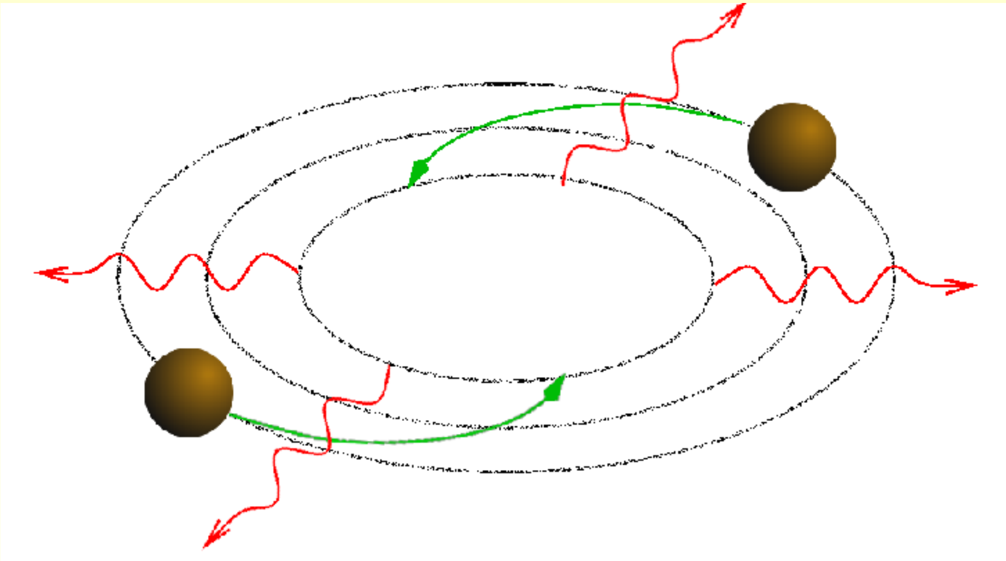
Evolution with the conformal field equations

Jörg Frauendiener
Institut für Astronomie und Astrophysik
Universität Tübingen

Introduction



- study a self–gravitating system
  - formation of singularities, horizons
  - extract radiation
  - global structure
- modelled as isolated system
- space–time is asymptotically flat $\Longleftrightarrow$ can be conformally compactified

Conformal compactification of Minkowski space–time

$$\tilde{g} = dt^2 - dr^2 - r^2 d\omega^2, \qquad d\omega^2 = \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

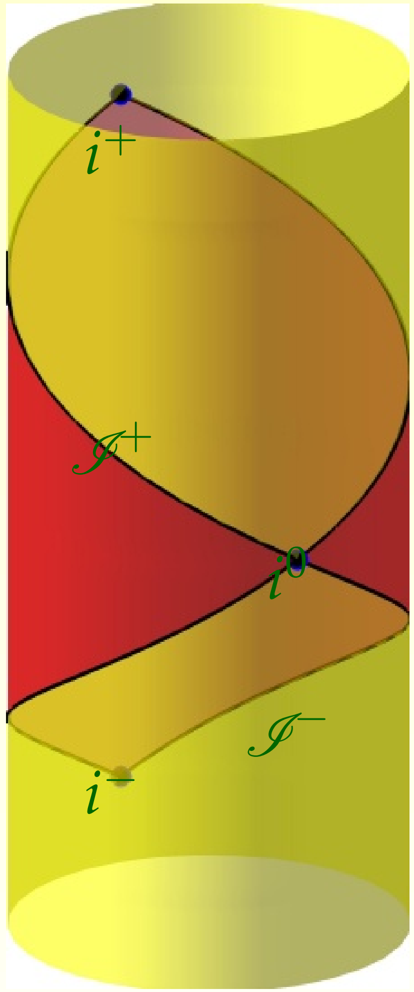
$$= \frac{1}{(\cos T + \cos R)^2}\left(dT^2 - dR^2 - \sin^2 R\, d\omega^2\right),$$

has the form

$$\tilde{g} = \Omega^{-2} g \iff g = \Omega^2 \tilde{g}.$$

- $\mathscr{I}^+$ and $\mathscr{I}^-$ form a 3–D boundary
- points on $\mathscr{I}$ are at infinity wrt $\tilde{g}$
- conformal factor $\Omega = \cos T + \cos R = 0$ on $\mathscr{I}$
- physical space–time $(\tilde{M}, \tilde{g})$ is smoothly embedded into a larger space–time (M, g) with the same light–cone structure.

$\rightarrow$ $\mathscr{I}$ is an abstract concept which allows us to talk about radiation



Conformal field equations

- express $\tilde{G}_{ab} = 0$ in terms of $(\mathcal{M}, \Omega, g_{ab})$
- conformal transformation

$$0 = \tilde{G}_{ab} = G_{ab} - \frac{2}{\Omega}\left(\nabla_a \nabla_b \Omega - g_{ab}\,\Box\Omega\right) - \frac{3}{\Omega^2}\, g_{ab}\, \nabla_c \Omega \nabla^c \Omega.$$

- read as equation for Ω ($G_{ab} = -2\Phi_{ab} - 6\Lambda g_{ab}$)

$$\nabla_a \nabla_b \Omega = \Omega \Phi_{ab} + g_{ab} S, \qquad S = \frac{1}{4}\Box\Omega$$

- need equation for $\Phi_{ab} \rightarrow$ Bianchi identities in M and $\tilde{M}$

$$\tilde{\nabla}_d C_{abc}{}^d = 2\nabla_{[a}\tilde{\Phi}_{b]c} + 2\tilde{g}_{c[a}\nabla_{b]}\tilde{\Lambda} = 0 \iff \nabla_a \underbrace{\left(\Omega^{-1} C_{abc}{}^d\right)}_{K_{abc}{}^d} = 0$$

$$\nabla_d C_{abc}{}^d = 2\nabla_{[a}\Phi_{b]c} + 2g_{c[a}\nabla_{b]}\Lambda \iff 2\nabla_{[a}\Phi_{b]c} = \nabla_d \Omega K_{abc}{}^d - 2g_{c[a}\nabla_{b]}\Lambda$$

Conformal field equations

$$\Gamma = \partial g \qquad \Big| \qquad \partial e = \gamma e$$

$$\partial^2 g + \partial g \partial g = \Omega K + \Phi + \Lambda \qquad \Big| \qquad \partial\gamma + \gamma\gamma = \Omega K + \Phi + \Lambda$$

$$\partial\Phi = \partial\Omega\; K - g\;\partial\Lambda$$

$$\partial K = 0$$

$$\partial^2\Omega = \Omega\;\Phi + g\;S$$

$$\partial S = \Phi\;\partial\Omega + \Omega\;\partial\Lambda + \Lambda\;\partial\Omega$$

gauge source functions

$$\Box x^\mu = F^\mu \qquad \Big| \qquad \Box x^\mu = F^\mu$$

$$\Big| \qquad \nabla_a(e_i \nabla^a e_k) = F_{ik}$$

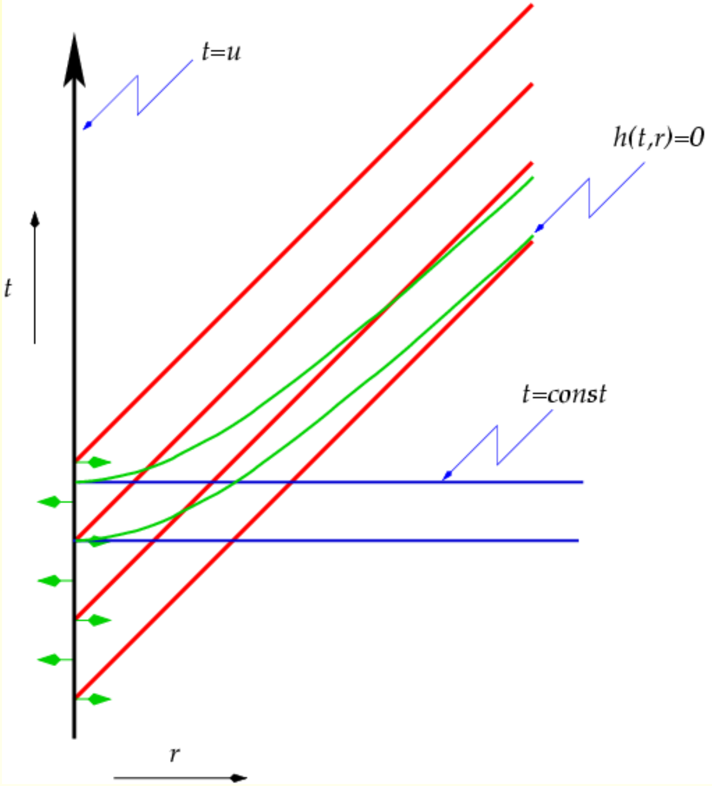
$$\Lambda \qquad \Big| \qquad \Lambda$$

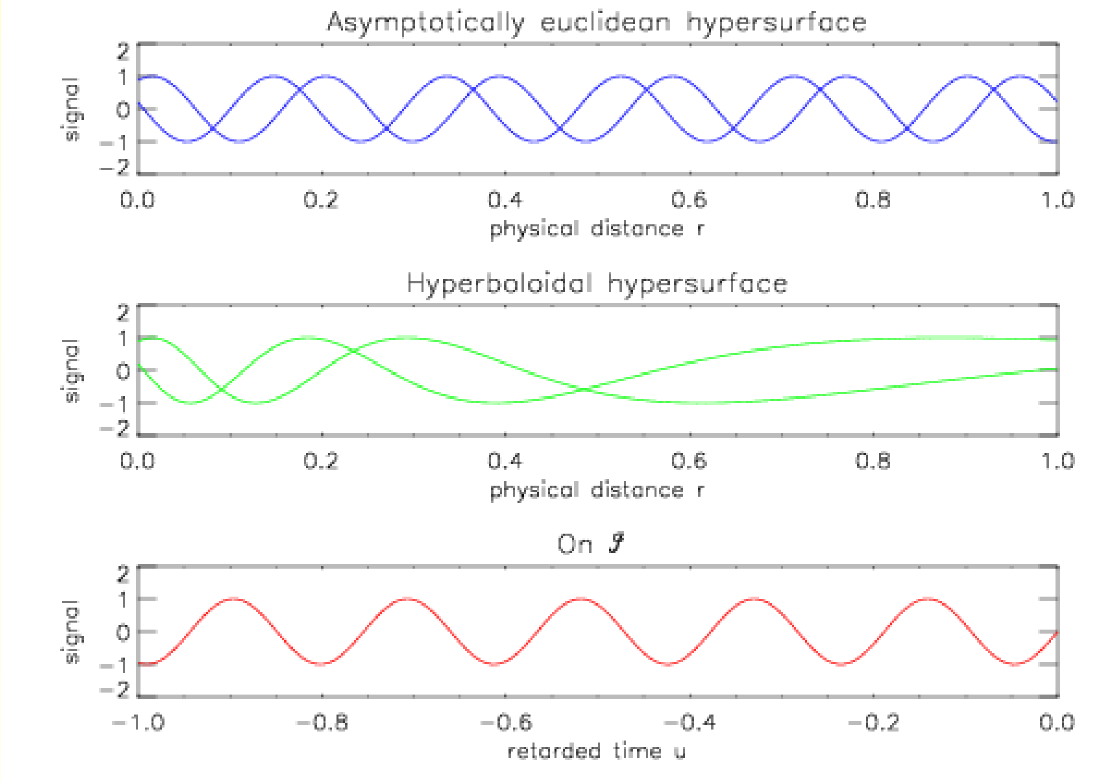
Analysis

Friedrich (1979–)

- Split into constraints and symmetric hyperbolic evolution equations
- evolution propagates the constraints
- hyperboloidal IVP is well posed
- (semi)–global existence for small hyperboloidal data
- gluing techniques give global existence for asymptotically flat manifolds
- initial data on hyperboloidal hypersurfaces can be constructed

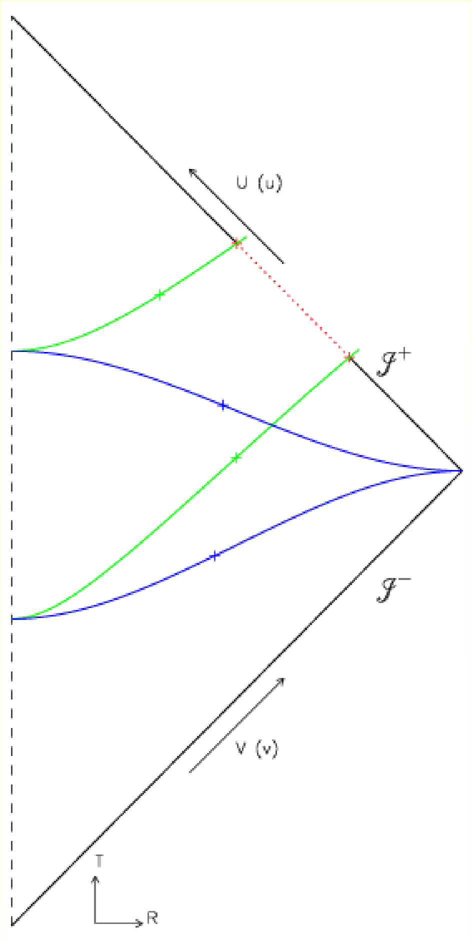
Waveforms on hypersurfaces in Minkowski space

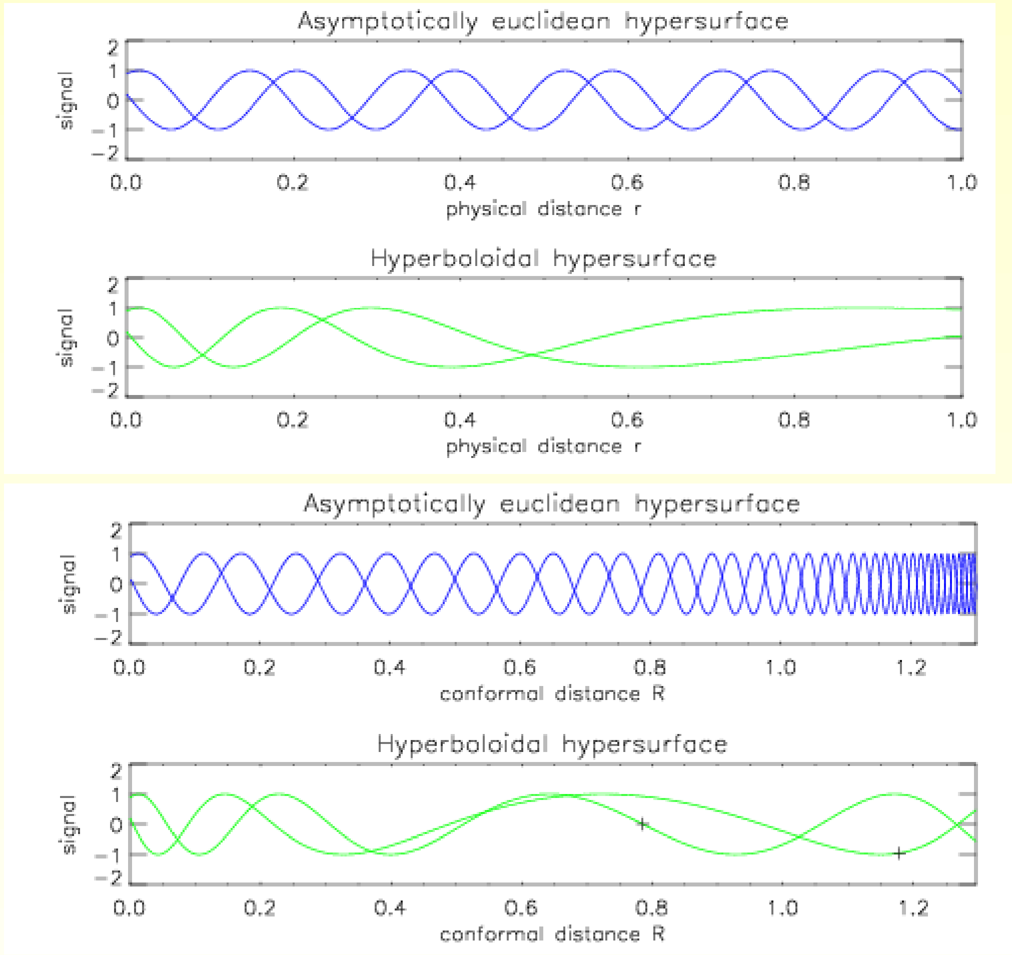




- $t = t_0$–surface 'collects' all waves emitted up to $t = t_0$
- hyp. surface 'selects' waves emitted in a finite interval before $t = t_0$
- $t = t_0$–surface is 'instantaneous', hyp. surface 'slows down'
- spatial succession turns into temporal succession

Waveforms on hypersurfaces in conformal Minkowski space





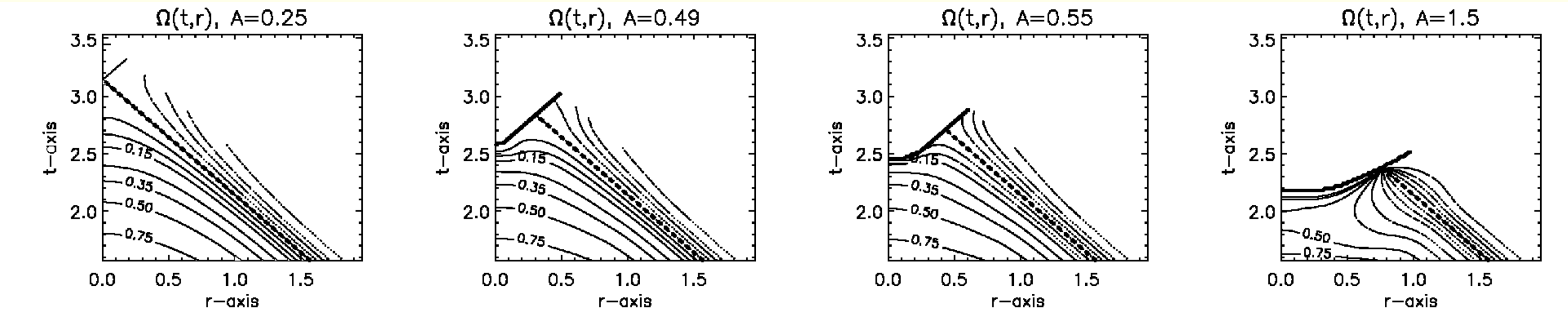
- piling up of oscillations on asymptotically euclidean hsf
  'all waves at once'

- smooth across null–infinity

Numerical implementations of CFE

P. Hübner (1994)

- spherical symmetry with conformally coupled scalar field
- parameter study of collapse
- numerical generation of conformal diagrams
- investigation of the causal nature of the singularity

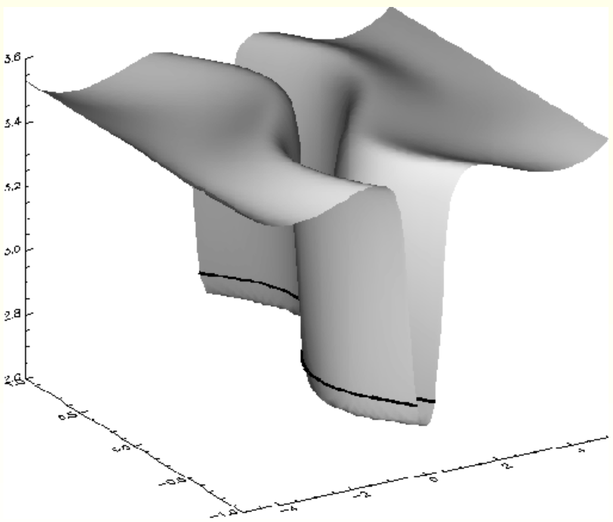


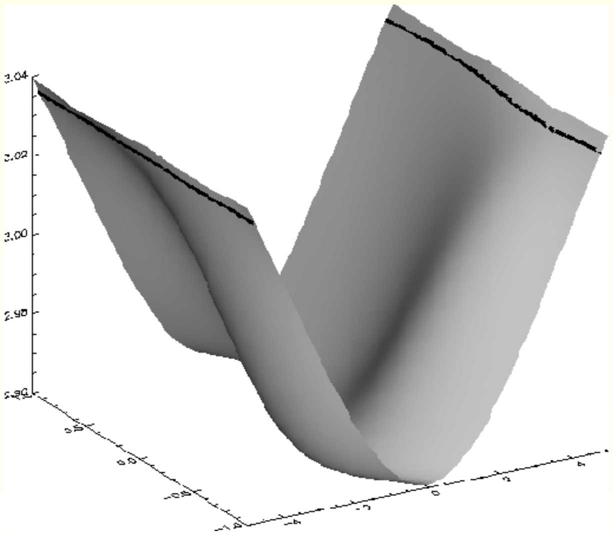
Numerical implementations of CFE

JF (1998)

- frame based vacuum code in $2+1$ dimensions
- space–like non–vanishing Killing vector with closed orbits: 'toroidal' $\mathscr{I}$
- 'generalized harmonic' time coordinate $\Box t = F$, shift components free
- Fermi transport of the frame
- Investigate radiation extraction

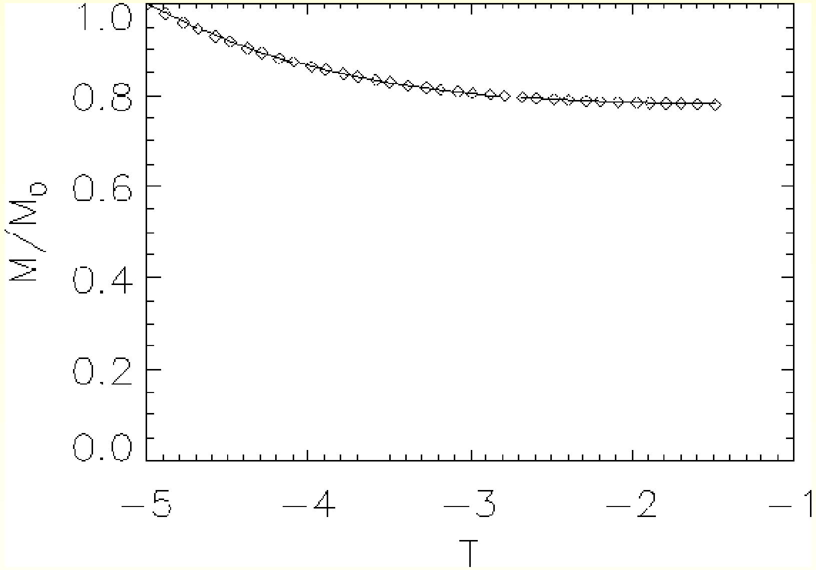
Several gauge choices, 'scri-freezing'

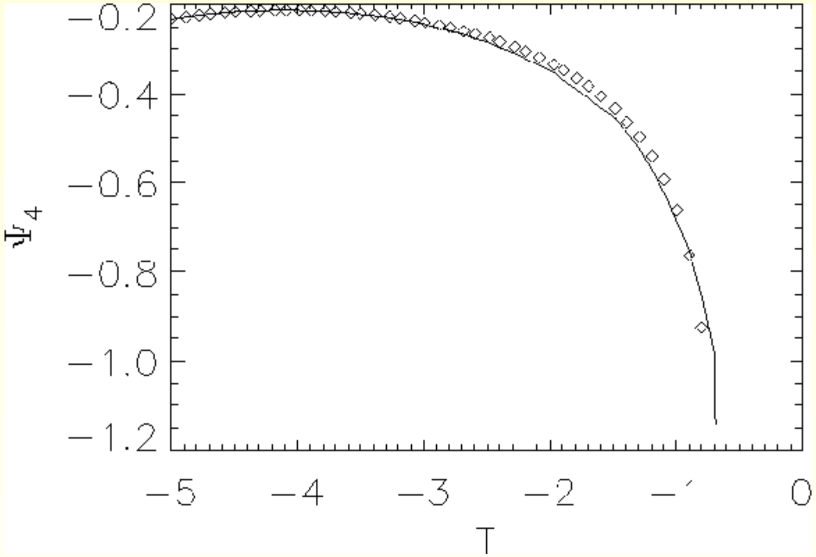




Numerical implementations of CFE

radiation extraction on $\mathscr{I}$





Numerical implementations of CFE

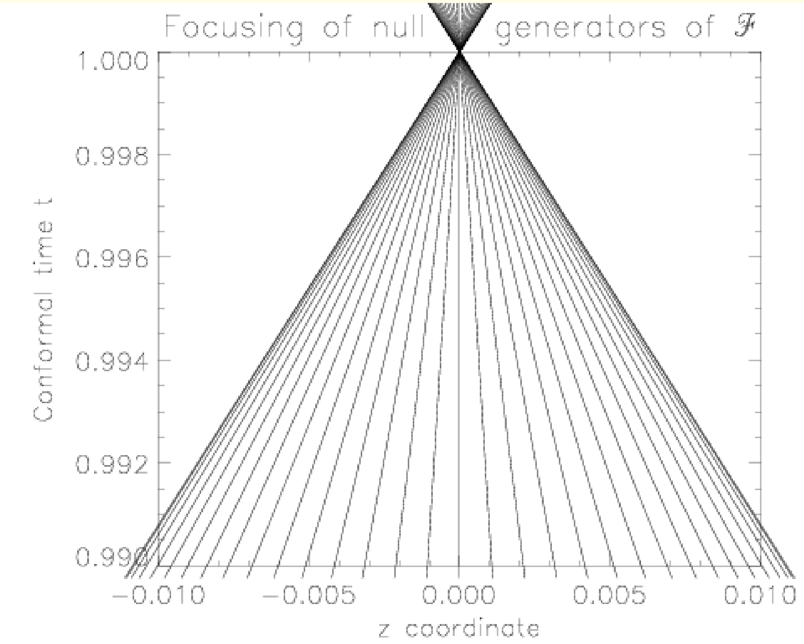
JF (1998)

- spectral solver for hyperboloidal initial data
  following Andersson, Chrusciel, Friedrich
- choose conformal 3-metric h_{ab}
- solve Yamabe equation for conformal factor so that $\tilde{h}_{ab} = \Omega^{-2} h_{ab}$ has $^3\tilde{R} = -6$
- all other initial data can be computed
- draw–backs: numerical differentiation, $0/0$-terms, nasty

Numerical implementations of CFE

P. Hübner (2000)

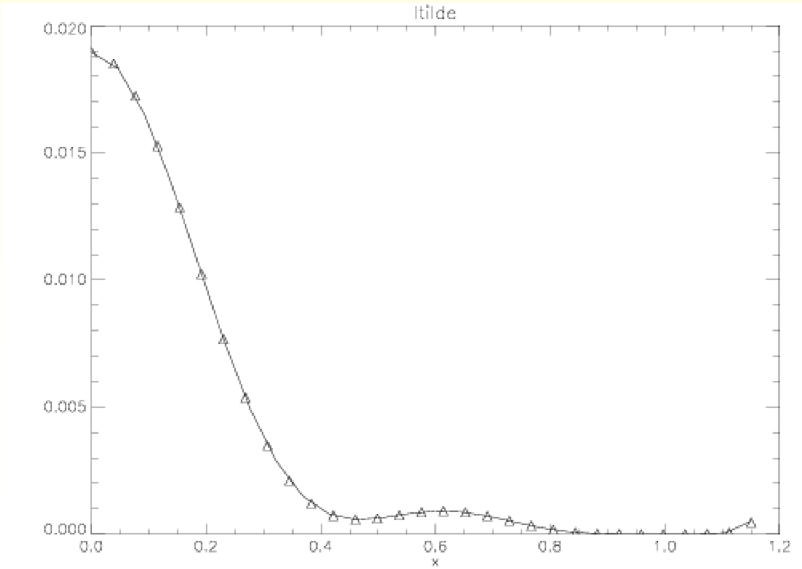
- Metric based 3+1 code, no symmetries
- MOL with 4th order space discretisation and 4th order Runge–Kutta
- Outer boundary outside $\mathscr{I}$ treated by transition zone to static state
- gauge sources: densitised lapse, shift and scalar curvature Λ
- numerically integrated generators of $\mathscr{I}$



Numerical implementations of CFE

S. Husa (2002)

- Further development with Hübner's code
- Parameter studies
- Variations in the gauge sources
- weak Brill data compared with perturbative analysis
- problems with transition zone



Numerical implementations of CFE

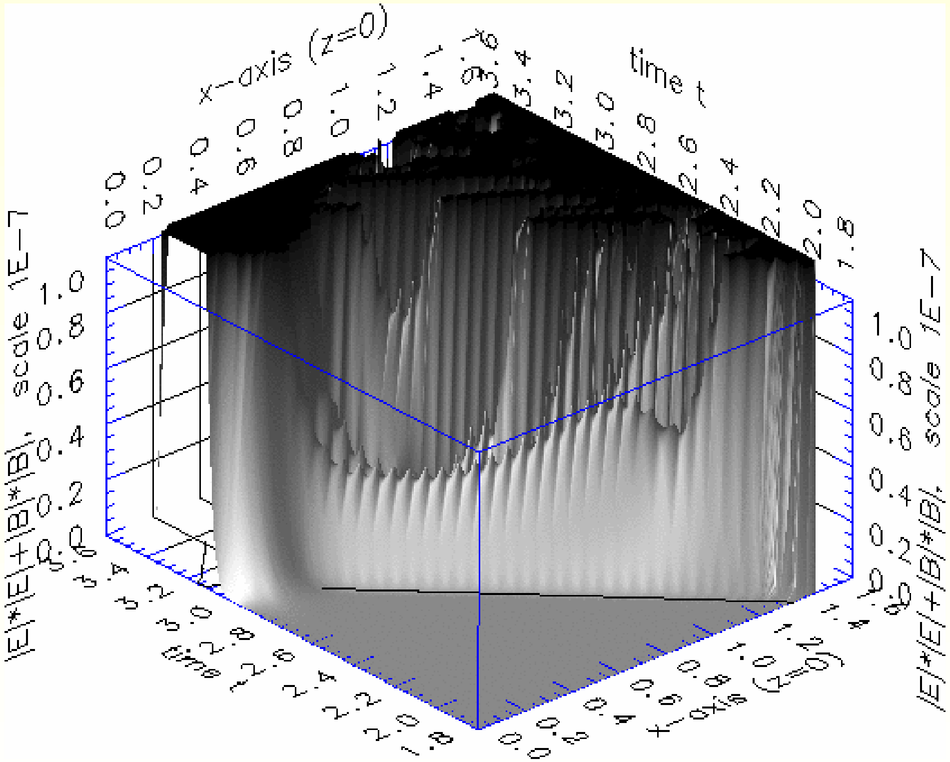
JF and M. Hein (2003)

- Frame based axi-symmetric code
- MOL with RK4 and space–discretisation using Cartoon method
- Outer boundary treated by eigen–field method
- gauge sources: F^{μ}, F_{ik}, Λ

Effect of $\mathscr{I}$ as a one-way membrane

Minkowski space-time with white noise as a boundary condition

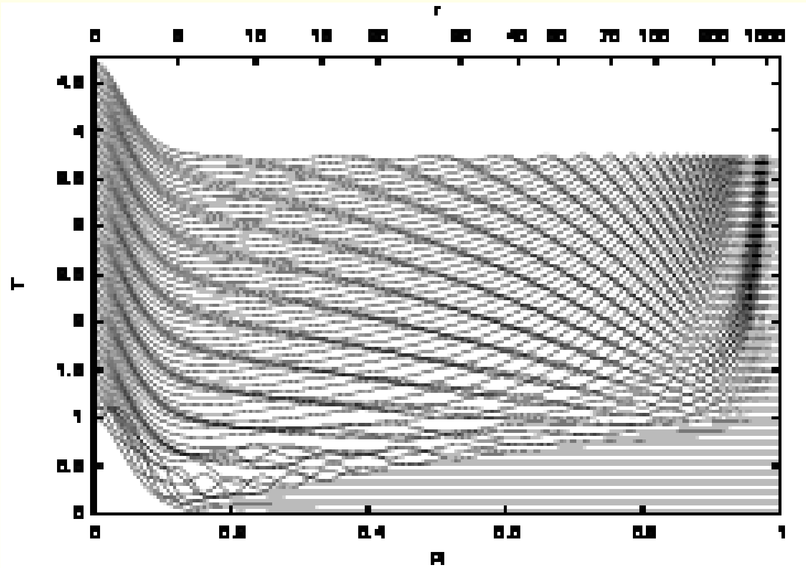
norm of the Weyl tensor

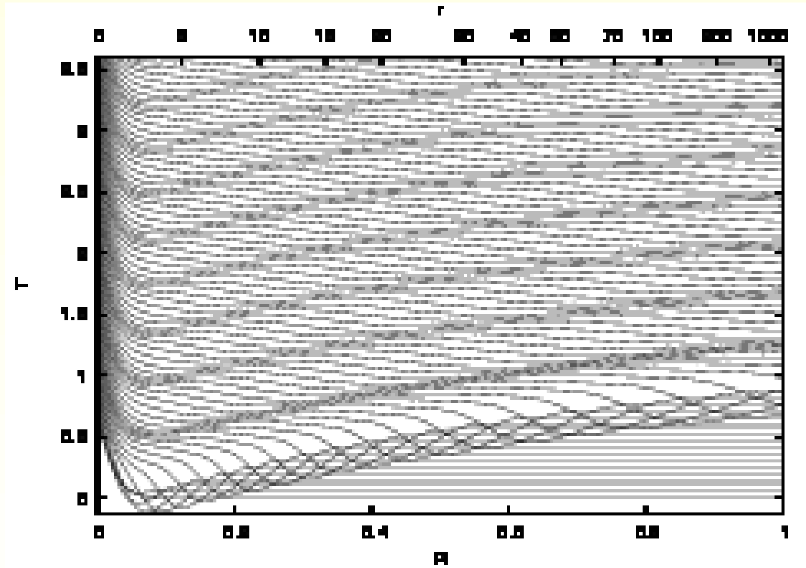


Numerical implementations of CFE

Fodor and Rácz (2005)

- Yang–Mills–Higgs system in spherical symmetry
- spherically symmetric perturbation of BPS monopole
- Conformal compactification and hperboloidal slicings
- breathing solution found
- good interaction with analytical community (Forgács, Volkov)





Other developments

- Development of 3+1 code within the Cactus framework
- Extension of Friedrich–Nagy boundary treatment to CFE
- Thin–sandwich approach for hyperboloidal initial data
- Initial data close to Kruskal data
- Perturbative analysis of constraint propagation for the Weyl system
- Preliminary work on the General CFE
- Exploration of conformal Gauss gauge on Schwarzschild and Kerr
- Need a way to solve the conformal constraints directly!