Next steps in the simulation/implementation of Einstein equations

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Overview:
disconnected pieces of (some) reality…

- Book of stories, old readings, new readings...
- Preface.
- Review chapter. What we already learned
- Book #5. Tale of a disillusion –coordinate conditions—
- Book #6. Tale of search –striving for accuracy—
- Book #7. Kid's book –and they lived happily--

Goal: obtain a robust implementation of Einstein equations

- Applicable to a large set of problems
- Independent of particular 'tweaks' *as much as possible*
- Able to yield accurate simulations (in theory & in practice)

Ingredients:

1. Formulation with good properties (sym/strongly hyperbolic) (IVP ok at continuum) [Choquet-Bruhat,Fisher-Marsden,Friedrich,Reula,York…]
   - But which formulation is better?
2. Boundary treatment (*IVBP OK*) [*Friedrich-Nagy*, Calabrese-Pullin-Reula-Sarbach-Tiglio, Winicour-Szliagyi, Scheel-Lindblom-Teukolsky….]
   - Good for max. dissipative boundary conditions, radiative BC's require additional care!
3. Initial data
   - Far from ideal scenario… but more than enough for now to testing the 'GOAL'
4. Realize/translate (1+2) to the discrete arena
5. Adopt 'useful' coordinate conditions
6. Ensure simulation is well resolved and errors are on target

Reminder…

- ‘continuum stability’ ➔ discrete stability
  - In FD, employ operators satisfying summation by parts so that

$$\int \dot{u}u \rightarrow \int u_{,x}u \rightarrow u^2 \,|_a^b \qquad\qquad \partial \rightarrow D \; s.t. \quad (u, Du) = -(Du, u) + u^2$$

  - Use these derivative operators + suitable time integrators (RK3, RK4..) + appropriate boundary conditions (Olsson method, SAT technique)

$$D_t\, u = A^i(t,x)\, D_i\, u + B^i(t,x)\, u + C(t,x)$$

    is stable, no tweaks or knobs required

  - This might still not be enough, Leibnitz rule isn’t satisfied at the discrete level… one can help out

$$u_{,t} = 2cDu \;\; "\neq" \; D(cu) \;\; + (Du) \;\, c_{,x} - (Dc) \;\, u$$

  - At the non-linear level, dissipation might be needed, dissipative operators have been constructed that do not spoil SBP

Combinations yield a strong starting point

[Calabrese,LL,Neilsen,Pullin,Reula,Sarbach,Tiglio] motivated by [Olsson,Tadmor,Kreiss,Carpenter,…]

Coordinate conditions

[Palenzuela,LL,Bona]

- A priori, requirements

Fundamental

- Artificial of coordinate singularities
  - Eg, ‘geodesic’ slicings $t_a = n_a$
- ‘symmetry-seeking’ (take advantage of possible approx symmetries)
  - Eg, adapted to the congruence defined by a Killing field
- If no symmetry, minimize the rate of change of appropriately defined geometrical quantities
  - Eg, minimal distortion of slices at t=const (York-Smarr)

Additional

- Suitable behavior near singularities
  - Eg. If no-excision singularity avoiding (Maximal slices), if excision be such that characteristics are all inflow at excision boundary $v = (\pm \alpha + \beta)$ ➔ $\alpha < \beta$
- Appropriate asymptotic behavior
- Covariantly defined if possible

God strongly suggested

- Combined with the main (evolution) eqns, whole system be at least strongly hyperbolic

Options

- Minimal distortion/strain

  Minimize action of the 'strain' of the hypersurface with respect to β

$$\Sigma_{ij} = 2\nabla_{(i}\beta_{j)} - 2\alpha K_{ij}; \tilde{\Sigma}_{ij} = \Sigma_{ij} - \lambda \Sigma^i_i$$

$$S = \int \tilde{\Sigma}_{ij}\tilde{\Sigma}^{ij} \to \delta S = 0 \to \nabla^j \tilde{\Sigma}_{ij} = 0$$

- Dynamical options:

$$\partial_t \alpha = -\alpha^2 Q$$

$$\partial_t \beta^i = -\alpha F^i$$

  – Need conditions for Q, F^i; for instance, for Harmonic coords

$$\alpha Q = -\beta^l \partial_l \ln \alpha + \alpha K^i_i$$

$$\alpha F^i = -\beta^l \partial_l \beta^i - \alpha^2 (g, \alpha, \ldots)$$

➔ Symm hyperbolic, but not symmetry seeking without source functions.

Lots of freedom…cutting to the chase..

- For lapse. Adopt…

$$\alpha Q = -a\beta^l \partial_l \ln\alpha + f(\alpha) K_i^i$$

a=0,1 (along the *t* flow or *n* flow); f=0,1,2/α (geodesic,time harmonic,1+log)

- For shift…. 'drive' the generalized minimal distortion condition

$$\partial_{tt}\beta^i = -\eta\alpha\partial_t\beta^i + g\alpha^2\nabla_k\widetilde{\Sigma}^{ki}$$

$$\Downarrow$$

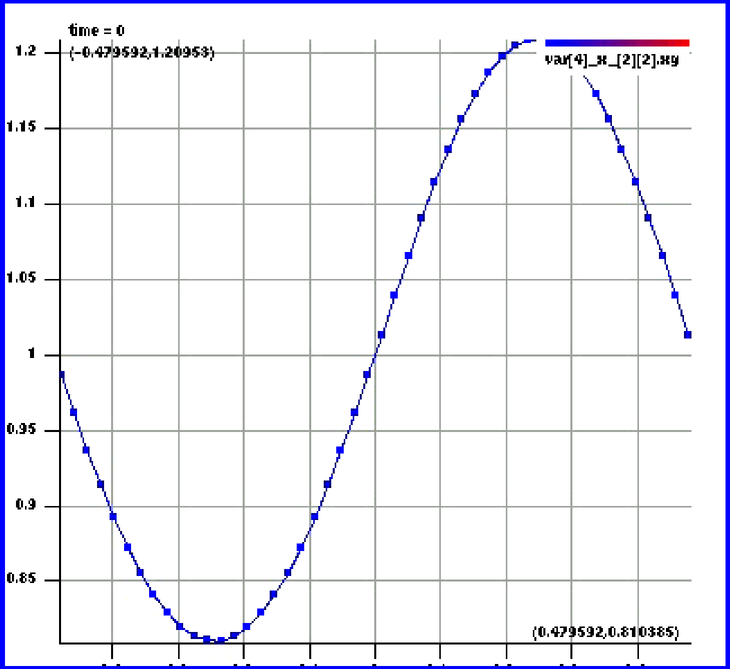
$$(\partial_t - a\beta^l\partial_l)\beta^i = -\alpha F^i \quad ; \quad (\partial_t - bL_\beta)F^i = -\eta F^i - \nabla_k(2g\alpha\widetilde{\Sigma}^{ki})$$

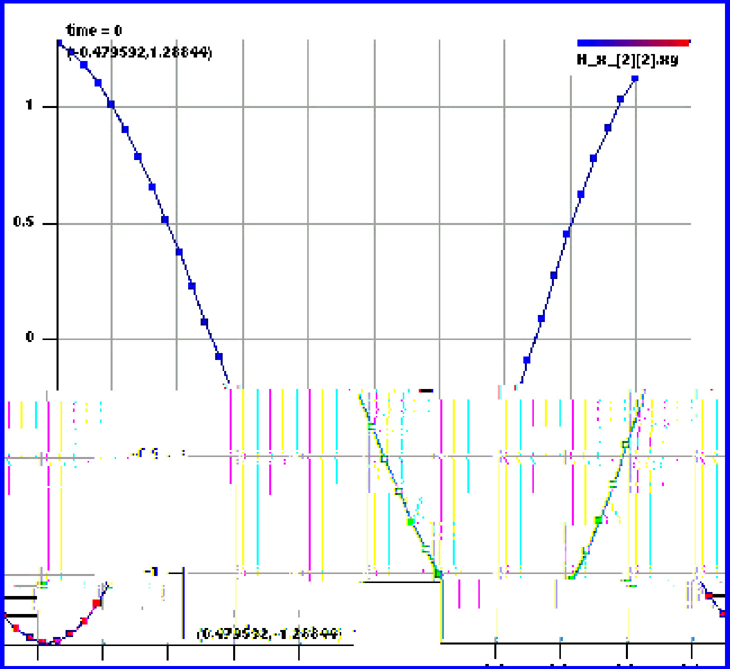
- Props, *FOR Z4 & Sarbach-Tiglio formulation and others…*
  - a=1,b=1. Strongly hyperbolic; though doesn't freeze without sources
  - a=0,b=1. Weakly hyperbolic if β = α. Freezes along the *n* flow?
  - a=0,b=0. Strongly hyperbolic, but needs boundary conditions at inner boundary, eigenvector mixes gauge & main variables. Freezes.

Good example

- GR eqns:
  - Z4 [Bona]. $R_{ab} + 2\nabla_{(a}Z_{b)} = 0$
  - & Sarbach-Tiglio formalism (R_{ab}=0 + constraints + judicial choice of pars)

- Gauge waves $ds^2 = F(t,x)(-dt^2 + dx^2) + dy^2 + dz^2$





- Not so good ones…
  - Schwarzschild with a perturbation: needed dissipation to deal with weakly hyperbolic pts.
    - During evolution hard to control that $\alpha<\beta$, when this is violated, instabilities arise.
    - Possibility… add suitable extra terms so that this condition is satisfied. But then… minimization might be broken

- Other alternatives?
  - Induce eqns from 'minimal' extended-Killing eqns, I.e. minimize

$$L = \xi_{(a;b)}\xi^{(a;b)} - 1/2(\nabla_a \xi^a)^2$$

  - Killing, Homotetic Killing and Affine Killing vectors satisfy the eqn
  - 'minimize' the Lagrangian → dynamical conditions for α,β
  - Resulting eqns are strongly hyperbolic except at $\alpha=\beta$!
  - When there is a killing timelike vector, nice evolutions. When not… could be good if 'source' terms are added.
    - Eg, in Gowdy, drive to trK.

Notes…

- How did we get in this mess?
  - Looked for a geometrical way of defining 'less spurious dynamics'. Dynamical (coupled to the main system) → elliptic equation
  - Adopted an evolution equation to obtain the solution, didn't want to hide 'under the rag' problems and stuck to the full equation.
    - → End up with options where freezing give nice conditions as long as BH's are not present.
    - → When BHs are present, either have to deal with weakly hyperbolic systems and/or need to give boundary conditions at the hole coupling main variables and gauge conditions

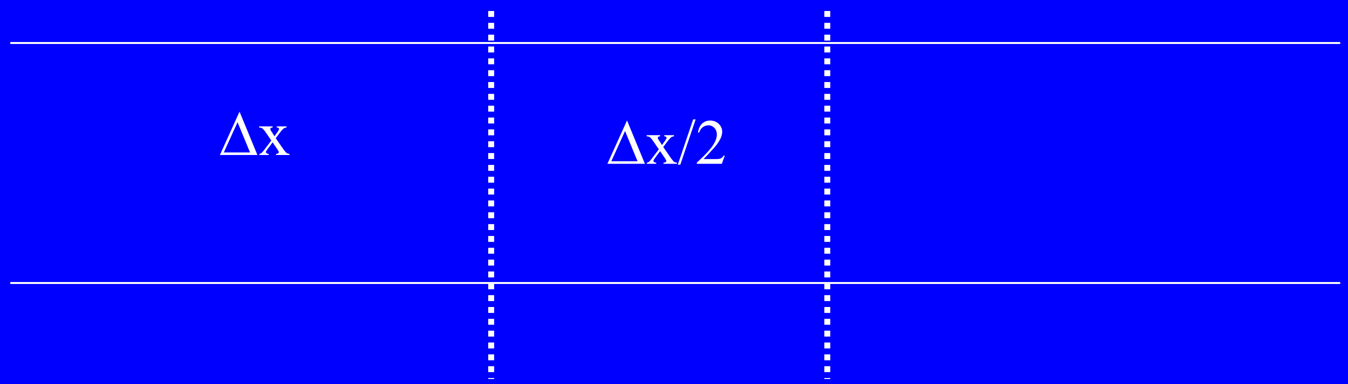
- How do we get out?
  - Stick to the elliptic problem and consider an elliptic-hyperbolic problem
    - Boundary conditions are known (e.g. Dain et.al.)
    - Elliptic gauge might be needed anyways (Sarbach's talk)
  - 'Abandon' symmetry seeking property
    - Strong/symmetric hyperbolicity ensured
    - Conditions seem to not be on the way of complicated problems (see Pretorius talk)

The search: Resolution issue [LL, Liebling, Reula]

- Need to guarantee a simulation resolves the solution accurately
  - Finer structure can develop without us being able to anticipate them
- Efficient use of (limited) resources

→ Need both adaptivity and higher accuracy operators.

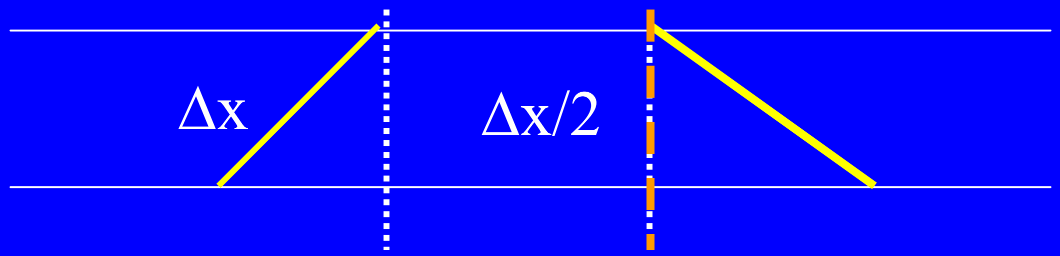
- Adaptive mesh refinement provides a way to do so.



Ground rules

1. Use what gives a stable *unigrid* evolution
2. Employ a 2-time-level scheme
3. Allow for higher order accuracy. Phase is especially delicate, LIGO needs it!!
   - Want error of ~ 1%? Need ~1%/(N_time steps) per time-step.
   - Per-wavelength, 2^{nd} order code, $\varepsilon=10^{-5} \rightarrow$ 816pts; 4^{th} order $\rightarrow$ 48pts; 6^{th} order $\rightarrow$ 19pts
   - Wavelength O(M) so…2^{nd} order off after a few 10’s of steps.. Need to go higher
   - Finer scale might develop, need to be adaptive

- What are the options?
  - Standard approach ‘Berger-Oliger’, provide boundary data by suitable interpolation at refinement boundaries to *all fields*
  - ‘Penalty approach’. Provide boundary data *only* to incoming modes via penalty method
  - Tapered boundary approach…. ‘fool-proof’ way, integrate on a larger domain and drop points affected by whatever was done at the boundary.



What is stable?

- No formal result available, intertwined evolutions so it isn't surprising.
- For lack of any better/smarter option, proceed as:
  - Employ SBP, boundary treatment + RK3/4 to guarantee stable unigrid part
  - Assess stability of parent ←→ child interaction on a model problem
  - Leave aside dissipation to begin with.

- Ingredients:
  - Child grid creation: J_{12} parent to child injection, *direct/interpolate to order (N+2)*
  - Update operator (RKs): U
  - Boundary data definition: B (involves interpolation from parent values at boundaries, if needed)
  - Parent fields redefined: J_{21}, child to parent, *direct injection*.
  - Projector operator: P = 1 within child grid region of interest, 0 otherwise

$$u_p^{n+1} = Q\, u_p^n$$

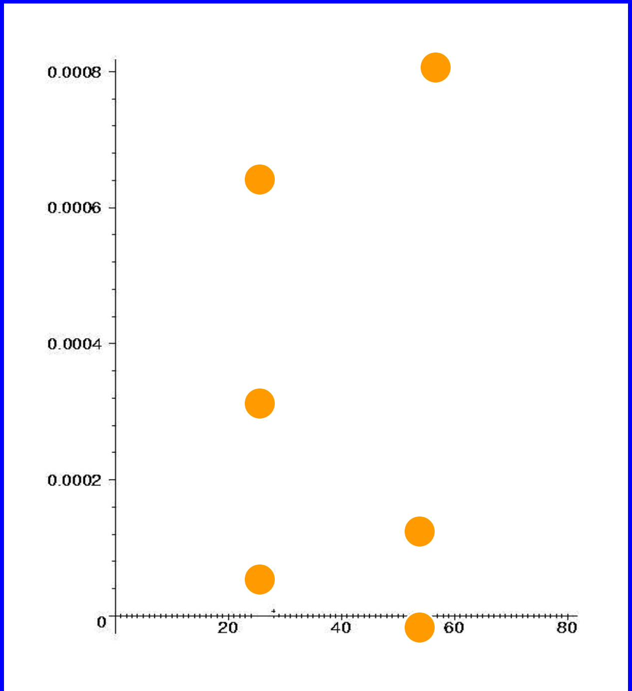
$$Q \equiv (1-P)U + P(J_{21}\; B\, U\; B\, U\; J_{12})$$

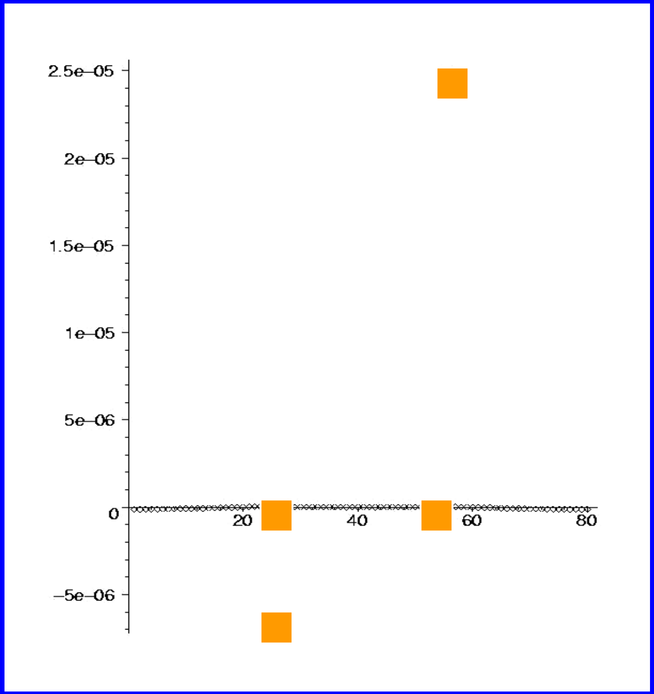
If eigenvalue(Q)>1, unstable.

Bottom line

- 2nd order, all obvious possibilities are stable. Tapered errors smaller, but not a big deal.
- 4th order
  - Standard: unstable for all obvious and not so obvious possibilities
    - Sideways derivatives, SBP derivatives, 2 ghost zones filled by interpolation (2nd or 3rd order accurate). ε
  - Penalty method: unstable for only 'allowed' possibilities (1 ghost zone filled by 2nd or 3rd order accurate interp) $\varepsilon/10$
  - Tapered boundary. Stable (obviously, and unfortunately?) $\varepsilon/100$.
- Cost assessment:
    - For target error ε, 2nd standard order vs 4th order tapered, assuming N and 2N pts used respectively. $C_2\, 3^3 \sim C_4$. ➔ 4th order 'wins' if $\varepsilon < 10^{-2}$
    - Dissipation helps… but errors are >> than those in tapered.

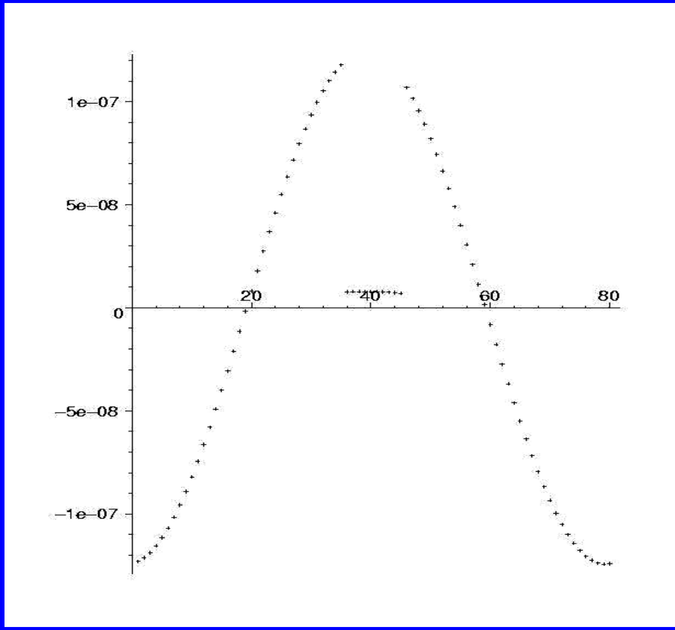
BO





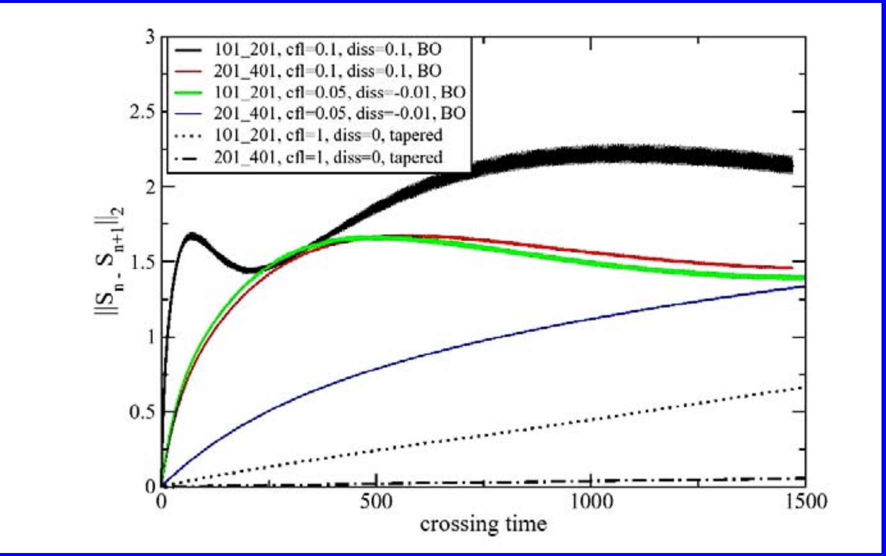
Penalty

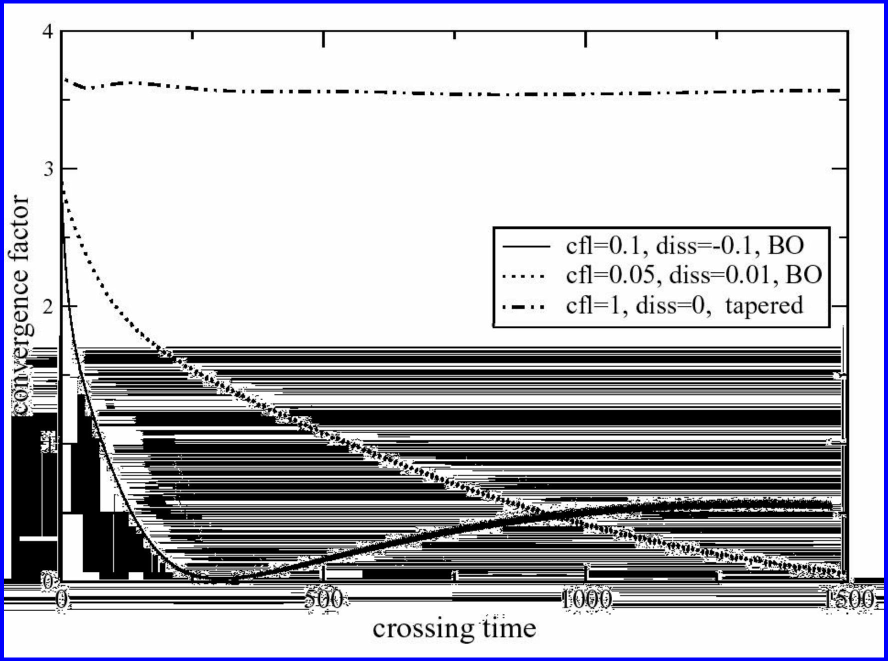
Errors for $u_{,t}=u_{,x}$ *after 1 step*



Tapered

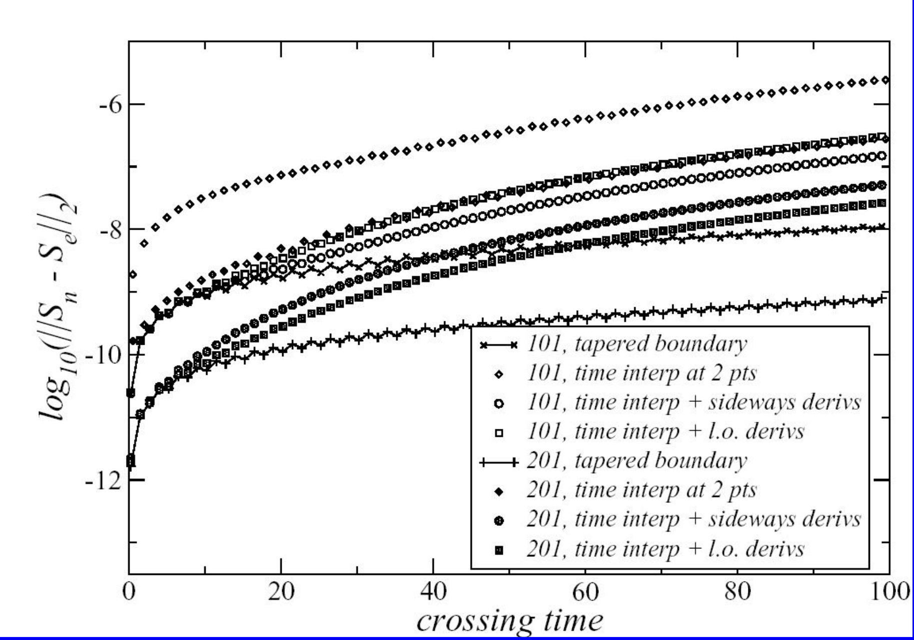
- Simple test; $u_{,t}=u_{,x}$
- Parent grid + 2 levels of refinement (fixed in place).
- Look for long term behavior with dissipation added.

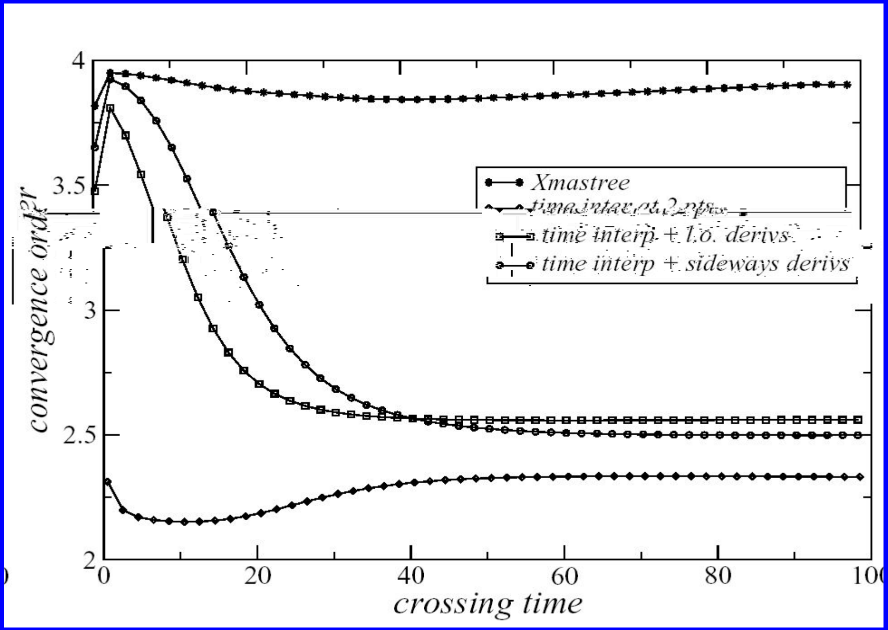




Linearized Einstein eqns over a gauge wave.

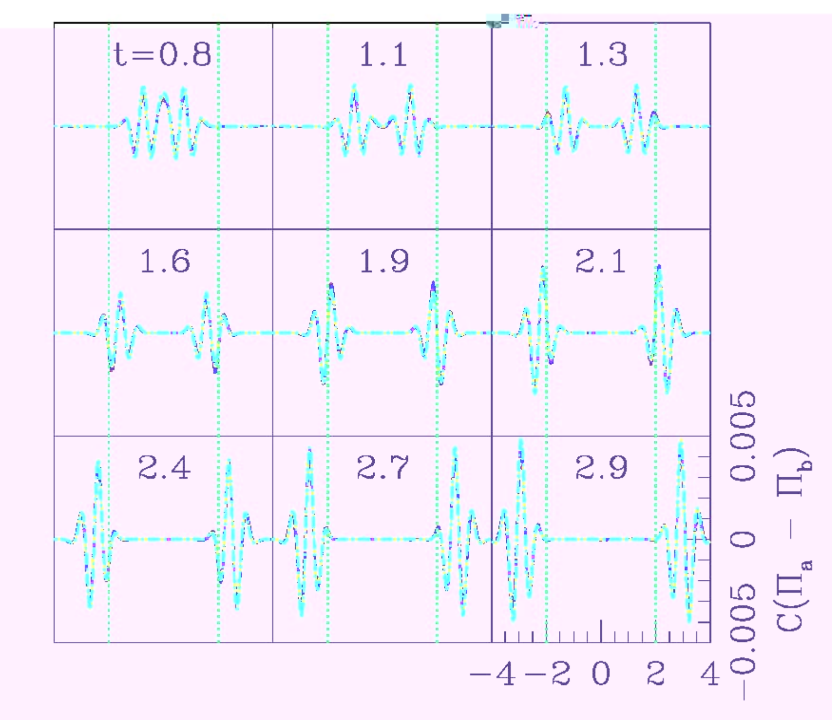
- First order in time/space symmetric hyperbolic eqn with time/space variable coefficients.
- Domain [-1,1], with base grid + 2 child grids [-.5/n,.5/n]; n=1,2.



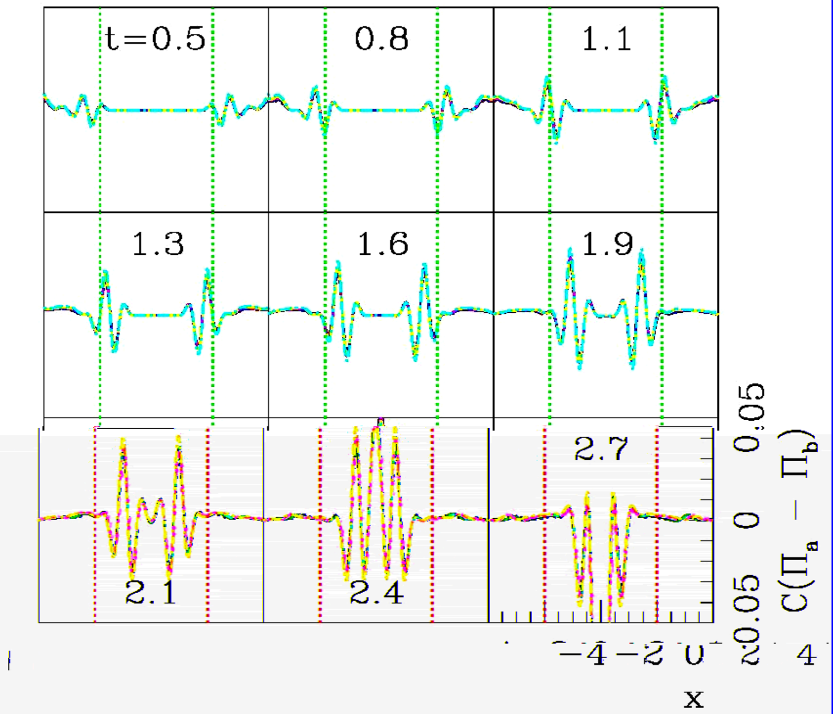


- <u>Non-linear wave eqn</u>
- $\Box u = k\, u^p$
- Reduced to first order in time & space.
- Implemented RK3, 4th order derivative.
- Solution converges to 3rd order
- For p=7, singularity forms

k=0

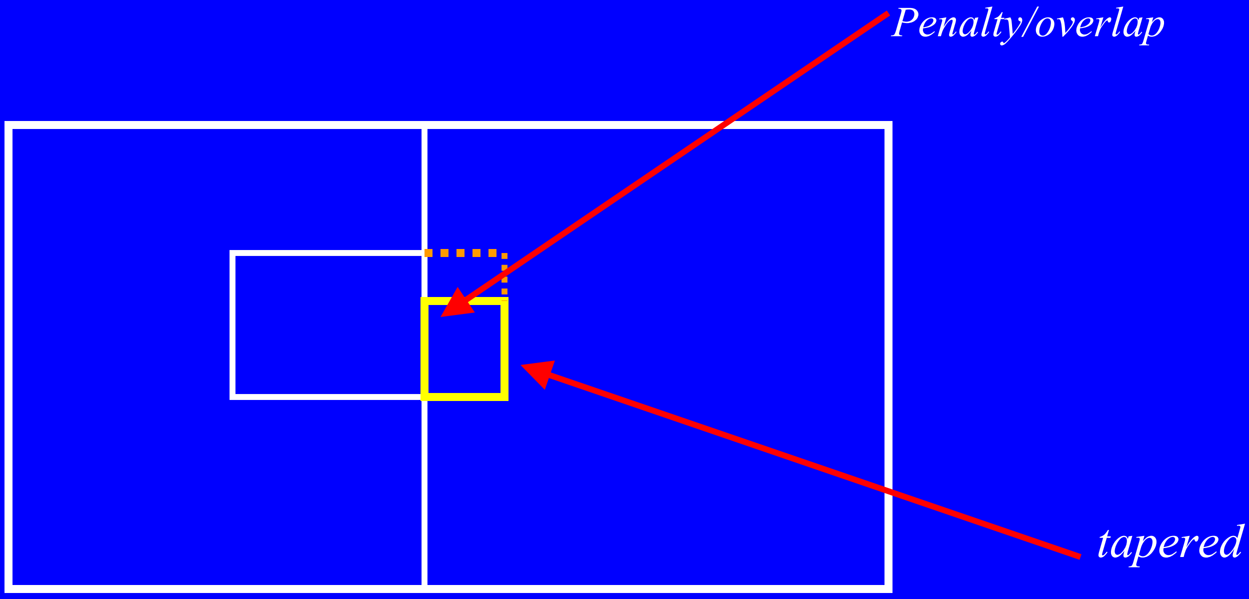


k=1



Notes (2)

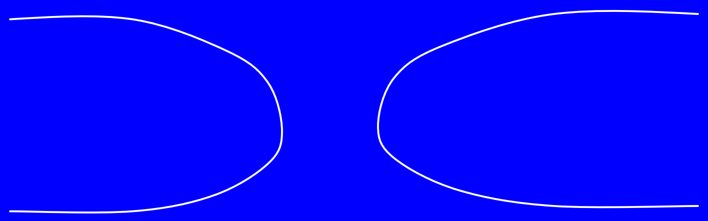
- For higher order AMR, tapered approach is a (the?) viable way
  - Extra cost involved offset by gains in accuracy, reflections and CFL allowed
- Is this the only way? Perhaps (hopefully) not, but certainly will required more involved options than those considered
- Easy to implement… in fact… it should be there already in all codes
- For multiple grids (Reula's talk), should be no problem.



Bubble spacetimes. A 'closed' story [Sarbach,LL]

Positive mass thm (Witten) requires existence of certain (asymptotically constant) spinors. In 5d Kaluza-Klein theory (asymptotically R^3xS^1) these spinors are not guaranteed.

1. Are there negative mass configurations?
2. Is cosmic censorship valid?



- Answer to 1. Yes, negative mass configurations found
  - Witten bubble (82): associated with instability of KK vacuum. More than 1 state with zero total energy.
  - Brill-Pfister (89): explicit solutions to 5D vacuum constraints with negative mass.
  - Brill-Horowitz (91): generalization to include 'gauge' fields.

- Qn: What's the space-time like?
  - Corley-Jacobson (94). Analyze area of the bubble, conclusion: It starts out expanding [collapsing], if this trend continues, unlikely to form a singularity.
    - Conjecture: It will keep expanding [collapsing] out (otherwise go through another moment of time symmetry).

- But….. This only from estimates at the initial hypersurface… what does really happen?… Need to solve the eqns…
  - Numerical effort (2000). Conclusion: negative mass bubbles expand but not forever…. At some point a naked singularity appears!!! (or does it?)

Revisiting the problem

- Consider: $ds^2 = -dt^2 + U(r)dz^2 + U(r)^{-1}dr^2 + r^2 d\Omega^2$

With *U(r) greater or =0 (*for $r=r_+$*)* a smooth function (*U*→*1* asymptotically)

Bubble is at r_+.

Electrovac case, consider $A_u dx^u = k(r_+^{-n} - r^{-n}) \quad k \in R; n = 2,3....$

Time symmetry (mom const =0); Hamiltonian constraint

$$U(r) = 1 - m/r + b/r^2 + \tilde{k}^2/r^{2n}$$

With *m,b* constants. In particular $M_{ADM}=m/4$*...but this* *can be negative*

Initial acceleration of the bubble's area [extending Corley-Jacobson]

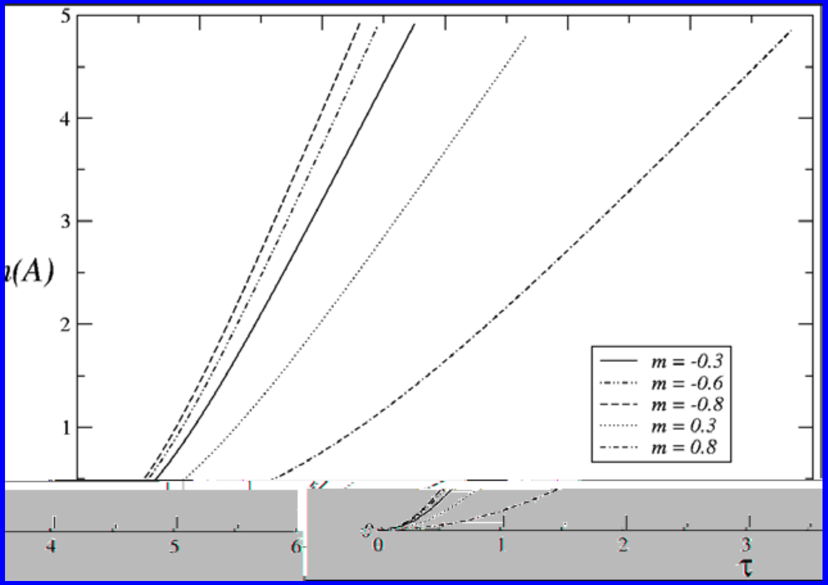
$$\ddot{A} \propto \left(1 - m - k[n-1][n-2]\right)$$

- *n=2. If m<0, bubble expands; m>0 both cases possible*
- *n>2. For k large, arbitrary negative acceleration with negative mass...sounds promising!*

Numerical evolution

- Variables functions of *(t,r)* only (1D evolution)
- At outer boundary: constraint preserving boundary conditions [a-la Calabrese,LL,Tiglio 02]
- At bubble, regularity conditions used.

- Proved well posedness at continuum level, translated to the numerical arena thanks to SBP in a first order formulation.
    - Improved resolution at bubble with a non-uniform radial coordinate.



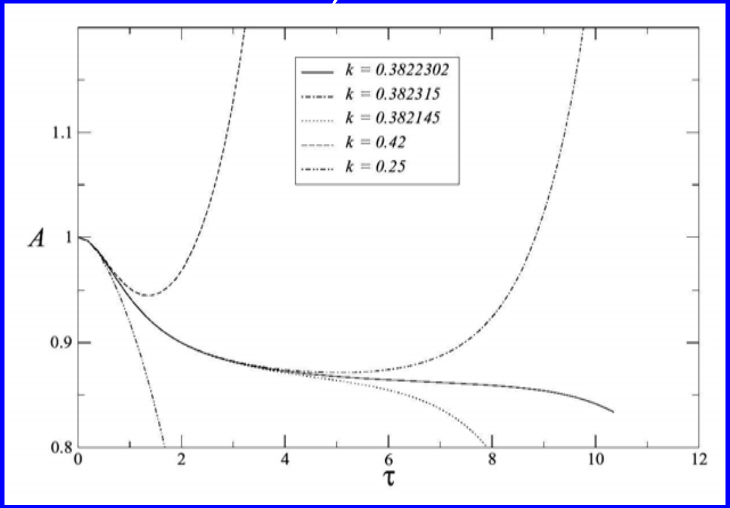
k=0

Case studied numerically previously, no naked singularity found, m<0 expands even faster than m>0

More than we asked for…

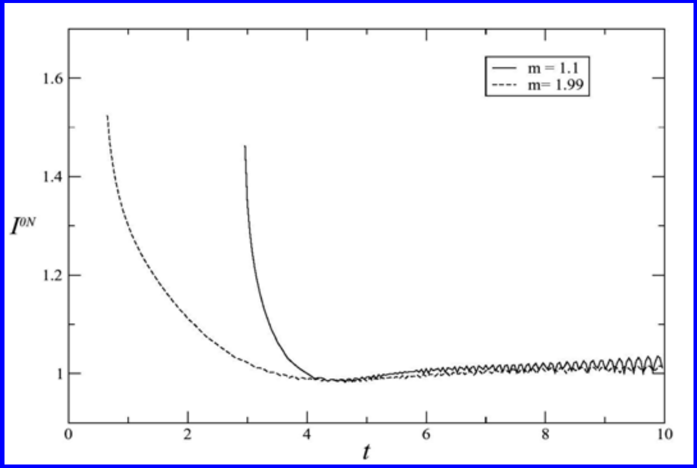
What happens with a non-zero gauge field?

• Choose $n=2$, and stick to cases where bubble starts out collapsing (positive mass)



Depending on field strength, the bubble either collapses ($k<k^*$) *to a black string* or bounces back to expand ($k>k^*$).

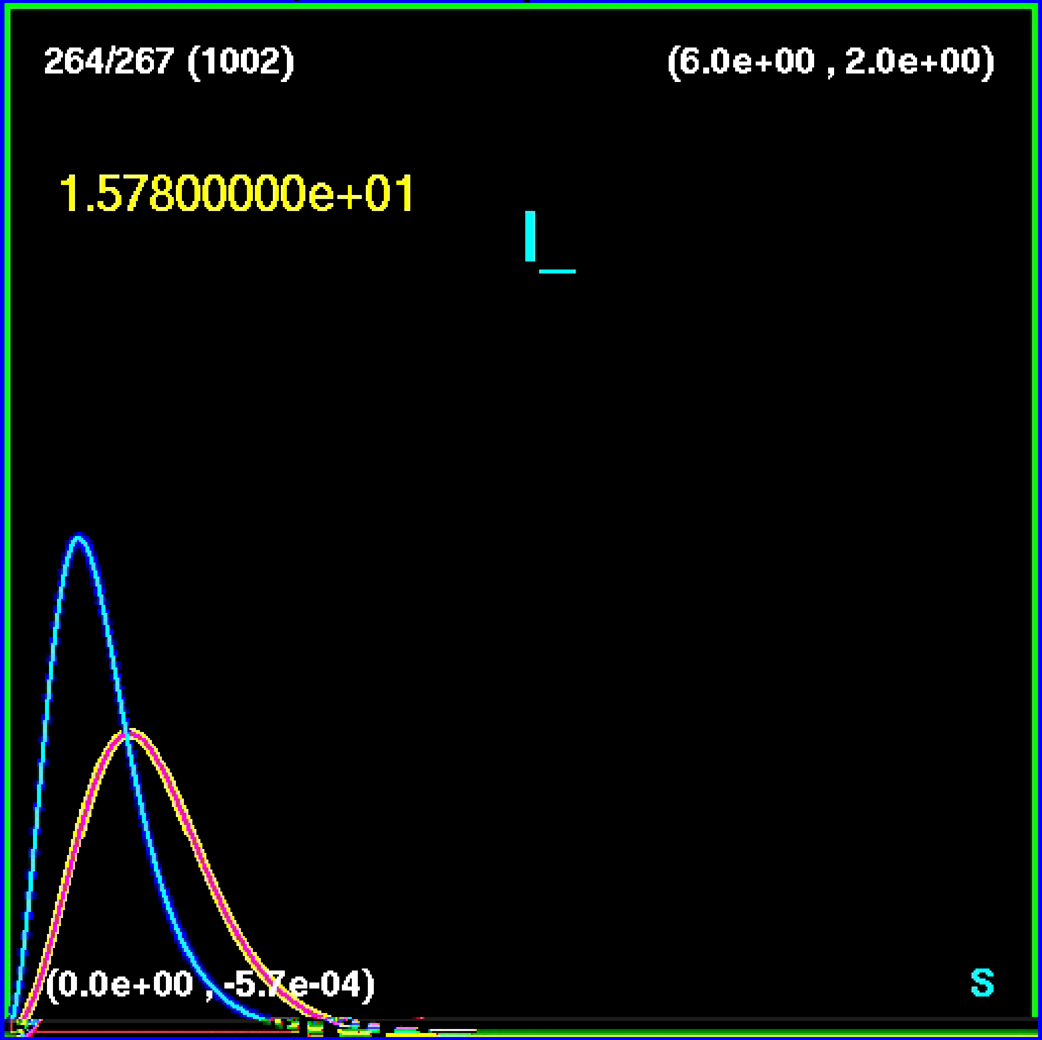
$$\Delta T \xrightarrow[k\to k^*]{} \infty$$



Changes behavior almost always without going through another moment of time symmetry

Last… it appears to approach a stationary solution… does it exist?

Curvature invariant, sub/supra critical behavior



Observation… there must be a static solution at the threshold

- Put static anzats, solve resulting constraint and…

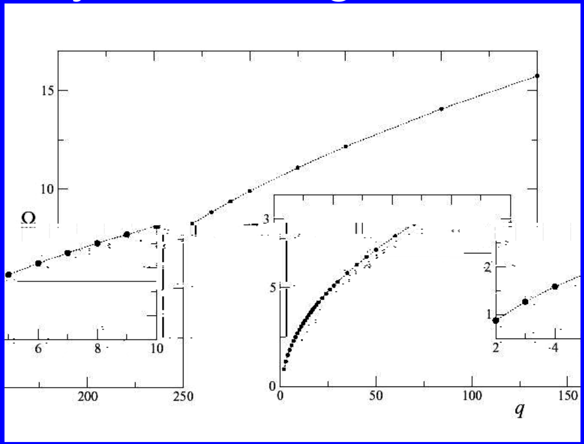
$$ds^2 = -V(r)dt^2 + V(r)/U(r)dr^2 + U(r)/V(r)^2 dz^2 + r^2 V(r) dz^2$$

$$A_a dx^a = \pm\frac{1}{2}\sqrt{3(r_+/r_- - 1)}dz/V(r)$$

With $V=(1-r_-/r)$; $U=(1-r_+/r)$. And the parameters are obtained from

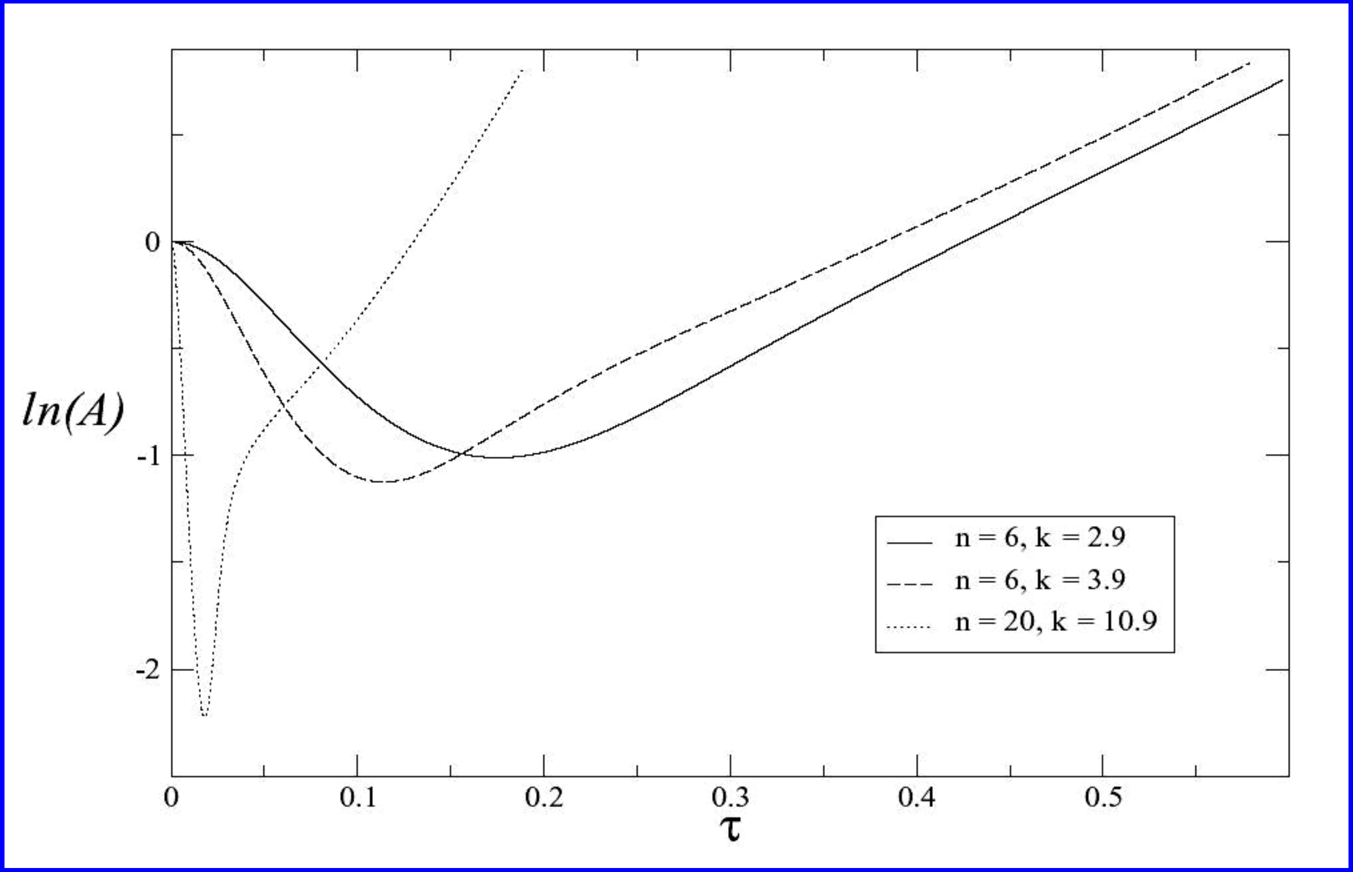
$$P=4\pi r_+(1-r_-/r_+)^{(3/2)} \text{ and } M=r_+/4.$$

- New solution?… nah… obtained by 'just' analytically continuing that of a charged black string….[found in Horowitz-Maeda 03]
  - Analyzed perturbation off this solution:
    - Obtained single growing mode



  - Work analytical continuation 'backwards'. Used to show a family of charged black strings becomes <u>*more unstable*</u> as charged is added (opposite to what was conjectured)

- What happened with the negative mass data that started contracting with arbitrary negative acceleration?
  - Bubble shrinks to arbitrarily small sizes, but ‘bounces’ back… cosmic censorship stood its ground



And they lived happily ever after the end…