Self-consistent orbital evolution of a particle around a Schwarzschild black hole

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Extreme Mass Ratio Inspirals have long been a promising source of gravitational waves for the LISA, the space based gravitational wave detector.

Accurate models are a critical component of any observation.

Even more true now that LISA is no more and there are proposals for eLISA/NGO which will have less sensitivity.
Solve the coupled system of equations for the motion of the particle and its retarded field.

Self-interaction of the particle with its retarded field, $\Phi^{\text{ret}}$.

$\Phi^{\text{ret}}$ diverges like $1/r$ on the world-line.

“Unphysical” divergence removed by appropriate regularization.

\[
\Box \Phi^{\text{ret}} = -4\pi q \int \frac{\delta^4(x - z(\tau))}{\sqrt{-g}} d\tau
\]

\[
\frac{Du^\alpha}{d\tau} = a^\alpha = \frac{\bar{q}}{m(\tau)} (g^{\alpha\beta} + u^\alpha u^\beta) \nabla_\beta \Phi^{\text{ret}}
\]

\[
\frac{dm}{d\tau} = -\bar{q} u^\beta \nabla_\beta \Phi^{\text{ret}}
\]
Effective source regularization

- Split retarded field into locally constructed field and “regularized” remainder.

- Derive an equation for $\Phi^R$.

- Always work with $\Phi^R$ instead of $\Phi^{\text{ret}}$.

- If $\Phi^S$ is chosen appropriately, then we can just replace $\Phi^{\text{ret}}$ with $\Phi^R$ in the equations of motion.

\[\Phi^{\text{ret}} = \Phi^S + \Phi^R\]

\[\square \Phi^R = \square \Phi^{\text{ret}} - \square \Phi^S\]

\[\frac{Du^\alpha}{d\tau} = a^\alpha = \frac{\bar{q}}{m(\tau)}(g^{\alpha\beta} + u^\alpha u^\beta)\nabla_\beta \Phi^R\]

\[\frac{dm}{d\tau} = -\bar{q}u^\beta \nabla_\beta \Phi^R\]
Effective source regularization

- If $\Phi^S$ is exactly the Detweiler-Whiting singular field, $\Phi^R$ is a solution of the homogeneous wave equation.

- If $\Phi^S$ is only approximately the Detweiler-Whiting singular field, then the equation for $\Phi^R$ has an effective source, $S$.

- $S$ is typically finite, but of limited differentiability on the world line.
Self-consistent Evolution

- Solve the coupled system of equations for the motion of the particle and its regularized field.

- $\Phi^R = \Phi^{\text{ret}}$ in the wave zone

- $\Phi^R$ finite and (typically) twice differentiable on the world-line
\[ \Phi^R(t) \]

\[
q = \frac{M}{32}
\]

\[
m = M
\]

\[
p_0 = 7.2
\]

\[
e_0 = 0.5
\]
\[ q = \frac{M}{32} \]
\[ m = M \]
\[ p_0 = 7.2 \]
\[ e_0 = 0.5 \]
Orbital motion

$q=0$

$q=1/32$
Orbital motion

- Parametrize orbits in terms of a dimensionless semilatus rectum $p$ and eccentricity $e$, such that $r_\pm = Mp / (1 \mp e)$.

- Separatrix, $p = 6 + 2e$, corresponds to unstable circular orbits and represents the boundary in $p$–$e$ space separating bound from plunging orbits.

\[
\frac{d\phi}{dt} = \left[ 1 - \frac{2Mr'}{r - 2M} \right] \times \frac{[p - 2 - 2e \cos(\chi - w)][1 + e \cos(\chi - w)]^2}{M \sqrt{p^3[(p - 2)^2 - 4e^2]}}
\]

\[
r(t) = \frac{Mp}{1 + e \cos(\chi - w)}
\]
Orbital evolution
Orbital evolution - “dissipative”

\[ r(t) = \frac{M_p}{1 + e \cos(\chi - w)} \]
Orbital evolution - “conservative”

\[
r(t) = \frac{M_p}{1 + e \cos(\chi - w)}
\]
Waveforms at $\mathcal{I}^+$