsymbol{B} - 4\pi \boldsymbol{M}$ and $\boldsymbol{M} \neq 0$

  (textbook result)

- Each vortex has a magnetic field supported by superconducting currents

- These "molecular" currents contribute to magnetization $\boldsymbol{M}$ (magnetic moment of the unit volume)

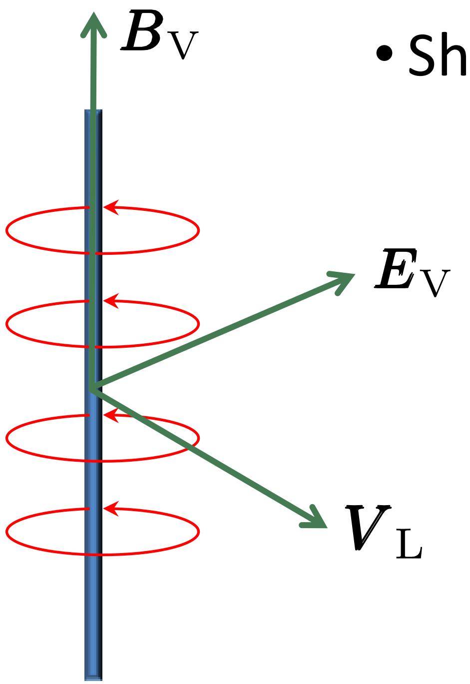
- It is straightforward to show: $|\boldsymbol{M}| = \dfrac{1}{4\pi} \phi_0 \, N_{\rm V}$

vortex magnetic flux

areal vortex density

SP-SFL mixture as a medium with $\boldsymbol{H} \neq \boldsymbol{B}$ and $\boldsymbol{D} \neq \boldsymbol{E}$

- Why $\boldsymbol{D} \neq \boldsymbol{E}$?   *Gusakov & Dommes'16*



- Short answer: Because $\boldsymbol{D} = \boldsymbol{E} + 4\pi \boldsymbol{P}$ and $\boldsymbol{P} \neq 0$
  (textbook result)

- Each ***moving*** vortex induces an electric field

$$\boldsymbol{E}_{\rm V} = -\frac{1}{c} \boldsymbol{V}_{\rm L} \times \boldsymbol{B}_{\rm V}$$

and electric charge: $\rho_{\rm c} = \frac{1}{4\pi} \,{\rm div}\, \boldsymbol{E}_{\rm V}$

- Using $\rho_{\rm c}$, one can calculate the electric polarization vector $\boldsymbol{P}$ (or the electric dipole moment of a unit volume) and find:

$$\boldsymbol{P} = -\frac{1}{c} \boldsymbol{V}_{\rm L} \times \boldsymbol{M} \neq 0$$

EM + vortex energy density

- By specifying the energy density we specify the energy-momentum tensor

- What is the contribution to the system energy density from the electromagnetic field and vortices?

$$d\varepsilon_{\mathrm{EM+vortex}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}^\mu_{(\mathrm{E}i)} d\mathcal{W}_{(\mathrm{E}i)\mu} + \mathcal{W}_{(\mathrm{M}i)\mu} d\mathcal{V}^\mu_{(\mathrm{M}i)}$$

EM + vortex energy density

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$$d\varepsilon_{\mathrm{EM+vortex}} = \underbrace{\frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu}_{\text{“electromagnetic” contribution}} + \mathcal{V}^\mu_{(\mathrm{E}i)} d\mathcal{W}_{(\mathrm{E}i)\mu} + \mathcal{W}_{(\mathrm{M}i)\mu} d\mathcal{V}^\mu_{(\mathrm{M}i)}$$

- has a standard form

- depends on the four-vectors which reduce to

$$E^\mu = (0,\, \boldsymbol{E}) \qquad D^\mu = (0,\, \boldsymbol{D})$$
$$B^\mu = (0,\, \boldsymbol{B}) \qquad H^\mu = (0,\, \boldsymbol{H})$$

in the comoving frame $u^\mu = (1,0,0,0)$,
moving with the normal liquid component

EM + vortex energy density

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$$d\varepsilon_{\mathrm{EM+vortex}} = \underbrace{\frac{1}{4\pi}E_\mu dD^\mu + \frac{1}{4\pi}H_\mu dB^\mu}_{\text{“electromagnetic” contribution}} + \mathcal{V}^\mu_{(\mathrm{E}i)} d\mathcal{W}_{(\mathrm{E}i)\mu} + \mathcal{W}_{(\mathrm{M}i)\mu} d\mathcal{V}^\mu_{(\mathrm{M}i)}$$

- Generally, one can say that it depends on two tensors:

electromagnetic tensor

$$F^{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

complementary tensor

$$G^{\alpha\beta} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{pmatrix}$$

EM + vortex energy density

- By specifying the energy density we specify the energy-momentum tensor
- What is the contribution to the system energy density from the electromagnetic field and vortices?

$$d\varepsilon_{\mathrm{EM+vortex}} = \frac{1}{4\pi}\,E_\mu dD^\mu + \frac{1}{4\pi}\,H_\mu dB^\mu \underbrace{+\mathcal{V}^\mu_{(\mathrm{E}i)}d\mathcal{W}_{(\mathrm{E}i)\mu} + \mathcal{W}_{(\mathrm{M}i)\mu}d\mathcal{V}^\mu_{(\mathrm{M}i)}}_{\text{“vortex” contribution}}$$

- Depends on the vorticity tensor $\mathcal{V}^{\mu\nu}_{(i)} = \partial^\mu V^\nu_{(\mathrm{s}i)} - \partial^\nu V^\mu_{(\mathrm{s}i)} + e_i F^{\mu\nu}$

which is related to the density of vortices (non-relativistic analogue: $m_i\,\mathrm{curl}\,\boldsymbol{V}_{(\mathrm{s}i)} + \frac{e_i}{c}\boldsymbol{B}$)

superfluid velocity

- Depends on a complementary tensor $\mathcal{W}^{\mu\nu}_{(i)}$

$$\begin{cases} \mathcal{V}^\mu_{(\mathrm{E}i)} \equiv u_\nu \mathcal{V}^{\mu\nu}_{(i)} \\ \mathcal{V}^\mu_{(\mathrm{M}i)} \equiv \frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,u_\nu\,\mathcal{V}_{(i)\alpha\beta} \end{cases} \qquad \begin{cases} \mathcal{W}^\mu_{(\mathrm{E}i)} \equiv u_\nu \mathcal{W}^{\mu\nu}_{(i)} \\ \mathcal{W}^\mu_{(\mathrm{M}i)} \equiv \frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,u_\nu\,\mathcal{W}_{(i)\alpha\beta} \end{cases}$$

EM + vortex energy-momentum tensor

- Electromagnetic and vortex contributions to the second law of thermodynamics induce corrections to the energy-momentum tensor

$$d\varepsilon_{\mathrm{EM+vortex}} = \frac{1}{4\pi} E_\mu dD^\mu + \frac{1}{4\pi} H_\mu dB^\mu + \mathcal{V}^{\mu}_{(\mathrm{E}i)} d\mathcal{W}_{(\mathrm{E}i)\mu} + \mathcal{W}_{(\mathrm{M}i)\mu} d\mathcal{V}^{\mu}_{(\mathrm{M}i)}$$

electromagnetic correction

$$\mathcal{T}^{\mu}_{(\mathrm{E})\,\nu} = \frac{1}{4\pi}\left({}^{\parallel}G^{\mu\alpha}F_{\nu\alpha} + u^\mu u^\gamma \perp_{\nu\beta} F^{\alpha\beta}G_{\alpha\gamma} + g^{\mu}{}_{\nu} D_\alpha E^\alpha\right)$$

$$\mathcal{T}^{\mu}_{(\mathrm{M})\,\nu} = \frac{1}{4\pi}\left({}^{\perp}G^{\mu\alpha}F_{\nu\alpha} - u^\mu u^\gamma \perp_{\nu\beta} G^{\alpha\beta}F_{\alpha\gamma}\right)$$

Related to Abraham tensor of ordinary electrodynamics

vortex correction

$$\mathcal{T}^{\mu}_{(\mathrm{VE})\ \nu} = {}^{\parallel}\mathcal{W}^{\mu\alpha}_{(i)}\mathcal{V}_{(i)\nu\alpha} + u^\mu u^\gamma \perp_{\nu\beta} \mathcal{V}^{\alpha\beta}_{(i)}\mathcal{W}_{(i)\alpha\gamma} + g^{\mu}{}_{\nu} \mathcal{W}_{(\mathrm{E}i)\alpha}\mathcal{V}^{\alpha}_{(\mathrm{E}i)}$$

$$\mathcal{T}^{\mu}_{(\mathrm{VM})\ \nu} = {}^{\perp}\mathcal{W}^{\mu\alpha}_{(i)}\mathcal{V}_{(i)\nu\alpha} - u^\mu u^\gamma \perp_{\nu\beta} \mathcal{W}^{\alpha\beta}_{(i)}\mathcal{V}_{(i)\alpha\gamma}$$

“Closing” the system of equations

- We have found that the second law of thermodynamics and energy-momentum tensor depend on the electromagnetic and vorticity tensors $F^{\mu\nu},\ \mathcal{V}^{\mu\nu}_{(i)}$, as well as on the complementary tensors $G^{\mu\nu},\ \mathcal{W}^{\mu\nu}_{(i)}$

- To close the system of equations we need to express the tensors $G^{\mu\nu},\ \mathcal{W}^{\mu\nu}_{(i)}$ through $F^{\mu\nu},\ \mathcal{V}^{\mu\nu}_{(i)}$. The relation between these tensors will **depend on a detailed microphysics model of a mixture.**

- This is in full analogy with the ordinary electrodynamics where, in order to close the system one needs to specify the relation between the tensors $F^{\mu\nu}$ and $G^{\mu\nu}$

e.g., $\boldsymbol{D} = \varepsilon\,\boldsymbol{E}$ and $\boldsymbol{B} = \mu\boldsymbol{H}$

$$F^{\alpha\beta} = \begin{pmatrix} 0 & \boxed{E_1 \quad E_2 \quad E_3} \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

$$G^{\alpha\beta} = \begin{pmatrix} 0 & D_1 & D_2 & D_3 \\ -D_1 & 0 & H_3 & -H_2 \\ -D_2 & -H_3 & 0 & H_1 \\ -D_3 & H_2 & -H_1 & 0 \end{pmatrix}$$

“Closing” the system of equations

- In the next slides we will discuss the simplified dynamic equations in the so called “MHD” approximation. In that case the complementary tensors can be expressed as:

$$
\begin{cases}
G^{\mu\nu} = 0 \\
\mathcal{W}^{\mu\nu}_{(n)} = 0 \\
\mathcal{W}^{\mu\nu}_{(p)} = \dfrac{H_{c1} c}{4\pi e_p B} \perp^{\mu\alpha} \perp^{\nu\beta} F_{\alpha\beta}
\end{cases}
$$

energy of neutron vortices is neglected

$$\perp^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$$

"MHD" approximation

Assumptions:

- protons form type-II superconductor
- vortex interactions are neglected
- diffusion of normal thermal excitations is suppressed

- In neutron stars: $B \approx B_{\mathrm{V}p} \gg B_{\mathrm{V}n},\ H,\ E,\ D$

$D \sim \nabla \mu_e / e$

magnetic field stored in proton vortices

This allows one to simplify substantially general equations describing superfluid-superconducting mixture

"MHD" approximation

1. One can discard the Maxwell's equations:

$$\mathrm{div}\,\boldsymbol{D} = 4\pi\rho_{\mathrm{free}}$$
$$\mathrm{curl}\,\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{J}_{\mathrm{free}} + \frac{1}{c}\frac{\partial \boldsymbol{D}}{\partial t} \quad \Longleftrightarrow \quad \partial_\alpha G^{\alpha\beta} = -4\pi\, J^\beta_{(\mathrm{free})}$$

and set to zero the four-current density of free charges in other equations:

$$J^\mu_{(\mathrm{free})} = e_j j^\mu_{(j)} = e_p(n_p - n_e)u^\mu + e_i\, Y_{ik} w^\mu_{(k)} = 0$$

In the absence of entrainment this means that protons approximately co-move with electrons.

“MHD” approximation

2. The electromagnetic + vortex contribution to the second law of thermodynamics simplifies

$$d\varepsilon_{\mathrm{EM+vortex}} \approx \frac{H_{\mathrm{c1}}}{4\pi B}\, B_\mu\, dB^\mu$$

(neglect contribution from neutron vortices)

$$\mathcal{V}^\mu_{(\mathrm{M}p)} \approx \frac{e_p}{c} B^\mu$$

$$H_{\mathrm{c1}} = \frac{4\pi E_{\mathrm{V}p}}{\phi_{p0}}$$

vortex areal density

which is simply the statement: $d\varepsilon_{\mathrm{EM+vortex}} \approx E_{\mathrm{V}}\, dN,$ because $N \propto |B^\mu|$

vortex energy per unit length

NOTE: This expression for EM+vortex energy density corresponds to the following choice of complementary tensors

$$\begin{cases} G^{\mu\nu} = 0 \\ \mathcal{W}^{\mu\nu}_{(n)} = 0 \\ \mathcal{W}^{\mu\nu}_{(p)} = \dfrac{H_{\mathrm{c1}} c}{4\pi e_p B} \perp^{\mu\alpha} \perp^{\nu\beta} F_{\alpha\beta} \end{cases}$$

$$\perp^{\mu\nu} \equiv g^{\mu\nu} + u^\mu u^\nu$$

Full system of MHD equations

Particle and energy-momentum conservation:

$$\partial_\mu j^\mu_{(j)} = 0$$
$$\partial_\mu T^{\mu\nu} = 0$$

$$j^\mu_{(i)} = n_i u^\mu + Y_{ik} w^\mu_{(k)}$$
$$j^\mu_{(e)} = n_e u^\mu$$
$$T^{\mu\nu} = (P+\varepsilon)\, u^\mu u^\nu + P g^{\mu\nu} + Y_{ik} \left(w^\mu_{(i)} w^\nu_{(k)} + \mu_i\, w^\mu_{(k)} u^\nu + \mu_k\, w^\nu_{(i)} u^\mu \right) + \Delta T^{\mu\nu}$$

$$\Delta T^{\mu\nu} = \frac{H_{\rm c1}}{4\pi B} \left({}^\perp F^{\mu\alpha}\, {}^\perp F^\nu_{\ \alpha} + u^\nu\, {}^\perp F^{\mu\alpha}\, E_\alpha + u^\mu\, {}^\perp F^{\nu\alpha}\, E_\alpha \right)$$

$${}^\perp F^{\mu\nu} \equiv \epsilon^{\alpha\beta\mu\nu} u_\beta B_\alpha$$

Second law of thermodynamics:

$$d\varepsilon = T\, dS + \mu_i\, dn_i + \mu_e\, dn_e + \frac{Y_{ik}}{2}\, d\left(w^\alpha_{(i)} w_{(k)\alpha} \right) + \frac{H_{\rm c1}}{4\pi B}\, B_\mu\, dB^\mu$$

“Superfluid” equations for neutrons and protons:

$$u^\nu \mathcal{V}_{(n)\mu\nu} = \mu_n n_n\, f_{(n)\mu}$$
$$E_\mu = \mu_p n_p\, f_{(p)\mu}$$

$$f^\mu_{(n)} = \alpha_n \perp^{\mu\nu} \mathcal{V}_{(n)\nu\lambda} W_{(n)\delta} \perp^{\lambda\delta} + \frac{\beta_n}{\mathcal{V}_{(\mathrm{M}n)}} \perp^{\mu\eta} \perp^{\nu\sigma} \mathcal{V}_{(n)\eta\sigma} \mathcal{V}_{(n)\lambda\nu} W_{(n)\delta} \perp^{\lambda\delta}$$

$$f^\mu_{(p)} = \alpha_p \perp^{\mu\nu} F_{\nu\lambda} W_{(p)\delta} \perp^{\lambda\delta} + \frac{\beta_p}{B} \perp^{\mu\eta} \perp^{\nu\sigma} F_{\eta\sigma} F_{\lambda\nu} W_{(p)\delta} \perp^{\lambda\delta}$$

$$\partial_\alpha {}^\star F^{\alpha\beta} = 0$$
$$J^\mu_{(\mathrm{free})} = e_j j^\mu_{(j)} = e_p (n_p - n_e) u^\mu + e_i Y_{ik} w^\mu_{(k)} = 0$$

Electromagnetic sector

Evolution equation for the magnetic field

- The MHD approximation discussed above allows one to obtain a simple nonrelativistic evolution equation for the magnetic field (see also *Konenkov & Geppert'01*):

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathrm{curl}\,(\boldsymbol{B} \times \boldsymbol{v}_{\mathrm{L}p}) = 0$$

magnetic field transport by vortices

$$\boldsymbol{v}_{\mathrm{L}p} = \boldsymbol{V}_{\mathrm{norm}} - \alpha_p\, m_p n_p \boldsymbol{W}_p - \frac{\beta_p}{B}\, m_p n_p\, \boldsymbol{B} \times \boldsymbol{W}_p$$

velocity of proton vortices

normal velocity

$$\boldsymbol{W}_p = \frac{c}{4\pi e\, n_p}\, \mathrm{curl}\left(\frac{H_{c1}\boldsymbol{B}}{B}\right)$$

vanishes in the absence of vortex tension and buoyancy

$$\begin{cases} \alpha_p = -\dfrac{1}{m_p n_p}\dfrac{1}{1+\mathcal{R}_p^2} \\ \beta_p = \dfrac{1}{m_p n_p}\dfrac{\mathcal{R}_p}{1+\mathcal{R}_p^2} \end{cases}$$

correct at $T = 0$

$\mathcal{R}_p$ drag coefficient

Evolution equation for the magnetic field

- This equation ***differs*** from the similar equation derived in *Glampedakis et al.'11*, *Graber et al.'15* under the same assumptions:

$$\partial_t B^i = \epsilon^{ijk}\nabla_j \left[\epsilon_{klm}\left(v_e^l B^m\right) - \frac{H_{c1}B}{4\pi a_p \rho_p}\frac{\mathcal{R}}{1+\mathcal{R}^2}\left(\mathcal{R}\,\hat{\kappa}_p^l \nabla_l \hat{\kappa}_k^p + \epsilon_{klm}\hat{\kappa}_p^l \hat{\kappa}_p^s \nabla_s \hat{\kappa}_p^m\right)\right]$$

- Magnetic field here is not transported with the velocity of vortices (although it is the magnetic field of flux tubes) – puzzling result.

- In the weak-drag limit, $\mathcal{R} \to 0$, magnetic field is transported with velocity:

Glampedakis et al.'11, Graber et al.'15: ⇒ $\boldsymbol{V}_{\rm sp}$

Our result: ⇒ $\boldsymbol{V}_{\rm sp} + \dfrac{c}{4\pi e\, n_p}\,\mathrm{curl}\left(\dfrac{H_{c1}\boldsymbol{B}}{B}\right)$

Our result:

$$\boldsymbol{v}_{\mathrm{Lp}} = \boldsymbol{V}_{\mathrm{s}p} + \cancel{\frac{c}{4\pi e n_p} \mathrm{curl}\left(\frac{H_{c1}\boldsymbol{B}}{B}\right)}$$

This result is easy to understand; it follows from the balance of forces acting upon vortex in the weak-drag regime:

$$\boldsymbol{F}_{\mathrm{Magnus}} + \boldsymbol{F}_{\mathrm{Tension}} + \boldsymbol{F}_{\mathrm{Buoyancy}} = 0$$

The second term vanishes only if: $\boldsymbol{F}_{\mathrm{Tension}} + \boldsymbol{F}_{\mathrm{Buoyancy}} = 0$

Conclusions and some comments

- A set of fully relativistic finite-temperature equations is derived for superfluid-superconducting npe-mixture.

- Neutron and proton vortices, both types of proton SP and various dissipative corrections are allowed for; buoyancy force (i) is contained in our equations (no need to introduce it "by hands"); (ii) differ from the "standard" usually used expression.

- In comparison to MHD of *Glampedakis et al'11* we:

(i) take into account the relativistic and finite-temperature effects;

(ii) provide a general framework allowing one to incorporate new physics into the existing dynamic equations (relation between $G^{\mu\nu}$, $\mathcal{W}^{\mu\nu}_{(i)}$ and $F^{\mu\nu}$, $\mathcal{V}^{\mu\nu}_{(i)}$);

(iii) demonstrate that the displacement field is not equal to the electric field; and

(iv) obtain a **different** evolution equation for the magnetic field in the MHD limit.

Our equations does not reduce to those of GAS11 in the nonrelativistic limit

- The proposed dynamic equations can be used, e.g., to study evolution of the NS magnetic field.

- However, for sufficiently hot neutron stars, for which $T \sim T_{ci}$ the effects of particle diffusion (more precisely, diffusion of thermal excitations) may become important.

- These effects are ignored in the proposed MHD.

- Now we work to take them into account properly.

(*Dommes & Gusakov''16, in preparation*).

More details:

- Gusakov M.E. , **PRD** (2016)
- Gusakov M.E., Dommes V.A. , **arXiv: 1607.01629** (submitted to **PRD**)

Preliminary result:
Magnetic field evolution equation in the presence of diffusion

- Magnetic field evolution equation will remain formally unchanged

$$\frac{\partial \boldsymbol{B}}{\partial t} + \mathrm{curl}\,(\boldsymbol{B} \times \boldsymbol{v}_{\mathrm{L}p}) = 0$$

But vortex velocity will be different:

$$\boldsymbol{v}_{\mathrm{L}p} = \boldsymbol{V}_{\mathrm{norm}} - \alpha_p\, m_p n_p\, \boldsymbol{W}_p - \frac{\beta_p}{B}\, m_p n_p\, \boldsymbol{B} \times \boldsymbol{W}_p$$

$$\boldsymbol{W}_p = \frac{c}{4\pi e\, n_p}\,\mathrm{curl}\left(\frac{H_{c1}\boldsymbol{B}}{B}\right) + \boxed{\frac{c}{n_p}(\alpha_{pk} - \alpha_{ek})\left[\boldsymbol{\nabla}\left(\frac{\mu_k}{T}\right) - \frac{e_k \boldsymbol{E}}{T}\right]}$$

diffusion-induced term

Dommes & Gusakov''16, in preparation