Precession-like oscillations in a magnetic star

Sam Lander

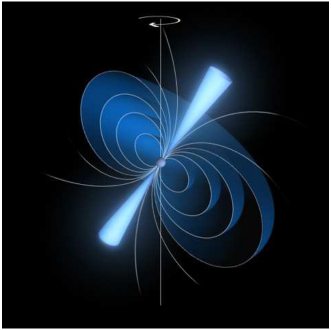
with Ian Jones

UNIVERSITY OF Southampton

Southampton
13th September 2016

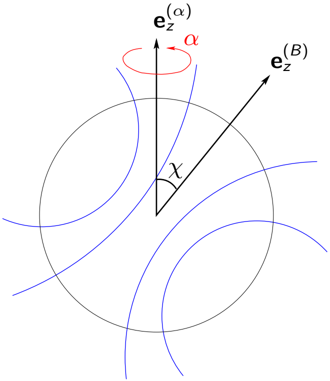
Overview

Neutron stars are not aligned rotators



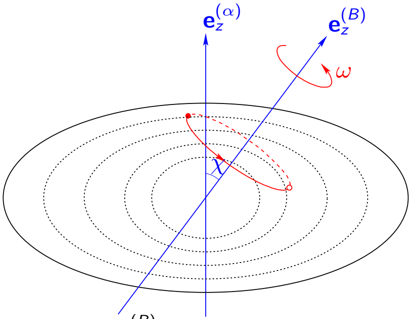
- typical to model NSs as axisymmetric, aligned rotators ('human frailty')
- not terribly realistic since pulsars... pulse
- exterior (magnetosphere) models finally moved away from aligned case
- how do we extend interior B-field models?

Simplest possible misaligned model



- rotating, magnetised, self-gravitating fluid star
- rotational ϵ_α and magnetic ϵ_B distortions $\ll 1$
- B gives the star 'rigidity', since ϵ_B is a distortion misaligned with α-axis
- angular momentum conservation $\Longrightarrow$ star must precess (Spitzer 1958, Mestel & Takhar 1972)
- angular velocity now $\mathbf{\Omega} = \alpha \mathbf{e}_z^{(\alpha)} + \omega \mathbf{e}_z^{(B)}$
- precession frequency $\omega = \alpha \epsilon_B \cos \chi$
- how do we account for non-rigidity of star?

Mestel and Takhar's argument



- in coordinates referred to $\mathbf{e}_z^{(B)}$-axis, density distribution is $\rho(r,\theta,\phi,t) = \rho_0(r) + \delta\rho_B(r,\theta) + \delta\rho_\alpha(r,\theta,\phi+\omega t)$
- fluid elements slowly (rate ω) dragged through different ρ contours
- continuity equation $\rightarrow$ non-rigid response from additional velocity $\dot{\boldsymbol{\xi}}$:

$$\frac{\partial}{\partial t}(\delta\rho_\alpha) = -\nabla\cdot(\rho\dot{\boldsymbol{\xi}}) \approx -\nabla\cdot(\rho_0\dot{\boldsymbol{\xi}}).$$

Velocity of a fluid element seen from inertial frame is:

$$(\alpha\mathbf{e}_z^{(\alpha)} + \omega\mathbf{e}_z^{(B)}) \times \mathbf{r} + \dot{\boldsymbol{\xi}}.$$

ξ-motions

How can one calculate these additional non-rigid motions?

Mestel's approach:

- continuity is only one equation for the three components of $\dot{\boldsymbol{\xi}}$
- appeal to some additional physics so that $\nabla \cdot \dot{\boldsymbol{\xi}} = 0$
- *still* short of one equation!
  - 1st idea: assume $\dot{\xi}_\phi = 0$ (Mestel & Takhar 1972)
  - 2nd idea: minimise kinetic energy of $\dot{\boldsymbol{\xi}}$ (Mestel et al. 1981)
- afterwards, can put obtained $\dot{\boldsymbol{\xi}}$ into induction equation to get $\delta \mathbf{B}$

Problems: why do we need extra physics? Where has B gone?

Conclude: need to go to higher perturbative order to solve for $\dot{\boldsymbol{\xi}}, \delta \mathbf{B}$.

General equations for a precessing star

Recall: inertial-frame velocity $= \boldsymbol{\Omega} \times \mathbf{r} + \dot{\boldsymbol{\xi}} = (\alpha \mathbf{e}_z^{(\alpha)} + \omega \mathbf{e}_z^{(B)}) \times \mathbf{r} + \dot{\boldsymbol{\xi}}$.
Work in co-precessing frame $\rightarrow$ only velocity we see is $\dot{\boldsymbol{\xi}}$.

The equations of motion for our *non-rigidly* rotating star are:

$$\ddot{\boldsymbol{\xi}} + (\dot{\boldsymbol{\xi}} \cdot \nabla)\dot{\boldsymbol{\xi}} + 2\boldsymbol{\Omega} \times \dot{\boldsymbol{\xi}} + \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\nabla H - \nabla \Phi + \frac{1}{4\pi\rho}(\nabla \times \mathbf{B}) \times \mathbf{B},$$

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \dot{\boldsymbol{\xi}}),$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\dot{\boldsymbol{\xi}} \times \mathbf{B}),$$

$$\nabla^2 \Phi = 4\pi G \rho,$$

$$H = H(\rho),$$

$$\nabla \cdot \mathbf{B} = 0.$$

We perturb these by writing all quantities in the form:

$$\begin{array}{ccccccccc} \rho & = & \rho_0 & + & \delta\rho_\alpha & + & \delta\rho_B & + & \delta\rho_{\alpha B} & + & \dots \\ & & \mathcal{O}(1) & & \mathcal{O}(\epsilon_\alpha) & & \mathcal{O}(\epsilon_B) & & \mathcal{O}(\epsilon_\alpha \epsilon_B) & & \end{array}$$

Lower-order equations

We need to solve, successively, a series of perturbation problems:

$\mathcal{O}(1)$ → spherical background star, with $\rho_0(r) \sim (\sin r)/r$ for $\gamma = 2$ polytrope
$\mathcal{O}(\epsilon_B)$ → 'magnetic mountain' $\delta\rho_B(r,\theta)$, ellipticity ϵ_B → gives us ω
$\mathcal{O}(\epsilon_\alpha)$ → centrifugal bulge $\delta\rho_\alpha(r,\theta,\phi+\omega t)$ moving slowly around $\mathbf{e}_z^{(B)}$

All provide input into $\mathcal{O}(\epsilon_\alpha\epsilon_B)$ equations, in which we find $\dot{\boldsymbol{\xi}}$ and $\delta\mathbf{B}$:

$$\frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} + \omega \mathbf{e_z} \times (\boldsymbol{\Omega} \times \mathbf{r}) + \boldsymbol{\Omega} \times (\omega \mathbf{e}_z \times \mathbf{r}) = -\nabla \delta H_{\alpha B} - \nabla \delta \Phi_{\alpha B}$$
$$- \frac{\delta\rho_\alpha}{4\pi\rho_0^2}(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 + \frac{1}{4\pi\rho_0}[(\nabla \times \delta\mathbf{B}) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \delta\mathbf{B}],$$

$$\frac{\partial \delta\rho_\alpha}{\partial t} = -\nabla \cdot (\rho_0 \dot{\boldsymbol{\xi}}),$$
$$\frac{\partial \delta\mathbf{B}}{\partial t} = \nabla \times (\dot{\boldsymbol{\xi}} \times \mathbf{B}_0),$$
$$\nabla^2 \delta\Phi_{\alpha B} = 4\pi G \delta\rho_{\alpha B},$$
$$\delta H_{\alpha B} = \delta H_{\alpha B}(\delta\rho_{\alpha B}, \delta\rho_\alpha \delta\rho_B),$$
$$\nabla \cdot \delta\mathbf{B} = 0.$$

Getting ODEs out of the $\mathcal{O}(\epsilon_\alpha \epsilon_B)$ equations

How can we solve these equations? Let's take the curl of the Euler equation.

Then the full system separates:

Solve these for $\delta\mathbf{B}$:

$$\nabla \times \left\{ \frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} \right\} = \nabla \times \left\{ -\frac{\delta\rho_\alpha}{4\pi{\rho_0}^2}(\nabla \times \mathbf{B}_0) \times \mathbf{B}_0 + \frac{1}{4\pi\rho_0}[(\nabla \times \delta\mathbf{B}) \times \mathbf{B}_0 + (\nabla \times \mathbf{B}_0) \times \delta\mathbf{B}] \right\},$$

$$\nabla \cdot \delta\mathbf{B} = 0.$$

Afterwards, solve these for $\dot{\boldsymbol{\xi}}$:

$$\frac{\partial \delta\rho_\alpha}{\partial t} = -\nabla \cdot (\rho_0 \dot{\boldsymbol{\xi}}),$$

$$\frac{\partial \delta\mathbf{B}}{\partial t} = \nabla \times (\dot{\boldsymbol{\xi}} \times \mathbf{B}_0).$$

Final ODEs

Next, need to do a lot of algebra:

- Toroidal-poloidal split of curled Euler
- decompose $\delta\mathbf{B} = \sum_{l,m} U_l^m(r) Y_l^m \mathbf{e}_r + V_l^m(r) \nabla Y_l^m + W_l^m(r) \mathbf{e}_r \times \nabla Y_l^m$
- $\nabla \cdot \delta\mathbf{B} = 0 \implies$ can eliminate V_l^m in favour of U_l^m
- Y_l^m orthogonality relations turns one big infinite sum (r, θ, ϕ) into an infinite set of unsummed DEs in r alone

Our equations for $\delta\mathbf{B}$ can then be reduced to two coupled ODEs (actually DAEs) in the two radial functions, for each l and m:

$$sources_1 = f(U''_{l-2}, U''_l, U''_{l+2}, U'_{l-2}, U'_l, U'_{l+2}, U_{l-2}, U_l, U_{l+2}, W'_{l-1}, W'_{l+1}, W_{l-1}, W_{l+1})$$
$$sources_2 = g(U'_{l-3}, U'_{l-1}, U'_{l+1}, U'_{l+3}, U_{l-3}, U_{l-1}, U_{l+1}, U_{l+3}, W_{l-2}, W_l, W_{l+2})$$

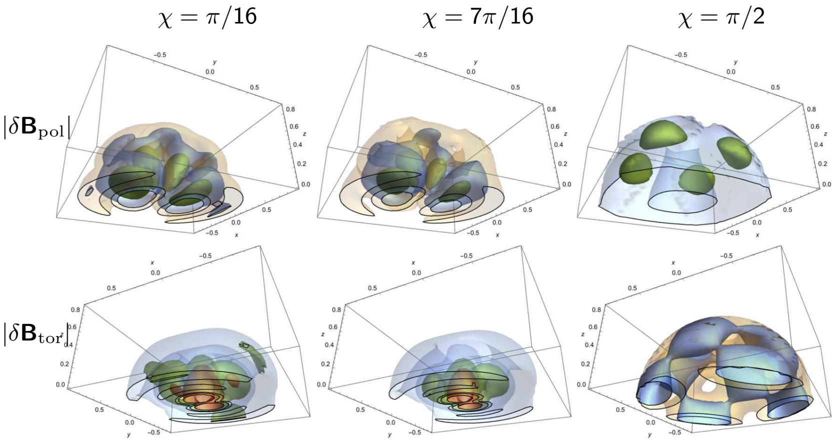
Can find closed-form expressions for both $\delta\mathbf{B}$ and $\dot{\boldsymbol{\xi}}$ in terms of U, W, so after solving the above two equations we are done.

The headache page

$$
\begin{aligned}
0 =& \frac{4\pi}{\Lambda}\left[mr\tilde{\Psi}_l - i(l-1)Q_l(r\tilde{\Upsilon}'_{l-1} + \tilde{\Upsilon}_{l-1}) + i(l+2)Q_{l+1}(r\tilde{\Upsilon}'_{l+1} + \tilde{\Upsilon}_{l+1})\right] \\
&+ Q_{l-1}Q_l\left\{-\frac{2}{(l-1)}r^2\bar{U}''_{l-2} + \left[\frac{2(l-5)}{(l-1)} + \frac{r\rho'_0}{\rho_0}\right]r\bar{U}'_{l-2} + \left[\frac{2(l-3)}{(l-1)} + \frac{r\rho'_0}{\rho_0} + r\left(\frac{r\rho'_0}{\rho_0}\right)'\right]\bar{U}_{l-2}\right\} \\
&+ \frac{(m^2 + 2(l+1)Q_l^2 - 2lQ_{l+1}^2)}{l(l+1)}r^2\bar{U}''_l + \left[\frac{6m^2}{l(l+1)} - 2 + \frac{2(l+4)Q_l^2}{l} + \frac{2(l-3)Q_{l+1}^2}{(l+1)} + \left(\frac{m^2}{l(l+1)} - 1 + Q_l^2 + Q_{l+1}^2\right)\frac{r\rho'_0}{\rho_0}\right]r\bar{U}'_l \\
&+ \left[\frac{6m^2}{l(l+1)} - m^2 - 2 + \frac{2(l+2)Q_l^2}{l} + \frac{2(l-1)Q_{l+1}^2}{(l+1)} + \left(\frac{2m^2}{l(l+1)} - 1 + Q_l^2 + Q_{l+1}^2\right)\frac{r\rho'_0}{\rho_0} - (1 - Q_l^2 - Q_{l+1}^2)r\left(\frac{r\rho'_0}{\rho_0}\right)'\right]\bar{U}_l \\
&+ Q_{l+1}Q_{l+2}\left\{\frac{2}{(l+2)}r^2\bar{U}''_{l+2} + \left[\frac{2(l+6)}{(l+2)} + \frac{r\rho'_0}{\rho_0}\right]r\bar{U}'_{l+2} + \left[\frac{2(l+4)}{(l+2)} + \frac{r\rho'_0}{\rho_0} + r\left(\frac{r\rho'_0}{\rho_0}\right)'\right]\bar{U}_{l+2}\right\} \\
&+ (l-3)mQ_lrX'_{l-1} + mQ_l\left[(3l-5) + (l-1)\frac{r\rho'_0}{\rho_0}\right]X_{l-1} - (l+4)mQ_{l+1}rX'_{l+1} - mQ_{l+1}\left[(3l+8) + (l+2)\frac{r\rho'_0}{\rho_0}\right]X_{l+1},
\end{aligned}
\tag{161}
$$

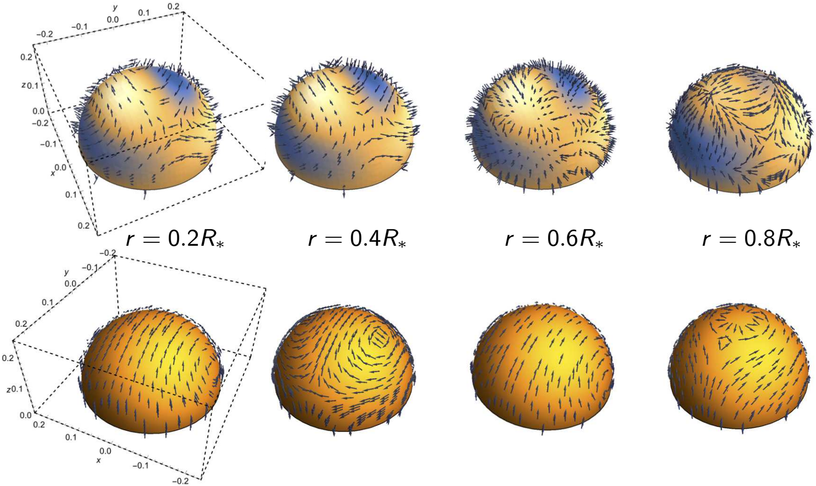
$$
\begin{aligned}
0 =& \frac{4\pi i}{\Lambda}\left\{-(l-1)(l-2)Q_{l-1}Q_l\tilde{\Upsilon}_{l-2} + [m^2 + (l-1)(l+1)Q_l^2 + l(l+2)Q_{l+1}^2]\tilde{\Upsilon}_l - (l+2)(l+3)Q_{l+1}Q_{l+2}\tilde{\Upsilon}_{l+2}\right\} \\
&- (l-1)Q_{l-2}Q_{l-1}Q_l\left[\frac{2}{(l-2)}r\bar{U}'_{l-3} - \left(\frac{2(l-4)}{(l-2)} + \frac{r\rho'_0}{\rho_0}\right)\bar{U}_{l-3}\right] + \frac{Q_l}{l}\left\{2\left[\frac{m^2}{(l-1)} + lQ_{l-1}^2 - (l-1)Q_l^2 + (l+2)Q_{l+1}^2\right]r\bar{U}'_{l-1}\right. \\
&\left. + \left[\frac{4m^2}{(l-1)} - 2l[l-1-(l+1)Q_{l-1}^2] + 2(l-2)[(l-1)Q_l^2 - (l+2)Q_{l+1}^2] - l[(l-1)(1 - Q_{l-1}^2 - Q_l^2) + (l+2)Q_{l+1}^2]\frac{r\rho'_0}{\rho_0}\right]\bar{U}_{l-1}\right\} \\
&+ \frac{Q_{l+1}}{(l+1)}\left\{2\left[\frac{m^2}{(l+2)} + (l-1)Q_l^2 - (l+2)Q_{l+1}^2 + (l+1)Q_{l+2}^2\right]r\bar{U}'_{l+1} + \left[\frac{4m^2}{(l+2)} + 2(l+1)[l+2-lQ_{l+2}^2]\right.\right. \\
&\left.\left. + 2(l+3)[(l-1)Q_l^2 - (l+2)Q_{l+1}^2] + (l+1)[(l+2)(1 - Q_{l+1}^2 - Q_{l+2}^2) + (l-1)Q_l^2]\frac{r\rho'_0}{\rho_0}\right]\bar{U}_{l+1}\right\} \\
&- (l+2)Q_{l+1}Q_{l+2}Q_{l+3}\left[\frac{2}{(l+3)}r\bar{U}'_{l+3} + \left(\frac{2(l+5)}{(l+3)} + \frac{r\rho'_0}{\rho_0}\right)\bar{U}_{l+3}\right] + mQ_{l-1}Q_l[(l-2)(l+1) - 2(l-1)]X_{l-2} \\
&+ m\left\{-l(l+1) + [(l-2)(l+1) - 2(l-1)]Q_l^2 + [l(l+3) + 2(l+2)]Q_{l+1}^2\right\}X_l + mQ_{l+1}Q_{l+2}[l(l+3) + 2(l+2)]X_{l+2}.
\end{aligned}
\tag{162}
$$

$\delta\mathbf{B}$: magnitude



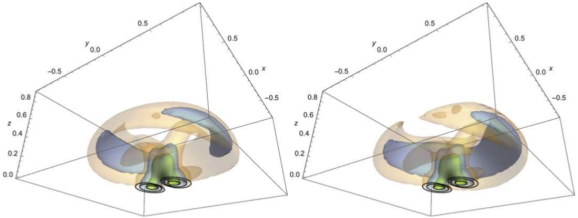
- contours ordered brown, blue, green, red (each twice strength of last)
- near-aligned and near-orthogonal results similar
- oscillates: pattern rotates around z-axis with period $2\pi/\omega$

$\delta\mathbf{B}$: direction

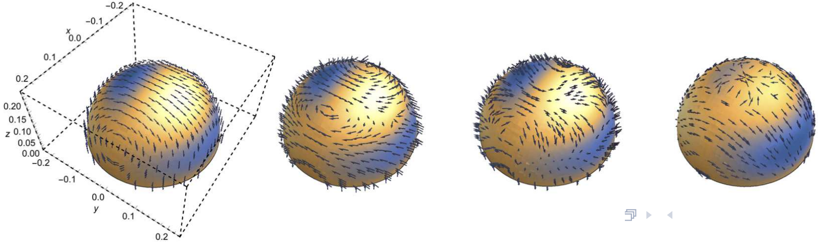


- upper: $\delta\mathbf{B}_{\rm pol}$, lower: $\delta\mathbf{B}_{\rm tor}$
- for upper plots, blue: $\delta\mathbf{B}$ points inwards, white: points out

ξ-motions



- near-aligned (left) and near-orthogonal (right) results again similar
- pattern again rotates around z-axis at rate ω
- less variation; each contour here is only $\sqrt{2}$ times the last



Conclusions

This work: finding solutions for $\delta\mathbf{B}, \dot{\boldsymbol{\xi}}$

- probably simplest possible oblique rotator model: rigid rotation, $m = 0, l = 1$ (dipole) toroidal magnetic field
- ended up with highly complex, multipolar $\delta\mathbf{B}, \dot{\boldsymbol{\xi}}$
- highly compressible motions, $\nabla \cdot \dot{\boldsymbol{\xi}} \neq 0$ (Mestel analysis invalid)

Future work: dissipation of these perturbations

- original motivation for study
- as $\delta\mathbf{B}, \dot{\boldsymbol{\xi}}$ dissipate, inclination angle $\chi \to 0$ (poloidal field) or $\chi \to \pi/2$ (toroidal field)
- interesting for distribution of inclination angles, pulsar death line, apparent absence of precession in NSs, gravitational waves
- our results are promising for rapid dissipation