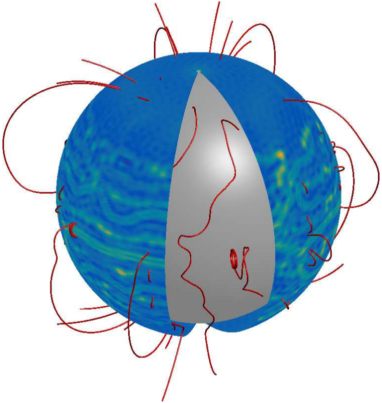
Magneto-thermal instability and resistive tearing instability in a neutron star crust

Toby Wood
Newcastle University



Why study the crust?

Magnetic instabilities in the crust could lead to:

- magnetic spots / hotspots at the surface
- enhanced magnetic dissipation

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Three known magnetic instabilities:

- Density-shear instability (requires density gradient)
  (Gordeev & Rudakov 1969, Rheinhardt et al. 2004, Wood et al. 2014)
- Resistive tearing instability (requires resistivity)
  (Gordeev 1970, Rheinhardt & Geppert 2002, Wood et al. 2014)
- Thermo-electric instability (requires temperature gradient)
  (Blandford et al. 1983, Urpin et al. 1986)

Electron MHD / Hall MHD

- Assume the crust is a rigid ionic lattice
- Charge and heat carried entirely by electrons.

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\begin{aligned}
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\mathbf{J} &= \frac{c}{4\pi}\boldsymbol{\nabla} \times \mathbf{B} \ \big(= -en\mathbf{v}\big) \\
C_V \frac{\partial T}{\partial t} &= -\boldsymbol{\nabla} \cdot \mathbf{Q} + \mathbf{E} \cdot \mathbf{J}
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$$\mathbf{E} + \boldsymbol{\nabla}\mu/\mathrm{e} = \boldsymbol{\sigma}^{-1} \cdot \mathbf{J} + \boldsymbol{G} \cdot \boldsymbol{\nabla} T$$

$$\mathbf{Q} + \frac{\mu}{\mathrm{e}}\mathbf{J} = T\boldsymbol{G} \cdot \mathbf{J} - \boldsymbol{k} \cdot \boldsymbol{\nabla} T$$

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Simplest model has:

$$(\boldsymbol{\sigma}^{-1})_{ij} = \sigma^{-1}\delta_{ij} + \frac{\epsilon_{ijk}B_k}{\mathrm{e}cn}$$
$$\boldsymbol{k} = \frac{\pi^2 k_{\mathrm{B}}^2 T}{3\mathrm{e}^2}\,\boldsymbol{\sigma}$$
$$(\boldsymbol{G})_{ij} = -\frac{S}{\mathrm{e}}\delta_{ij}\,.$$

The magneto-thermal evolution model

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times \left[\frac{c}{4\pi \mathrm{e} n} \underbrace{\mathbf{B} \times (\boldsymbol{\nabla} \times \mathbf{B})}_{\text{Hall}} - \underbrace{\eta \boldsymbol{\nabla} \times \mathbf{B}}_{\text{Ohm}} + \frac{c}{\mathrm{e}} \underbrace{S \boldsymbol{\nabla} T}_{\text{Biermann}} \right]$$

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$$C_V \frac{\partial T}{\partial t} = \boldsymbol{\nabla} \cdot (\boldsymbol{k} \cdot \boldsymbol{\nabla} T) + |\mathbf{J}|^2/\sigma - nT\mathbf{v} \cdot \boldsymbol{\nabla} S$$

$$k_{ij} = k \frac{\delta_{ij} + (\mathrm{e}c\tau/\mu)^2 B_i B_j - (\mathrm{e}c\tau/\mu)\epsilon_{ijk} B_k}{1 + (\mathrm{e}c\tau/\mu)^2 |\mathbf{B}|^2}$$

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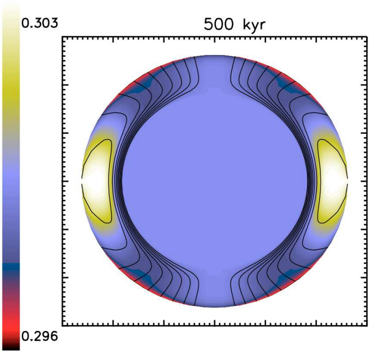
- A strong field inhibits heat flow across field lines:

$$\boldsymbol{k} \cdot \boldsymbol{\nabla} T = k \, (\hat{\mathbf{b}} \cdot \boldsymbol{\nabla} T) \, \hat{\mathbf{b}}$$

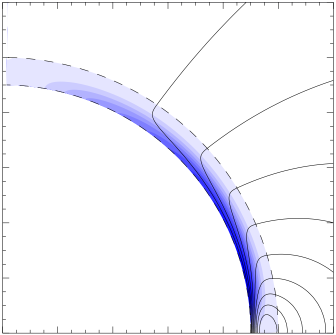
- But a weak field encourages heat flow across field lines!

$$\boldsymbol{k} \cdot \boldsymbol{\nabla} T = k \boldsymbol{\nabla} T + k(\mathrm{e}c\tau/\mu) \mathbf{B} \times \boldsymbol{\nabla} T$$

So what happens?



Pons et al. (2009)



Wood & Hollerbach (2015)

Thermoelectric instability

The principal effect leading to field generation can be understood as follows. Suppose that there is a small horizontal component of magnetic field of strength B. Hot electrons from below will be deflected horizontally by the field; cooler electrons from above will be deflected slightly less in the opposite direction. The net effect is to produce a horizontal heat flux, $F_\perp \sim (e\tau/\mu)\vec{B} \times \vec{F}$, where $e = |e|$ is the magnitude of the electronic charge, τ is the electron collision time, μ is the electron chemical potential and $\vec{F}$ is the vertical heat flux (we use units in which $c = k_B = \hbar = 1$). Fourier components of the magnetic field with horizontal wavelength comparable with the depth z create horizontal temperature gradients $\sim F_\perp/\kappa$, where κ is the thermal conductivity. The pressure of a degenerate, relativistic free electron gas is $P(n_e, T) = P(n_e, T=0) + (\pi^2/6)n_e T^2/\mu$. Hence there is an additional pressure gradient $\sim n_e T \nabla T/\mu$, which must be balanced by a thermoelectric field $E \sim T\nabla T/\mu e \sim BF/\mu n_e$. This field has a non-vanishing curl and so $-\vec{\nabla} \times \vec{E} = \partial\vec{B}/\partial t = \Gamma_+ \vec{B}$, where $\Gamma_+ \sim F/\mu n_e z$. The growth rate Γ_+ is positive when the heat flows down the density gradient.

Blandford et al. (1983)

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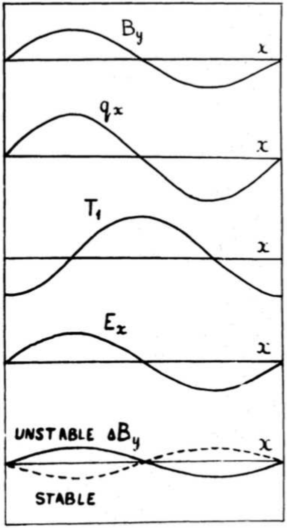

Urpin et al. (1986)

Thermoelectric instability

For perturbations to a background with $\mathbf{B} = \mathbf{0}$:

$$\frac{\partial \mathbf{B}}{\partial t} = \boldsymbol{\nabla} \times \left[-\eta \boldsymbol{\nabla} \times \mathbf{B} + \frac{c}{\mathrm{e}} S \boldsymbol{\nabla} T \right]$$
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$$0 = \boldsymbol{\nabla} \cdot (k \boldsymbol{\nabla} T') + (4\pi \mathrm{e} \tau k / \mu) \frac{\mathrm{d}T}{\mathrm{d}r} J_r$$

- Biermann battery generates $J_r \propto \nabla_{\mathrm{H}}^2 T'$ (analogous to buoyancy).
- Heat flux converges where $\mathbf{J} \cdot \boldsymbol{\nabla} T > 0$ (analogous to advection).

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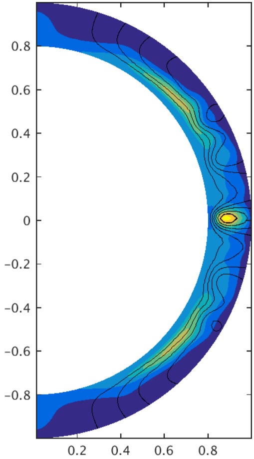
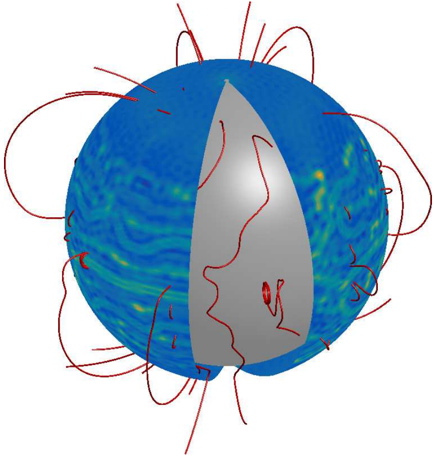
- Instability onset identical to convection in a porous medium!
- Max growth rate $\simeq \dfrac{3F}{n} \left(\dfrac{1}{\mu} \right)' \sim \dfrac{1}{10^5\,\mathrm{yr}}$ (Blandford et al. 1983).

Thermoelectric instability

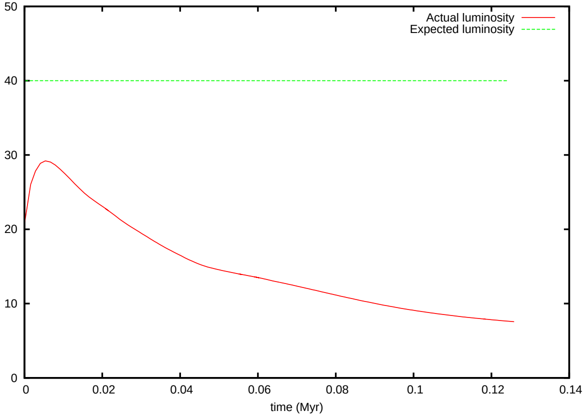
Do we actually get the instability?





Conclusions

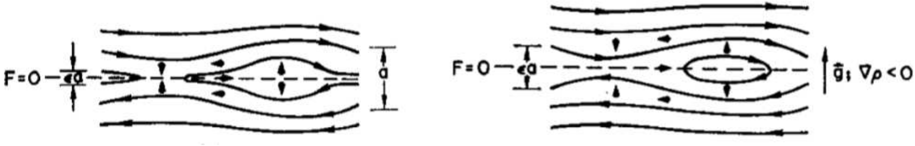
- Significant ohmic heating in the crust.
- Magnetic field reduces luminosity.



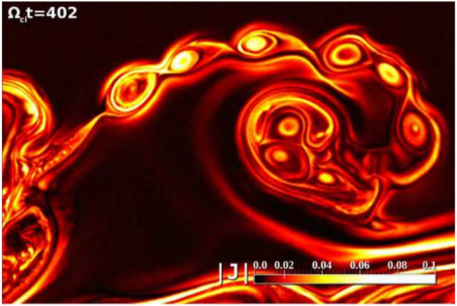
- No sign of thermo-electric instability (so far).

Part II

Tearing instablity



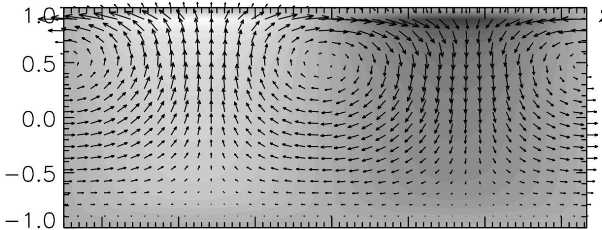
- Furth, Killeen & Rosenbluth (1963; 2515 citations)

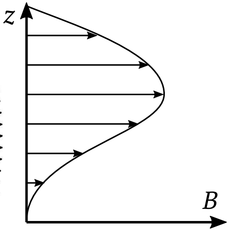


Karimabadi et al. (2013)

EMHD tearing instablity

Almost certainly seen by Rheinhardt & Geppert (2002)





- Localised near boundary (where $B = 0$)
- Growthrate $\sim \eta^{2/7}$ (Wood et al. 2014)

Tearing instablity

- "Normal" MHD gives the boundary-layer equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{t^2}{1+t^2}\frac{\mathrm{d}u}{\mathrm{d}t}\right) = \lambda^2 t^2 u$$

solved by Pegoraro & Schep (1986)

$$u = t^{-3/2}\left[W_{\frac{1}{2}-\frac{1}{4}\lambda,\frac{1}{4}}(\lambda t^2) - \tfrac{1}{4}(1-\lambda)W_{-\frac{1}{2}-\frac{1}{4}\lambda,\frac{1}{4}}(\lambda t^2)\right]$$

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Prospects for the crust

- Density-shear instability
  - Quite likely (Wood et al. 2014)
- Resistive tearing instability
  - Possible. But perhaps more relevant to the magnetosphere?
- Thermo-electric instability
  - Unlikely, except for particular field geometries