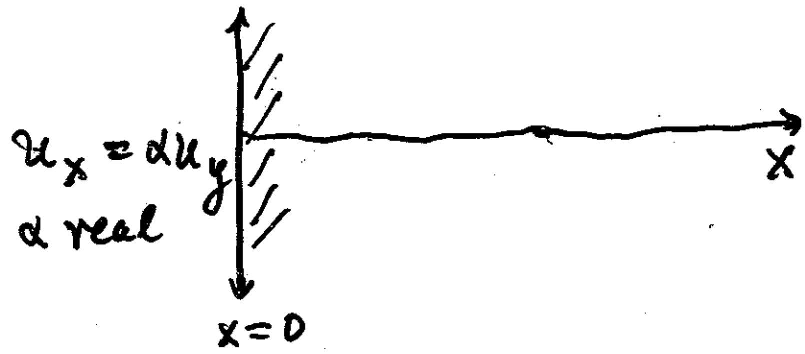
$$u_{tt} = u_{xx} + u_{yy} + 2a u_{xy}, \quad |a| < 1$$



Energy estimate

$$\frac{\partial}{\partial t}\left(\|u_t\|^2 + \|u_x\|^2 + \|u_y\|^2 + 2a(u_x, u_y)\right) = \langle u_t, u_x + a u_y \rangle$$

Energy estimate if $\alpha = -a$.

Well posed for other values of α ?

Mode analysis

$$u = e^{st + i\omega y}\hat{u}(x), \quad \mathrm{Re}\, s > 0, \quad |\hat{u}(x)|_\infty < \infty$$

Eigenvalue problem

$$(s^2 + \omega^2)\hat{u} = \hat{u}_{xx} + 2ai\omega \hat{u}_x$$

$$\hat{u}_x(0) = \alpha i \omega \hat{u}(0), \quad |\hat{u}|_\infty < \infty$$

(Fourier transform in y, Laplace transform in t)

<u>Lemma</u>. The problem is not well posed if there is an eigenvalue s with $\mathrm{Re}\, s > 0$.

<u>Proof</u> $\quad u^{(\gamma)} = e^{s\gamma t + i\omega\gamma y}\hat{u}(\gamma x)$

is a solution for any $\gamma > 0$.

General solution of (1)

$$\hat{u} = \sigma_1 e^{\varkappa_1 x} + \sigma_2 e^{\varkappa_2 x} \qquad (2)$$

$$\varkappa^2 + 2ai\omega\varkappa - (s^2+\omega^2) = 0$$

$$\varkappa_j = -ai\omega \pm \sqrt{s^2+(1-a^2)\omega^2}$$

$\mathrm{Re}\,\varkappa_1 > 0$, $\mathrm{Re}\,\varkappa_2 < 0$ for all real ω and all s, $\mathrm{Re}\,s>0$

$$\hat{u} = \sigma_2 e^{\varkappa_2 x}$$

$\hat{u}$ satisfies the boundary condition at $x=0$ if

$$\varkappa_2 = \alpha i\omega \qquad \text{no solution} \qquad (3)$$

No eigenvalue for $\mathrm{Re}\,s>0$ is only a necessary condition for well posedness.

Test1: Are there any eigensolutions of type

$$\hat{u}(x) = \sigma_2 e^{\varkappa_2 x} \quad \text{for } s=i\xi$$

($s=i\xi+\eta$, $\eta>0$, $\eta\to 0$). By (2), (3)

$$-ai\omega - \sqrt{-\xi^2+(1-a^2)\omega^2} = \alpha i\omega \qquad (4)$$

For any α, a there is a β such that $s=i\beta\omega$ solves (4)

<u>Test2</u>: strip problem

$$\hat{u} = \sigma_1 e^{\varkappa_1 x} + \sigma_2 e^{\varkappa_2 x}$$

If $\alpha \neq -a$ there are solutions

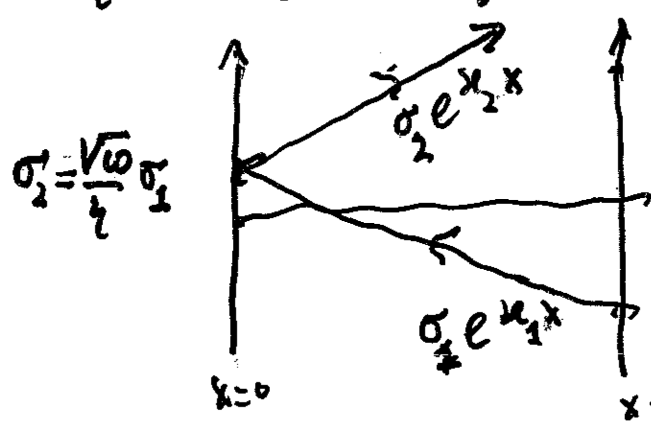
$$u = e^{st + i\omega y}\,\hat{u}(x) \qquad \mathrm{Re}\, s = \delta \log|\omega|,\ u \sim |\omega|^{t\delta}.$$

If $\alpha = -a$ energy estimate.

By (4)

$$\sqrt{-\xi^2 + (1-a^2)\omega^2} = 0 \quad (\text{double root},\ \varkappa_1 = \varkappa_2)$$

$$s = i\xi + \eta \qquad \varkappa_1 \simeq \sqrt{2i\xi\eta} \qquad \varkappa_2 \simeq -\sqrt{2i\xi\eta}\ ,\quad \xi = \sqrt{1-a^2}\,|\omega|$$



glancing wave

$\alpha = \beta_1 + i\beta_2,\ \beta_2 \neq 0,\ \beta_j$ real. Well posed for $\beta_1 = -a,\ \beta_2^2 + a^2 \leq 1$.

$$s^2 = -(\beta_1 + i\beta_2 + a)^2\omega^2 - (1-a^2)\omega^2$$

If $\beta_1 + a \neq 0$, $\mathrm{Re}\, s > \delta|\omega|$.

For $\beta_1 + a = 0$ $\quad s^2 = (\beta_2^2 + a^2 - 1)\omega^2$

$$\varkappa_j = -ai\omega \pm \sqrt{\beta_2^2\omega^2} \qquad \text{surface waves.}$$

What is well posed ?

1) Motivation

$$y_t = Ay \quad \text{i.e} \quad y(t) = e^{At} y_0$$

Assume $|e^{At}|_\infty \le K$, $s = i\xi + \eta$, $\eta > 0$

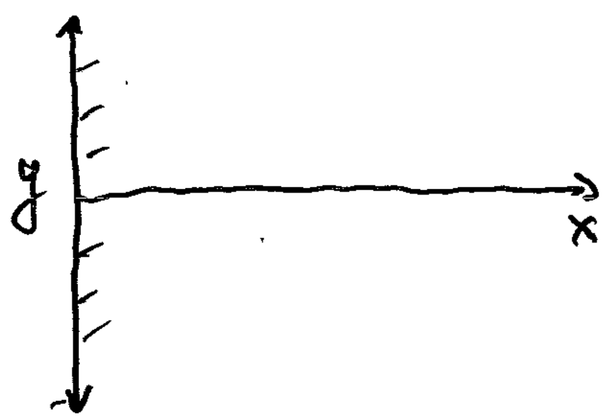
$$|(A - sI)^{-1}| = \left| \int_0^\infty e^{(A-sI)t} dt \right| \le |e^{At}|_\infty \cdot \int_0^\infty e^{-\eta t} dt \le \frac{K}{\eta}$$

Resolvent condition

$$|(A - sI)^{-1}| \le \frac{K}{\operatorname{Re} s}, \quad \operatorname{Re} s > 0.$$

$$u_t = Au_x + Bu_y + F$$

$$u(x,y,0) = 0 \qquad u^{I}(0,y,t) = S u^{II}(0,y,t)$$



Strongly hyperbolic system $A = \begin{pmatrix} -\Lambda^{I} & 0 \\ 0 & \Lambda^{II} \end{pmatrix}$, $\Lambda^{I}_{r \times 1}$

Laplace - Fourier transformation

$$(s\hat{u} - A\hat{u}_x - iB\omega)\hat{u} = \hat{F}$$

$$\hat{u}^{I}(0) = S\hat{u}^{II}(0)$$

Eigenvalue problem

$$s\varphi - A\varphi_x + iB\omega\varphi = 0 \quad \varphi^{I}(0) = S\varphi^{II}(0), \ |\varphi|_\infty < \infty$$

same properties as before: for $\operatorname{Re} s > 0$, $\operatorname{Re} \varkappa_j \neq 0$ the number of $\varkappa_j$ with $\operatorname{Re} \varkappa_j$ = number of ingoing characteristics. Ill posed if eigenvalue with $\operatorname{Re} s > 0$ exists.

If there are no eigenvalues for $\operatorname{Re} s > 0$ the resolvent equation has a unique solution.

We call the problem well posed if

$$\|\hat{u}(\cdot, \omega, s)\| \leq \frac{K}{\eta} \|\vec{F}(\cdot, \omega, s)\| + \frac{K}{\eta}\|\hat{g}(\omega, s)\|, \quad \eta = \operatorname{Re} s > 0$$

i.e.

$$\int_0^\infty e^{-2\eta t} \|u(\cdot, \cdot, t)\|^2 \leq \frac{K^2}{\eta^2} \int_0^\infty e^{-2\eta t} \|F(\cdot, \cdot, t)\|^2 dt + \frac{K^2}{\eta^2}\int_0^\infty \|g(\cdot, t)\|^2 dt$$

<u>Theorem</u> If for all ω there are no eigenvalues with $\operatorname{Re} s \geq 0$ then the problem is well posed. Necessary and sufficient.

<u>Theorem</u>. If there is an energy estimate the problem is well posed. Also stable against lower order terms. $(\eta \rightarrow \eta - \eta_0)$.

Therefore reduction to Cauchy problem and halfplan problems possible, "frozen coefficients".

The theorem is not directly useful for Neumann typ condition. $g \equiv 0$

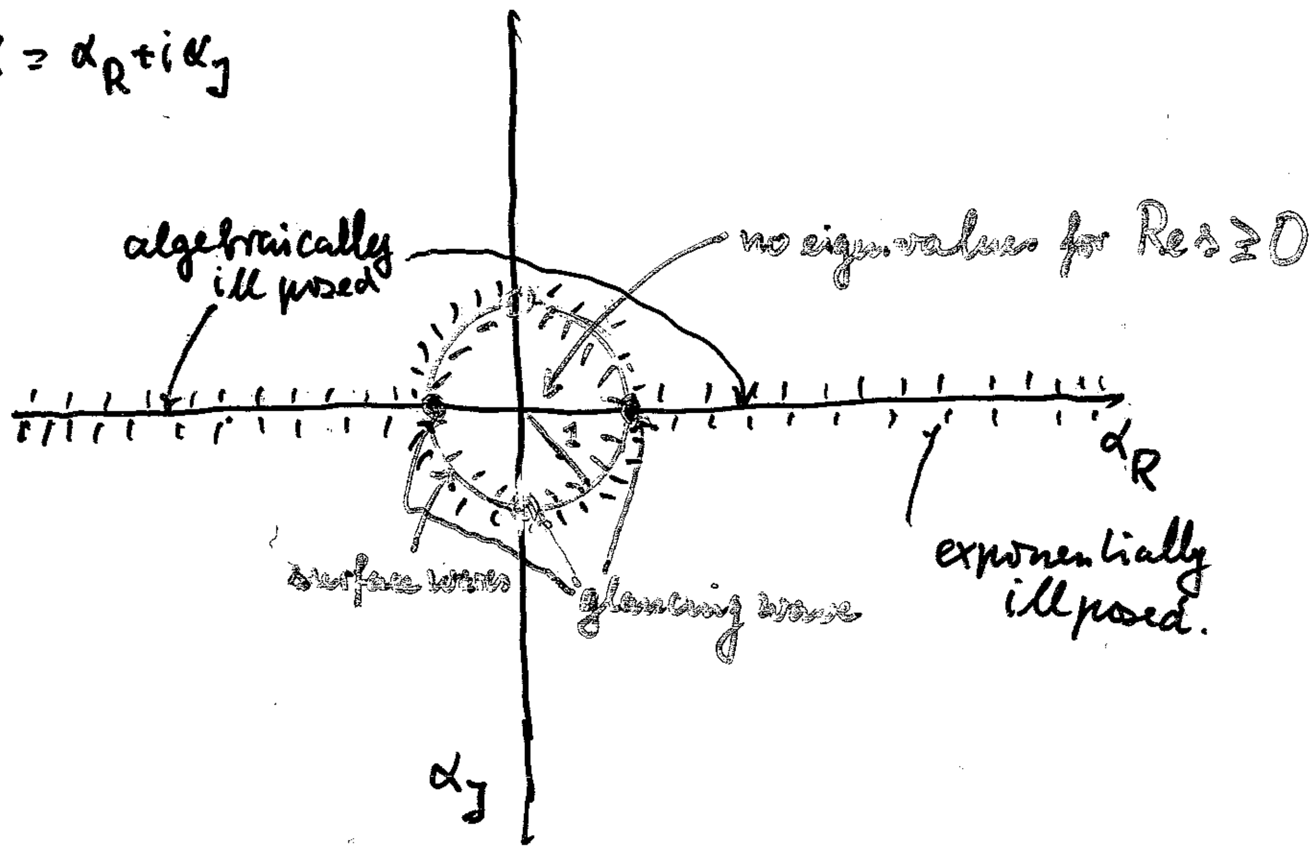
<u>Theorem</u> (Scalar equation) If the problem passes the two tests it is well posed.

Example

$$u_t = \begin{pmatrix} -1 & 0 \\ 0 & +1 \end{pmatrix} u_x + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} u_y$$

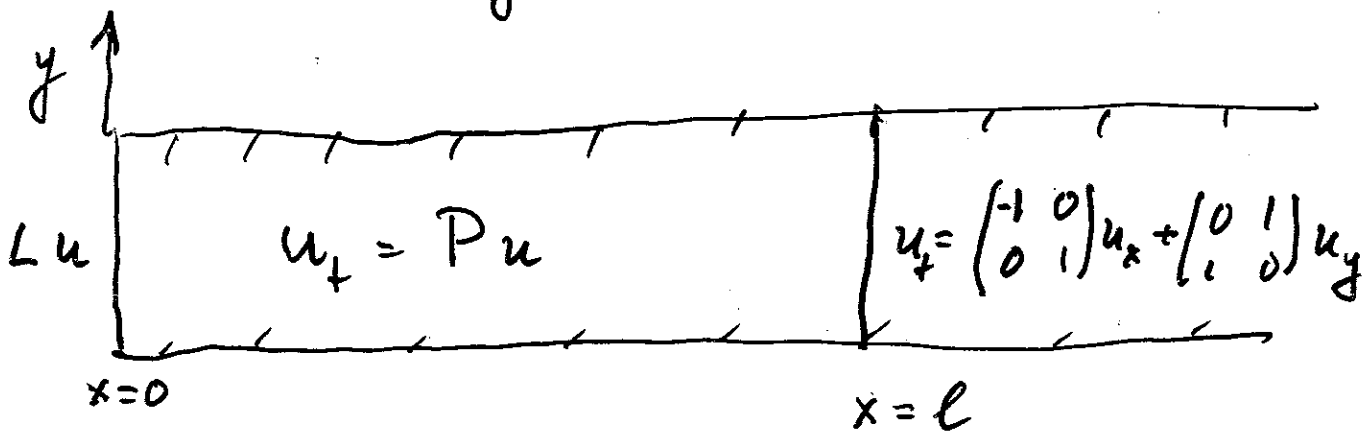
$$u^{I} = \alpha\, u^{II} \qquad \alpha \text{ complex.}$$

$\alpha = \alpha_R + i\alpha_J$



Convection dominated flow: no eigenvalue with Re s ≥ 0

Unbounded regions.



Boundary condition: at $x = l$

Laplace – Fourier transform

$$\hat{u}^{II}(l,\omega,s) = \frac{i\omega}{s+\sqrt{s^2+\omega^2}}\,\hat{u}_0^{I}(\omega,s)$$

$$|s| \gg |\omega| \;:\; s+\sqrt{s^2+\omega^2} = 2s + \frac{1}{2}\frac{\omega^2}{s} + \dots$$

Differential equation in the boundary.

Accuracy? Stability?

Rapid Numerical Implementation of Exact Radiation outer Boundary Conditions.
S.R. Lau J. Comp. Physics 199, 376–422 (2004)
Blackholes

Conformal Field equations

Work with A. Petersson and J. Nyström

3) Difference approximations for the wave equation.

$$u_{tt} = \Delta u =: u_{xx} + u_{yy}$$

$$u(\underline{x},0) = f_1 \quad u_t(\underline{x},0) = f_2$$

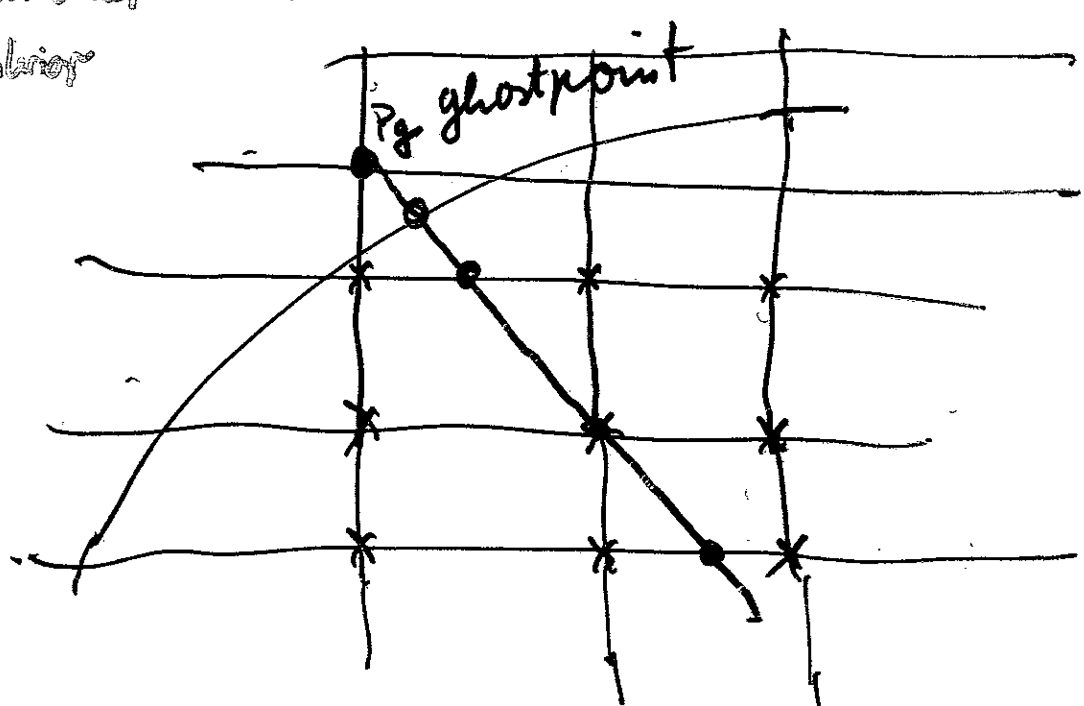
$u = 0$

or

$\partial u / \partial n = 0$ on Γ

We discuss only Neuman conditions i.e. our previous case with $\alpha = a = 0$

Standard second order methods in the interior



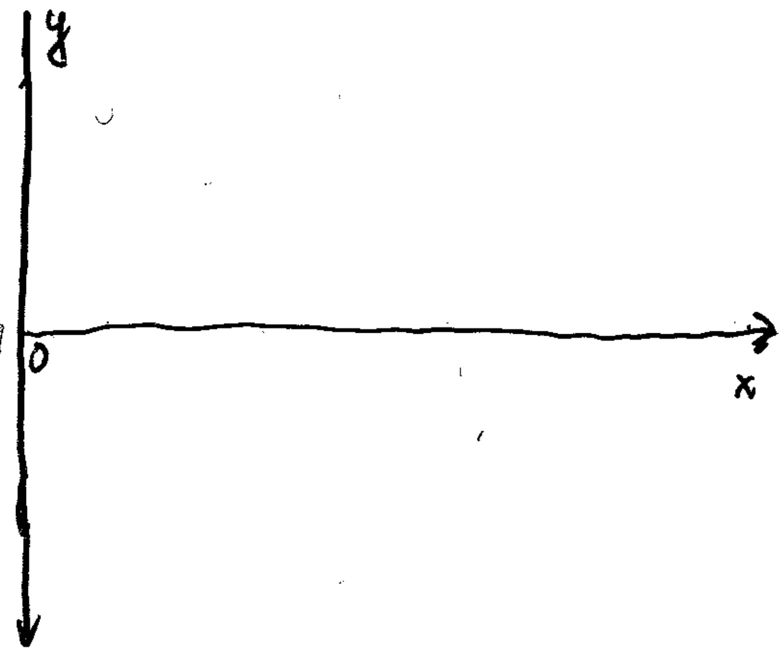
Relations between u at P_g and interior points approximate the boundary condition

Reduction to halfplan problem

Boundary condition

$u_x = \beta_1 h^2 u_{yyy} + \beta_2 h^3 u_{yyyy}$
$+ \gamma h^3 u_{yyyy}$

$\beta + \gamma > 0$



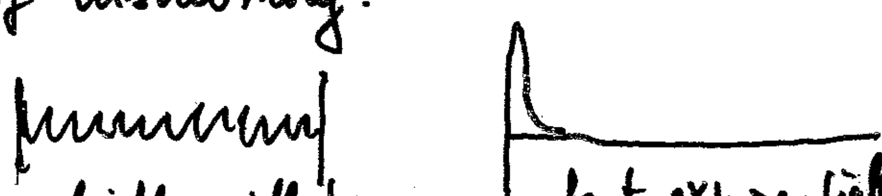
$$u_{tt} = \Delta u - \alpha h^3 \Delta(\varphi(x) \Delta u_t)$$

α:

1) If the mesh is aligned with the boundar (overlapping mesh) there is an energy estimate.

2) If the mesh is not aligned with the boundar we use the "modified equation" for analysis.

two types of instability:



Code is very robust. The amount of dissipation is very small and one calculate for long times. However, if there are corners we have to make ad h
doc decisions how to calculate tangential derivati

We eliminate the ghostpoints

$$\Delta_h u \longrightarrow A \underline{u}$$

$$\underline{u}_{tt} = A\underline{u} - \alpha h^3 A^* A \underline{u}_t$$

(Away from the boundary $A = \Delta_h$, $A^*A = \Delta_h^2$)

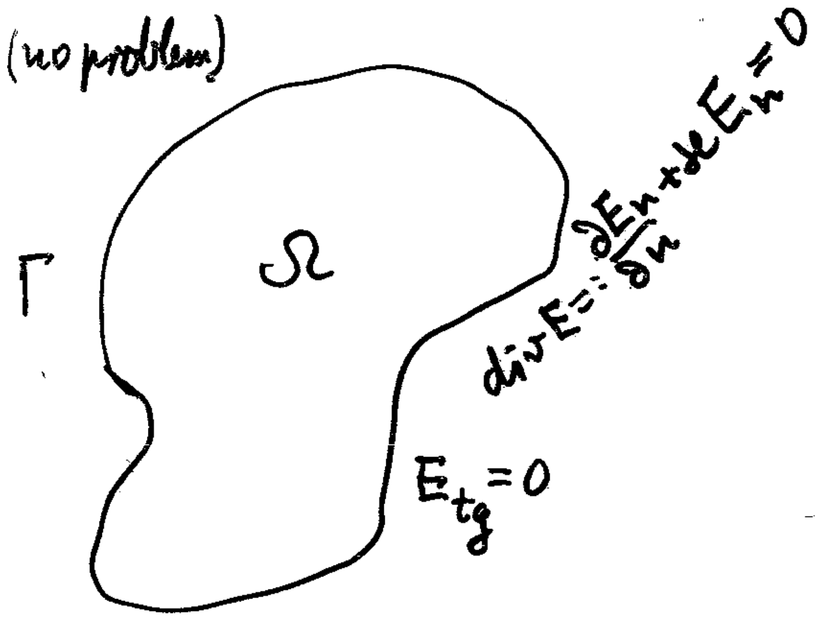
$$D_{+t}D_{-t}\underline{u}_t = A\underline{u} - \alpha h^3 A^* A D_{-t}\underline{u}$$

Higher order method away from the boundary.

Maxwell's equations.

$$E_{tt} = \Delta E, \quad E = \begin{pmatrix} E^{(x)} \\ E^{(y)} \end{pmatrix}$$

$\operatorname{div} E = 0$ (no problem)



Elastic wave equation. Model problem (halfplan)

$$u_{tt} = \Delta u + 2a u_{xy} \qquad |a| < 1, \; x \geq 0,$$

$$u_x + \alpha u_y = 0 \quad x = 0$$

Only well posed if $\alpha = a$.

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(4)

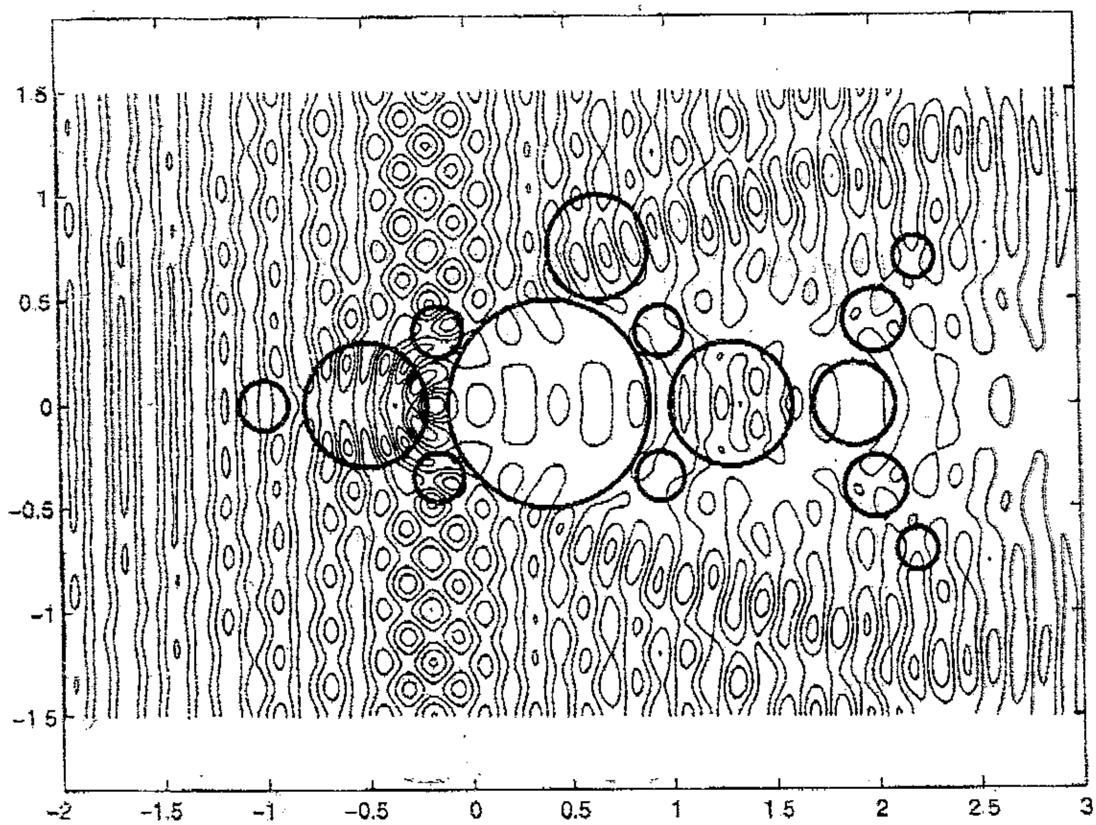

Figure 4: The solution after a plane wave has been scattered by a collection of bubbles with different wave propagation speeds. Note that the wave propagation speed in the top bubble equals the ambient speed and that the solution is symmetric around $y = 0$.