FACULTY OF MATHEMATICAL STUDIES

MATHEMATICS FOR PART I ENGINEERING

Lectures

MODULE 20 FURTHER CALCULUS II

1. Sequences and series
2. Rolle's theorem and mean value theorems
3. Taylor's and Maclaurin's theorems
4. L'Hopital's rule

1. Sequences and series

When obtaining a mathematical solution to an engineering problem it is possible to proceed in an obvious way but sometimes obtain the wrong answer, even though the initial model and all the algebraic manipulations are correct. In these situations the errors usually arise because some mathematical procedure has been *assumed* to be true, whereas a much closer inspection would have revealed a flaw in the argument. **Analysis** is the area of mathematics in which rigour is investigated. Very little on this topic is covered in your engineering course, but a few ideas are introduced in this module and in this section we consider sequences and series.

Consider a function f with domain $0, 1, 2, \ldots$, then the set of values $\{f(0), f(1), f(2), \ldots\}$ is called a **sequence**. The sequence is often written $\{f_0, f_1, f_2, \ldots\}$.

A sequence which does NOT end is called an **infinite sequence**,
a sequence which ends is called a **finite sequence**.

Denote a finite sequence by $\{f_k\}_{k=0}^{n}$, in which the first term is f_0 and the last term is f_n , and an infinite sequence by $\{f_k\}_{k=0}^{\infty}$.
(Note that the above notation is that used by **James**, although different notations appear in other books.)

The values of the terms in a sequence are often related. For example, the terms might satisfy $f_{k+1} = f_k + 1$ and this is called a **recurrence relation**.

There are two special types of sequence which almost all of you will have met before, and these are briefly considered below.

Arithmetic sequence − a sequence in which the difference between successive terms is constant

e.g. $\{4, 7, 10, 13\}$ or $\{1, -2, -5\}$.

The general form of an arithmetic sequence is $\{a + kd\}_{k=0}^{n-1}$, where a is the first term, d is the common difference and n is the number of terms.

Geometric sequence − a sequence in which the ratio of successive terms is constant

e.g. $\{4, 8, 16, 32, 64\}$ or $\{4, -2, 1, -\frac{1}{2}, \frac{1}{4}, \ldots\}$.

The general form of an geometric sequence is $\{a\,r^k\}_{k=0}^{n-1}$, where a is the first term, r is the common ratio and n is the number of terms.

Def. A **series** is the *sum* of terms in a sequence.

Arithmetic series − from the arithmetic sequence $a, a+d, a+2d, \ldots,$ add together the first n terms to give the arithmetic series with sum S_n, so that

$$S_n = a + (a + d) + (a + 2d) + \ldots + (a + (n-1)d)\,.$$

As you may know the formula for the sum can easily be derived by rewriting the above expression, reversing the order of the terms, so that

$$S_n = (a + (n-1)d) + (a + (n-2)d) + (a + (n-3)d) + \ldots + a\,.$$

Adding the corresponding terms in the latter two expressions then implies

$$2S_n = (2a + (n-1)d) + (2a + (n-1)d) + (2a + (n-1)d) + \ldots + (2a + (n-1)d), \qquad (n \text{ terms})$$

$$\rightarrow \qquad \underline{S_n = \frac{n}{2}\,(2a + (n-1)d) = \frac{n}{2}\,(\text{first term} + \text{last term})}\,,$$

where "first" and "last" refer to terms in the original series.

Geometric series – from the geometric sequence $a, ar, ar^2, \ldots,$ add together the first n terms to yield the **geometric series** with sum S_n,

$$S_n = a + ar + ar^2 + ar^3 + \ldots + ar^{n-1}\,.$$

To calculate the sum multiply everything in the expression by r

$$r\,S_n = ar + ar^2 + ar^3 + ar^4 \ldots + ar^n$$

and subtract from the first series giving

$$S_n - r\,S_n = (1-r)\,S_n = a - ar^n,$$

since almost all the terms on the right-hand side cancel. It follows immediately that the sum of a geometric series of n terms is

$$\underline{S_n = \frac{a(1-r^n)}{1-r}}\,.$$

The above results for the sums of arithmetic and geometric series with n terms should be remembered.

Ex 1. Calculate $4 - 2 + 1 - \frac{1}{2} + \frac{1}{4}$.

This is a geometric series, with 5 terms, first term $a = 4$, common ratio $r = -\frac{1}{2}$.
Hence, using the formula,

$$S_5 = \frac{4(1-(-1/2)^5)}{1-(-1/2)} = \frac{4\left(1+\frac{1}{32}\right)}{\frac{3}{2}} = \frac{4\left(\frac{33}{32}\right)}{\frac{3}{2}} = \frac{11}{4},$$

which can easily be verified by direct calculation.

Ex 2. How many terms in the series $11 + 15 + 19 + \ldots$ are needed to give a sum of 341?

This is an arithmetic series of n terms, for which $a = 11$ and $d = 4$. It is not difficult to deduce that the final term, the nth term, equals $11 + (n-1)4 = 7 + 4n$ and, therefore,

$$S_n = \frac{n}{2}\,(11 \;+\; (7+4n)) = 341.$$

Hence

$$\frac{n}{2}\,(18 + 4n) = 341,$$

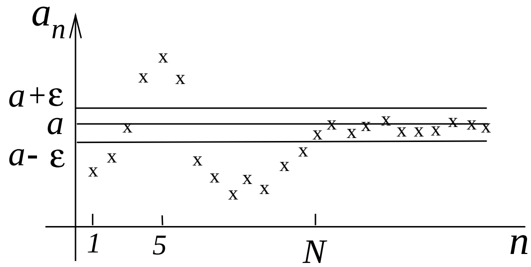
$$\rightarrow \qquad 4n^2 + 18n - 682 = 0, \quad \text{or} \quad 2n^2 + 9n - 341 = 0\,,$$

$$\text{i.e.} \qquad (n-11)(2n+31) = 0,$$

$$\rightarrow \qquad n = 11, \quad \text{or} \quad -31/2\,,$$

but the number of terms must be positive so the answer to the question is 11 terms.

Limit of a sequence   It is often important to know whether a sequence converges. The condition can be stated mathematically – but in words we say that a sequence $\{a_n\}_{n=0}^{\infty}$ **converges to a limit** a when for every small positive value of ϵ you choose there always exists a place in the sequence (at which $n = N$) beyond which EVERY term in the sequence lies between $a - \epsilon$ and $a + \epsilon$.



Properties of convergent sequences

If sequence $\{a_n\}$ has limit a and $\{b_n\}$ has limit b then

(i) $\{a_n + b_n\}$ has limit $a + b$;

(ii) $\{a_n - b_n\}$ has limit $a - b$;

(iii) $\left\{\dfrac{a_n}{b_n}\right\}$ has limit $\dfrac{a}{b}$;

(iv) $\{a_n\, b_n\}$ has limit $a\, b$.

Ex 3. Find the limits of $\{x_n\}_{n=0}^{\infty}$ defined by (a) $x_n = \dfrac{n}{n+1}$,

(b) $x_n = \dfrac{2n^2 + 3n + 1}{5n^2 + 6n + 2}$.

(a) Dividing the numerator and denominator by n and using properties (i) and (iii) above it follows that

$$x_n = \frac{\frac{n}{n}}{\frac{n+1}{n}} = \frac{1}{1 + \frac{1}{n}} \quad \rightarrow \quad \frac{1}{1+0} = \frac{1}{1} = 1\,.$$

(b) Using a similar method as above, but dividing here by n^2, leads to

$$x_n = \frac{2 + \frac{3}{n} + \frac{1}{n^2}}{5 + \frac{6}{n} + \frac{2}{n^2}} \quad \rightarrow \quad \frac{2+0+0}{5+0+0} = \frac{2}{5}$$

Convergence of series   You must be careful in deciding whether or not a series converges. For example, consider the **harmonic series**

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n}\,.$$

Each term gets smaller and approaches zero, but does the series converge since an infinite number of small terms may add to give something significant (i.e. does the sum approach a limit as $n \to \infty$?)
By splitting the fractional terms in the series into appropriate groups (the lengths of the groups being powers of 2^n, namely one, two, four, eight, ...) then it is easy to deduce that

$$\frac{1}{3} + \frac{1}{4} > 2 \times \frac{1}{4} = \frac{1}{2}\,,$$
$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 4 \times \frac{1}{8} = \frac{1}{2}\,,$$
$$\frac{1}{9} + \frac{1}{10} + \ldots + \frac{1}{16} > 8 \times \frac{1}{16} = \frac{1}{2}\,, \qquad \text{and so on.}$$

The sum of each group of terms is therefore greater than $\frac{1}{2}$. With an infinite number of terms in the original series the groups never stop (although successive ones contain an increasing number of terms) and so the sum of the series gets larger and larger, increasing by more than $\frac{1}{2}$ when each group of terms is added. The harmonic series does NOT converge, therefore, and the series is said to be **divergent**.

Convergence of geometric series It was shown earlier in this section that for a geometric series of n terms

$$a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

What happens as $n \to \infty$, when the series contains an infinite number of terms?
The above series clearly converges when $r^n \to 0$ and hence the series **converges** if and only if $|r| < 1$ and **diverges** if and only if $|r| \geq 1$.
For a converged infinite geometric series $\underline{S = \frac{a}{1-r}}$.

D'Alembert's ratio test is one very useful test for confirming, or otherwise, the convergence of infinite series. Suppose we consider the series $\sum_{k=0}^{\infty} u_k$, where all the u_k are positive, and suppose that the limit $\lim_{n\to\infty} \frac{u_{n+1}}{u_n}$ exists and equals ℓ. Then d'Alembert's ratio test states that the given series is convergent if $\ell < 1$ and divergent if $\ell > 1$
(if $\ell = 1$ then the test does not provide any conclusions about convergence).

N.B. A necessary condition for convergence of infinite series is that the general term in the series has limit zero as $n \to \infty$. Hence, if $u_n \to C$ as $n \to \infty$ and $C \neq 0$, then the series $\sum_{k=0}^{\infty}$ is divergent.

Ex 4. Determine whether the following series are convergent
(a) $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots,$ (b) $\sum_{k=0}^{\infty} \frac{2^k}{k!},$ (c) $\sum_{k=0}^{\infty} \frac{k}{k+1}.$

(a) This is a geometric series with $r = \frac{1}{2}$. The ratio is less than 1 and so the series converges to sum $S = \frac{1}{1-\frac{1}{2}} = 2$.

(b) All terms in the series are positive and

$$\frac{u_{k+1}}{u_k} = \frac{2^{k+1}/(k+1)!}{2^k/k!} = \frac{2^{k+1}}{(k+1)!}\,\frac{k!}{2^k} = \frac{2}{k+1} \quad \to 0 \quad \text{as } k \to \infty.$$

This limit is < 1 so d'Alembert's test implies that the series converges.

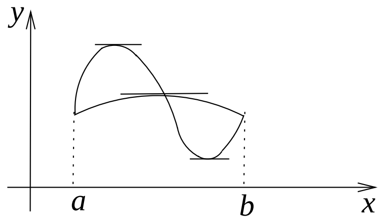
(c) In this case

$$u_k = \frac{k}{k+1} = \frac{\frac{k}{k}}{\frac{k+1}{k}} = \frac{1}{1+\frac{1}{k}} \quad \to \frac{1}{1+0} = 1 \quad \text{as } k \to \infty.$$

The terms do NOT tend to zero as k gets large and so the series diverges.

2. Rolle's theorem and mean value theorems

Consider the two functions $y = f(x)$ shown in the figure below. Clearly for each function there exists at least one point $x = c$ where the gradient of the curve is zero, i.e. there exists at least one point $x = c$ such that $f'(c) = 0$.



The above can be more precisely stated as

<u>Rolle's theorem</u>

If (i) $f(x)$ is continuous for $a \le x \le b$,
  (ii) $f(x)$ is differentiable for $a < x < b$ and
  (iii) $f(a) = f(b)$

then there exists at least one number c, where $a < c < b$, such that $f'(c) = 0$.

Condition (ii) above implies that the function f is not required to be differentiable at the end-points $x = a$ and $x = b$. The gradient at these points might be infinite, therefore, but the theorem still holds (see figure 2a).

Rolle's theorem does not hold for functions which are not differentiable within the interval (see figure 2b), or for functions which are not continuous throughout the interval (see figure 2c).

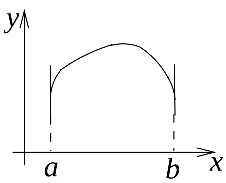


figure 2a

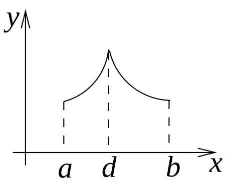


figure 2b

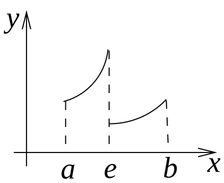


figure 2c

<u>N.B.</u> It is important to note that Rolle's theorem states the existence of c but does NOT tell you how to find its value!

<u>First mean value of integral calculus</u> states: if $f(x)$ is continuous in the closed interval $a \le x \le b$ then there exists at least one point c such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\, dx\,.$$

(Geometrically, above states that: area under graph = area of rectangle of height $f(c)$, see figure 2d).

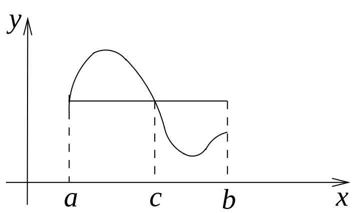


figure 2d

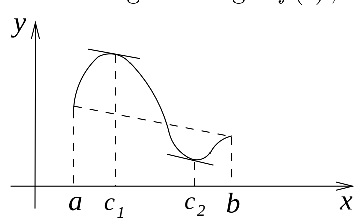


figure 2e

<u>First mean value of differential calculus</u> states: if $f(x)$ is continuous in the closed interval $a \le x \le b$ and differentiable in the open interval $a < x < b$ then there exists at least one point c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,.$$

(Geometrically, the above states that the slope of the tangent at the point $x = c$ is parallel to the straight line joining the end-points of the curve (see figure 2e))

3. Taylor's and Maclaurin's theorems

How accurately can we approximate functions, by using polynomials say? This is partly answered by the following theorem.

Taylor's theorem If $f(x),\ f'(x),\ \ldots, f^{(n)}(x)$ all exist and are continuous in $[a,\ x]$, and $f^{(n+1)}(x)$ exists in $(a,\ x)$ then

$$f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \ \ldots \ + \frac{(x-a)^n}{n!}\,f^{(n)}(a) + R_n(x)\,,$$

where

$$R_n(x) = \frac{(x-a)^{n+1}}{(n+1)!}\,f^{(n+1)}(c)\,, \qquad a < c < x\,.$$

(The above notation for R_n follows that in **James**, but it should be noted that the notation R_{n+1} is often used in the literature).

Comments

(i) The quantity θ, where $0 < \theta < 1$, is often used instead of c, giving $c = a + \theta(x-a)$.

(ii)The above theorem states the **Lagrange form of the remainder**. The remainder $R_n(x)$ denotes the difference between the original function $f(x)$ and the nth order polynomial that is used to approximate it. Taylor's theorem shows the surprising result that however many terms you choose to have in your polynomial approximation to $f(x)$, the difference between $f(x)$ and this polynomial can always be expressed in a comparatively simple form (Lagrange's form), although you do NOT know the value of c. Knowledge of the range of c, or θ, can often mean the maximum magnitude of the remainder can be *calculated*, and not *guessed*.

(iii) Infinite series $\sum\limits_{n=0}^{\infty} a_n x^n$, where the a_n are independent of x, are called **power series**.

Def. A power series has **radius of convergence** r if the series converges when $|x| < r$ and diverges when $|x| > r$.

There are many important power series. For example

$$e^x = 1 + x + \frac{x^2}{2!} + \ \ldots \ + \frac{x^r}{r!} + \ \ldots \qquad \text{for all } x\,, \qquad (\text{i.e. } r = \infty)$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \ \ldots \qquad \text{for all } x \qquad (\text{i.e. } r = \infty).$$

(iv) Choosing $a = 0$ in Taylor's theorem gives **Maclaurin's theorem**:

$$f(x) = f(0) + x\,f'(0) + \frac{x^2}{2!}\,f''(0) + \ \ldots \ + \frac{x^n}{n!}\,f^{(n)}(0) + R_n(x)\,,$$

where

$$R_n(x) = \frac{x^{n+1}}{(n+1)!}\,f^{(n+1)}(\theta x)\,, \qquad 0 < \theta < 1\,.$$

Taylor's and Maclaurin's series In Taylor's and Maclaurin's theorems if $R_n \to 0$ as $n \to \infty$ then the function $f(x)$ can be represented by the corresponding infinite series:

$$f(x) = f(a) + (x-a)\,f'(a) + \ \ldots \ + \frac{(x-a)^r}{r!}\,f^{(r)}(a) + \ \ldots \qquad (\text{Taylor's series})$$

$$f(x) = f(0) + x\,f'(0) + \ \ldots \ + \frac{x^r}{r!}\,f^{(r)}(0) + \ \ldots \qquad (\text{Maclaurin's series})$$

Ex 5. Use Maclaurin's theorem, with two terms and a remainder, to show that the error in writing $\sin x$ as x is less than 0.005 if $0 < x < 0.1$

With $n = 1$ Maclaurin's theorem states

$$f(x) = f(0) + x\,f'(0) + R_1(x), \qquad R_1(x) = \frac{x^2}{2!}\,f''(\theta x) \qquad (0 < \theta < 1)\,.$$

Now

$$\begin{aligned} f(x) &= \sin x\,, & f(0) &= 0 \\ f'(x) &= \cos x\,, & f'(0) &= \cos 0 = 1 \\ f''(x) &= -\sin x\,, \end{aligned}$$

and substituting into Maclaurin's theorem above gives

$$\sin x = 0 + x \times 1 + R_1 = x + R_1, \qquad \text{where} \quad R_1(x) = \frac{x^2}{2!}\,(-\sin(\theta x))\,, \quad 0 < \theta < 1.$$

Now

$$|R_1| = \frac{x^2}{2!}\,|\sin(\theta x)| \le \frac{x^2}{2}, \quad \text{since} \quad |\sin(\theta x)| \le 1,$$

and it then follows that if $0 < x < 1$ then

$$|R_1| < \frac{(0.1)^2}{2} = 0.005\,.$$

Thus replacing $\sin x$ by x has an error of maximum magnitude 0.005 (provided $0 < x < 1$).

Ex 6. Establish the convergence of Maclaurin's series for $\cos x$.

For this function

$$\begin{aligned} f(x) &= \cos x\,, & f(0) &= 1\,, \\ f'(x) &= -\sin x\,, & f'(0) &= 0\,, \\ f''(x) &= -\cos x\,, & f''(0) &= -1\,, \\ f^{(3)}(x) &= \sin x\,, & f^{(3)}(0) &= 0\,, \\ f^{(4)}(x) &= \cos x\,, & f^{(4)}(0) &= 1\,, \end{aligned}$$

and therefore

$$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^{n/2} & \text{if } n \text{ is even} \end{cases}$$

Maclaurin's theorem then yields

$$\begin{aligned} f(x) &= f(0) + x\,f'(0) + \ldots + \frac{x^r}{r!}\,f^{(r)}(0) + R_r \\ &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + \frac{x^{2n}}{(2n)!}(-1)^n + R_{2n+1}, \end{aligned}$$

where

$$R_{2n+1} = \frac{x^{2n+2}}{(2n+2)!}\,(-1)^{n+1}\,\cos(\theta x), \qquad 0 < \theta < 1$$

$$\rightarrow \qquad |R_{2n+1}| \le \left|\frac{x^{2n+2}}{(2n+2)!}\right| = \frac{|x|^{2n+2}}{(2n+2)!}\,.$$

The latter term tends to zero as $n \to \infty$ for all x, because however large x is then as n increases the denominator will always become bigger than the numerator. Thus the remainder tends to zero and the series converges giving, for all x,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots + (-1)^n\,\frac{x^{2n}}{(2n)!} + \ldots$$

4. L'Hopital's rule

It is often necessary to obtain limits of quotients as $x \to a$. For instance, as $x \to 0$ then $\dfrac{\sin x}{x} \to \dfrac{0}{0} = ?$ How do you discover whether a limit exists? One useful result is stated below.

L'Hopital's rule If (i) $f(a) = g(a) = 0$ and (ii) f and g are differentiable at $x = a$ then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(a)}{g'(a)}, \qquad \text{provided } g'(a) \neq 0.$$

If $\dfrac{f'(a)}{g'(a)} = \dfrac{0}{0}$ then keep differentiating the two functions, one derivative at a time, until at least one of the derivatives $f^{(n)}(a)$ and $g^{(n)}(a)$ is non-zero (see **Ex 9** below).

Ex 7. Find $\lim_{x\to 0}\left(\dfrac{\sin x}{x}\right)$.

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \frac{\sin 0}{0} = \frac{0}{0} =?$$

Using l'Hopital's rule

$$\lim_{x\to 0}\left(\frac{\sin x}{x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{1}\right) = \frac{\cos 0}{1} = \frac{1}{1} = 1$$

Ex 8. Find $\lim_{x\to\infty}\left(\dfrac{x}{e^x}\right)$.

$$\lim_{x\to\infty}\left(\frac{x}{e^x}\right) = \frac{\infty}{\infty} =?$$

Using l'Hopital's rule

$$\lim_{x\to\infty}\left(\frac{x}{e^x}\right) = \lim_{x\to\infty}\left(\frac{1}{e^x}\right) = 0$$

Ex 9. Find $\lim_{x\to 0}\left(\dfrac{1-\cos x}{x^2}\right)$.

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) = \frac{1-\cos 0}{0^2} = \frac{1-1}{0} = \frac{0}{0} =?$$

Using l'Hopital's rule in this case leads to

$$\lim_{x\to 0}\left(\frac{1-\cos x}{x^2}\right) = \lim_{x\to 0}\left(\frac{\sin x}{2x}\right) = \frac{\sin 0}{2(0)} = \frac{0}{0} =?$$

Here we have to differentiate again

$$\lim_{x\to 0}\left(\frac{\sin x}{2x}\right) = \lim_{x\to 0}\left(\frac{\cos x}{2}\right) = \frac{\cos 0}{2} = \frac{1}{2},$$

which is the required answer.

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