Finite difference approximations of first order in time, second order in space hyperbolic systems

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Outline

- Motivation: Why second order systems?
- Discretization of second order systems
  - No boundaries
    - Mixture of D_+D_- with D_0 can cause difficulties
    - Stability and choice of discrete norm
    - Examples: gKWB, NOR, ADM, Z4
  - Boundary treatment
    - (Limitations of the) discrete energy method
    - Laplace transform method

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Motivation

- Second order systems have fewer variables, fewer constraints, and typically smaller errors.
- They are used by several groups (e.g. BSSN).
- First order systems are better understood.
  - Improve our understanding of properties of (finite difference approx of) 2nd order systems;
  - identify stable discretizations;
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- Notation: h space step, k time step, $D_+ v_j = (v_{j+1} - v_j)/h$, $D_- v_j = (v_j - v_{j-1})/h$, $D_0 v_j = (v_{j+1} - v_{j-1})/(2h)$, $D_+ D_- v_j = (v_{j+1} - 2v_j + v_{j-1})/h^2$.

Some difficulties

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- Standard discretization of well-posed problems can give rise to unstable schemes.
  - Take the wave equation $\partial_t^2 \phi = \partial_x^2 \phi$ and change coordinates ($x' = x - \beta t$)

$$\partial_t^2 \phi = 2\beta \partial_t \partial_x \phi + (1 - \beta^2) \partial_x^2 \phi$$

(*shifted wave equation*)

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The discretization ($D_0 u_j = \frac{u_{j+1} - u_{j-1}}{2h}$, $D_+ D_- u_j = \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$)

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is unstable for $|\beta| > 1$. Who is to blame?

First order systems

- Continuum: A first order system $\frac{\partial u}{\partial t} = P(\partial_x)u$ is *strongly hyperbolic* iff

$$K^{-1} \le \hat{H}(\omega) = \hat{H}^*(\omega) \le K$$
$$\hat{H}(\omega)\hat{P}(i\omega) + \hat{P}^*(i\omega)\hat{H}(\omega) \le 2\alpha\hat{H}(\omega)$$

  - The Cauchy problem is *well-posed*.
  - Estimate in L_2 follows

$$\|u(t,\cdot)\| \le Ke^{\alpha t}\|u(0,\cdot)\|$$

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- Discrete: The scheme $v^{n+1} = Qv^n$ is *stable* iff

$$K^{-1} \leq \hat{H}(\xi) = \hat{H}^*(\xi) \leq K$$
$$|\hat{Q}(\xi)|_{\hat{H}} \leq e^{\alpha k}$$

  - Estimate in discrete L_2-norm follows

$$\|v^n\|_h \leq K e^{\alpha t} \|v^0\|_h$$

    where $\|v\|_h^2 = \sum_j v_j^2 h$.
  - Von Neumann necessary condition: $\sigma(\hat{Q}(\xi)) \leq e^{\alpha k}$.

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Simple sufficient condition

- Assumptions:
  - Method of lines: $\frac{d}{dt}v_j = Pv_j$. In F. space $\frac{d}{dt}\hat{v} = \hat{P}(\xi)\hat{v}$.
  - 3RK, 4RK, or ICN time integrators: $\hat{Q} = \mathcal{P}(k\hat{P})$.
- If there exists a *discrete symmetrizer* $\hat{H}(\xi)$ of $\hat{P}(\xi)$

$$K^{-1} \leq \hat{H}(\xi) = \hat{H}^*(\xi) \leq K$$
$$\hat{H}(\xi)\hat{P}(\xi) + \hat{P}^*(\xi)\hat{H}(\xi) = 0$$

(i.e. a conserved energy for the semi-discrete system in F. space) then the von Neumann condition

$$\sigma(k\hat{P}) \leq \alpha_0 \qquad \text{(e.g. } \alpha_0 = \sqrt{8} \text{ for 4RK)}$$

is necessary and sufficient for stability ($\|v^n\|_h \leq K\|v^0\|_h$).

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Second order systems

- Standard discretization of second order system of the form

$$\partial_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} A^i D_i^{(1)} + B & C \\ D^{ij} D_{ij}^{(2)} + E^i D_i^{(1)} + F & G^i D_i^{(1)} + J \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

  E.g. $D_i^{(1)} = D_{0i}$, $D_{ij}^{(2)} = D_{0i} D_{0j}$ if $i \neq j$, $D_{ij}^{(2)} = D_{+i} D_{-i}$ if $i = j$.

- If $\partial_t \hat{\boldsymbol{v}} = \hat{P}' \hat{\boldsymbol{v}}$, where $\hat{P}'$ is the *principal symbol* of the semi-discrete system, admits a conserved energy $\hat{\boldsymbol{v}}^* \hat{H} \hat{\boldsymbol{v}}$ and

$$K^{-1} I_\Omega \leq \hat{H} \leq K I_\Omega, \quad I_\Omega \equiv \begin{pmatrix} \Omega^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \Omega^2 = \sum_{i=1}^{d} |\hat{D}_{+i}|^2$$

  then, provided that $\sigma(k\hat{P}') \leq \alpha_0$, the fully discrete scheme is stable wrt $\|\boldsymbol{v}\|_{h,D_+}^2 \equiv \|u\|_h^2 + \|v\|_h^2 + \sum_{i=1}^d \|D_{+i} u\|_h^2$.

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Results

- Generalized KWB system

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\begin{aligned}
\partial_t A_i &= -E_i \\
\partial_t E_i &= -\partial^k \partial_k A_i + r \partial_i \partial^k A_k + \partial_i G \\
\partial_t G &= r \partial^k E_k
\end{aligned}
$$

  - Continuum: Cauchy problem is well-posed for $r \in \mathbb{R}$.
  - Discrete: stability wrt D_+-norm only for $r < 1$; for $r > 1$ the von Neumann condition is violated.

- NOR formulation of GR has similar properties

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\begin{aligned}
\partial_t \gamma_{ij} &= -2K_{ij} \\
\partial_t K_{ij} &= -\frac{1}{2} \partial^k \partial_k \gamma_{ij} + \frac{r}{2} \partial_i \partial_j \gamma_{kk} + \partial_{(i} f_{j)} \\
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Results

- Other systems analyzed: ADM, Z4.
- The approximation

$$\frac{d}{dt}\phi_j(t) = \Pi_j(t), \qquad \frac{d}{dt}\Pi_j(t) = D_+D_-\phi_j(t)$$

is stable wrt $\|\phi\|_h^2 + \|\Pi\|_h^2 + \|D_+\phi\|_h^2$. What about using D_0^2 instead of D_+D_- in the scheme, or D_0 instead of D_+ in the norm?

  - The D_0^2-scheme is unstable wrt the D_+-norm.
  - Similarly, the standard 2nd o.a. discretization is unstable wrt the D_0-norm.
  - D_0^2 in the scheme and D_0 in the norm is ok, but one has to be careful.

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$$\frac{d}{dt}\phi_j(t) = \Pi_j(t), \qquad \frac{d}{dt}\Pi_j(t) = D_+D_-\phi_j(t)$$

is stable wrt $\|\phi\|_h^2 + \|\Pi\|_h^2 + \|D_+\phi\|_h^2$. What about using D_0^2 instead of D_+D_- in the scheme, or D_0 instead of D_+ in the norm?

  - The D_0^2-scheme is unstable wrt the D_+-norm.
  - Similarly, the standard 2nd o.a. discretization is unstable wrt the D_0-norm.
  - D_0^2 in the scheme and D_0 in the norm is ok, but one has to be careful.

Testing stability

- For a linear scheme with *no forcing terms* a stability test should be aimed at establishing the existence of K and α, such that

$$\|v^n\| \leq K e^{\alpha t_n} \|v^0\| \qquad \text{for } h \leq h_0$$

where, for the NOR system, for example, the norm is

$$\sum_{i,j=1}^{3} \|\gamma_{ij}\|_h^2 + \sum_{i,j=1}^{3} \|K_{ij}\|_h^2 + \sum_{k,i,j=1}^{3} \|D_{+k}\gamma_{ij}\|_h^2 + \sum_{i=1}^{3} \|f_i\|_h^2.$$

- In the non linear case, however, this wouldn't work!
- Ultimately, we want convergence. Suggestion:

Test for convergence with consistent (but not exact!) initial data.

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Boundary treatment

- Although the scheme

$$\frac{d^2}{dt^2}\phi_j = 2\beta\frac{d}{dt}D_0\phi_j + (1-\beta^2)D_+D_-\phi_j$$

is unstable for $|\beta| > 1$, the approximation

$$\begin{aligned} \frac{d}{dt}\phi_j &= \beta D_0\phi_j + \Pi_j \\ \frac{d}{dt}\Pi_j &= \beta D_0\Pi_j + D_+D_-\phi_j \end{aligned}$$

is stable for any $\beta \in \mathbb{R}$.

- We will use the second approximation in the interior and look for appropriate discrete boundary prescriptions which are
  - consistent with those of the continuum problem;
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Boundary treatment (continuum)

- Quarter space ($x \geq 0$, $t \geq 0$) for the *shifted wave equation*:
  - Evolution equations:

$$\begin{aligned} \partial_t \phi &= \beta \partial_x \phi + \Pi + F^{\phi} \\ \partial_t \Pi &= \beta \partial_x \Pi + \partial_x^2 \phi + F^{\Pi} \end{aligned}$$

  - Initial data: $\phi(x,0) = f^{\phi}(x)$, $\Pi(x,0) = f^{\Pi}(x)$
  - Boundary data: $\Pi(0,t) - \partial_x \phi(0,t) = g(t)$ if $|\beta| < 1$; no BCs in the outflow case ($\beta \geq 1$)

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- Strong stability

$$\|u(\cdot,t)\|^2 \leq K(t) \left(\|f\|^2 + \int_0^t (\|F(\cdot,\tau)\|^2 + \delta |g(\tau)|^2) d\tau \right)$$

where $\delta = 0, 1$, and $u(x,t) = (\phi(x,t), \Pi(x,t), \phi_x(x,t))^T$.

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The Laplace transform method

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$$h^{q_2+1} D_+^{q_2+1} \phi_{-1} = g^{\phi}, \quad h^{q_1} D_+^{q_1} \Pi_{-1} = g^{\Pi}$$

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- Perform a discrete reduction to first order:

$$X_j = D_+ \phi_j$$

The Laplace transform method

- Reduced quarter space semi-discrete problem ($\beta > 1$, $j \geq 0$):

$$\frac{d}{dt} X_j = \beta D_0 X_j + D_+ \Pi_j + D_+ F_j^{\phi}$$

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$$X_j(0) = D_+ f_j^{\phi}, \quad \Pi_j(0) = f_j^{\Pi}$$

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- Result: the scheme is stable and second order convergent if $q_1, q_2 \geq 2$. Minimum order of extrapolation is

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- Result: the scheme is stable and second order convergent if $q_1, q_2 \geq 2$. Minimum order of extrapolation is

$$\phi_{-1} = 3\phi_0 - 3\phi_1 + \phi_2, \qquad \Pi_{-1} = 2\Pi_0 - \Pi_1$$

Proof of strong stability

- Three main parts of the proof:
  1. Verifying the Kreiss condition to obtain an estimate for the $F = 0$, $f = 0$ case.
  2. Estimate the solution of the problem with modified BCs in terms of f and F.
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  3. Put things together to derive estimate for the original problem.
- The strong stability estimate

$$\|v(t)\|_{D_+}^2 \leq K(t) \left(\|f\|_{D_+}^2 + \int_0^t (\|F(\tau)\|_{D_+}^2 + |g(\tau)|^2) d\tau \right)$$

where $\|v(t)\|_{D_+}^2 = \|\phi\|_h^2 + \|\Pi\|_h^2 + \|D_+\phi\|_h^2$, can be used to prove convergence (i.e. derive estimates for the error).

The Kreiss condition $(*)$

- Solve the Laplace transformed problem for $(f = 0$ and $F = 0)$ and express the solution in terms of the boundary data:

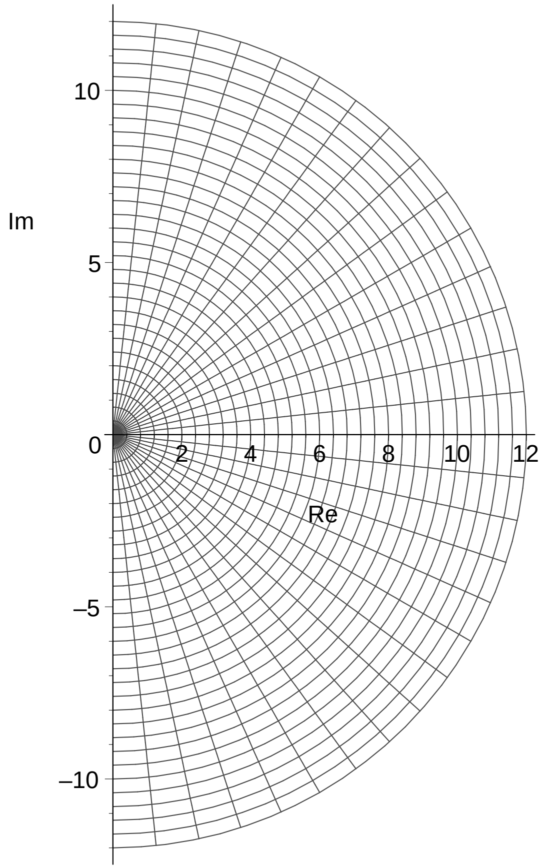
$$\hat{\Pi}_j = \sum_{k=\Pi,X} c^{\Pi}_{jk}\hat{g}^k, \quad \hat{X}_j = \sum_{k=\Pi,X} c^{X}_{jk}\hat{g}^k$$

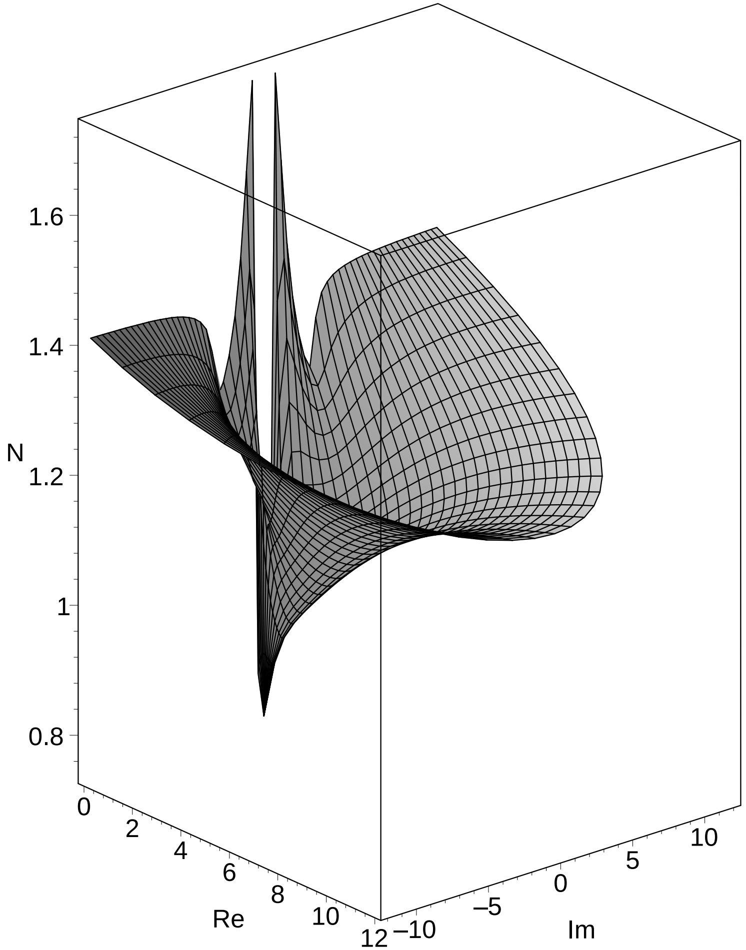
Verify the Kreiss condition $(|\hat{\Pi}_j|^2 + |\hat{X}_j|^2 \leq K(|\hat{g}^{\Pi}|^2 + |\hat{g}^X|^2))$ by plotting

$$N = \left(\sum_{\substack{j = -1, 0 \\ k = \Pi, X}} (|c^{\Pi}_{jk}|^2 + |c^{X}_{jk}|^2) \right)^{1/2}$$

The Kreiss condition ($*$)

...and verifying that it is bounded ($\beta = 2$, $q_1 = q_2 = 2$)





Other cases

- Similar result holds for the boundary conditions ($|\beta| < 1$)

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- Fourth order accuracy
  - Outflow case:

$$h^5 D_+^5 \phi_{-1} = 0 \qquad h^4 D_+^4 \Pi_{-1} = 0$$
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  - Time-like case:

$$\Pi_0 - D_0\left(1 - \frac{h^2}{6} D_+ D_-\right)\phi_0 = g \qquad h^4 D_+^4 \Pi_{-1} = 0$$
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Conclusion

- Subtle difficulties arise in the discretization of first order in time, second in space systems
  - Standard discretization of well-posed problems can give rise to unstable schemes. Not just β's fault!
  - With the standard discretization the discrete norm better contain D_+ operators.
  - Testing stability
- Boundary treatment
  - Limitations of the discrete energy method.
  - Strong stability proofs for the 2nd and 4th order accurate case.

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- Subtle difficulties arise in the discretization of first order in time, second in space systems
  - Standard discretization of well-posed problems can give rise to unstable schemes. Not just β's fault!
  - With the standard discretization the discrete norm better contain D_+ operators.
  - Testing stability
- Boundary treatment
  - Limitations of the discrete energy method.
  - Strong stability proofs for the 2nd and 4th order accurate case.