Implementing High Order Touching Grids in Numerical Relativity

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Collaborators

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Introduction/Motivation

- Want to solve a problem with very different time and space scales.
- Need to integrate for a long period of time to get to the physical situation we want to model.
- And cover a good size of space in order to extract the wave signal and minimize boundary effects.
- The solutions we are seeking have singularities.

Long time integration

- We need to integrate until we get rid of all spurious initial radiation.
- We need to integrate further until the radiation gets far enough from the mergers and into the far zone.
- We need very stable formulations
- We need very stable codes
- We need time integrators which do not dissipate too much of the solution

Big size in space

- Need to get far from the sources so that we reach the wave zone out of which we can be confident that the field perturbations would really move as waves and reach infinity without much distortion.
- Need to propagate the waves without much distortion.
- Need high accuracy difference operators, at least fourth order.
- Need FMR/AMR of higher order.

Singularities

- We expect singularities to form or even we start the simulation with singularities on the initial data.
- Numerically we can not handle singularities.
- We need to excise the region where the singularities are. [Punctures?].

Singularities

- Need to be able to handle integration regions which do not have trivial topology, S^2 x R, etc.
- Need appropriate difference operators. (SBP)
- Need to work with several grid patches at the same time. [Touching vs. Multi]

Introduction/Motivation

- Want to deliver very accurate and robust long time solutions. Where we can estimate the errors.
- Want to be able to solve this problems in reasonable time so that we can fill out the template space.
- Want to proceed on firm ground, solving first model problems, and using sound and proven methods.

Theory: Method of lines

- It means we discretize in space and then treat the discretized system as a system of O.D.E.'s and solve it.
- If the semi-discrete system is stable, then if solved with some (not every) standard method like Runge-Kuta third of fourth order, then the evolution is stable and convergent.

Theory: Method of lines

- You only have to check stability of the semi-discrete system.
- In implementations you can have at your disposal several O.D.E.'s integrators.
- You can treat the space discretization in a more modular way. (have several difference operators available)

Theory: SBP

$$\partial u = \partial \; u \qquad u(t, x = 1) = g(t), u(t = 0, x) = f(x)$$

$$\mathrm{E} = \int u^2 dx$$

$$\frac{\mathrm{E}}{\quad} = 2\int \;\; \partial_t \qquad = 2\int \;\; \partial_x \qquad = \int \partial_x(\;\; ^2)$$

$$= u^2(t, x = 1) - u^2(t, x = 0) \le g^2(t)$$

Theory: SBP

$$E(t) \le E(0) + \int_0^t g^2(s)\,ds$$

$$\le \int_0^1 f^2(x)\,dx + \int_0^t g^2(s)\,ds$$

This gives existence and uniqueness

Theory: SBP

$$\partial_t U = A^i(U)\partial_i U + B(U)$$

$$U(t=0,x) = F(x)$$

$$P_0^+ U(t,x=0) = G_0(t)$$

$$P_1^+ U(t,x=1) = G_1(t)$$

$$\mathrm{E} = \int < U, HU > dx$$

$$\frac{d\mathrm{E}}{dt} = \int [< U, HA^i \partial_i U > + < A^i \partial_i U, HU >] dx$$

symmetry

$$= \int [\partial_i < U, HA^i U > - < U, \partial_i (HA^i) U >]$$

Leibniz

$$= \oint < U, HA^i n_i U > ds - \int < U, \partial_i (HA^i) U > dx$$

$$\leq \| \partial_i (HA^i) \|_\infty \mathrm{E} + \oint < U, \lambda_i HP_i^+ U > ds$$

$$\leq \| \partial_i (HA^i) \|_\infty \mathrm{E} + \oint < G, \lambda_i HP_i^+ G > ds$$

Theory: SBP

$$(v,v)_\Sigma = \sum_{i=1}^{N} \sigma_i v_i\, v_i$$

$$(v,Dv)_\Sigma + (Dv,v)_\Sigma = v_N{}^2 - v_1{}^2$$

- SBP depends on the region and the order of accuracy of the difference operator
- D and Σ are given as a pair
- Pairs (D, Σ) are known for each order for squared grids *
- Some pairs are also known for grids with missing squares
- 2-1, 4-2, 6-3, 8-4, 4-3, 6-5, 8-7

Theory: SBP

[Kreiss and Scherer '74, Strand '94, Mattsson, Svard and Nordstrom 2004]

$$(u,v)_\Sigma = \Delta x \sum_{i,j} < u_i, v_j > \sigma_{ij}$$

•Diagonal and full restricted norms.

The norm is diagonal if $\sigma_{ij} = \sigma_i \delta_{ij}$, full restricted if $\sigma_{0i} = 0 \quad for \quad i \neq 0$

•
In the diagonal (full restricted) case, the order of the derivative is 2n in the interior and n (2n-1) at and close to boundaries.

•There are some issues in the non-diagonal case.

•Derivatives with minimum bandwith are not necessarily the optimal ones, as they might have a large spectral radius associated -> severe restrictions on the Courant limit.

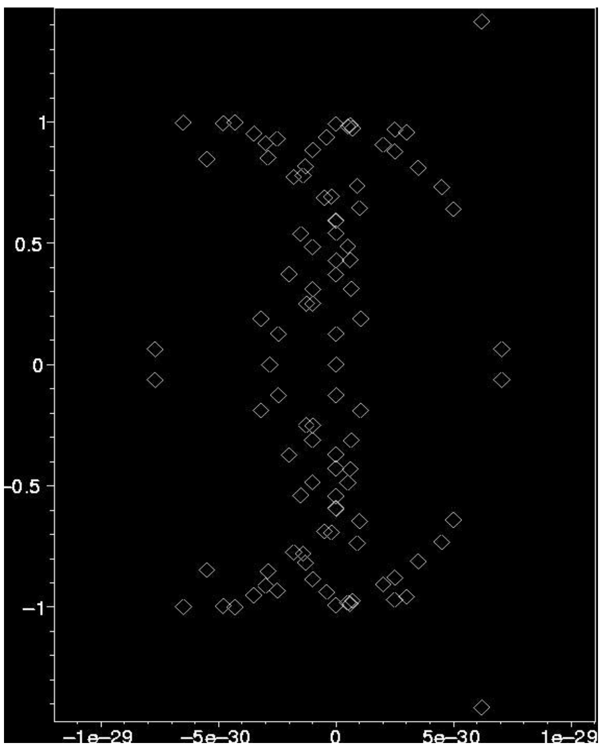
•Inventory of high order derivatives we have analyzed/whose spectral radius we have "minimized" (notation: order in the interior – order at and close to boundaries):

* 2-1, 4-2, 6-3, 8-4 (diagonal case)
* 4-3, 6-5, 8-7 (full restricted case)

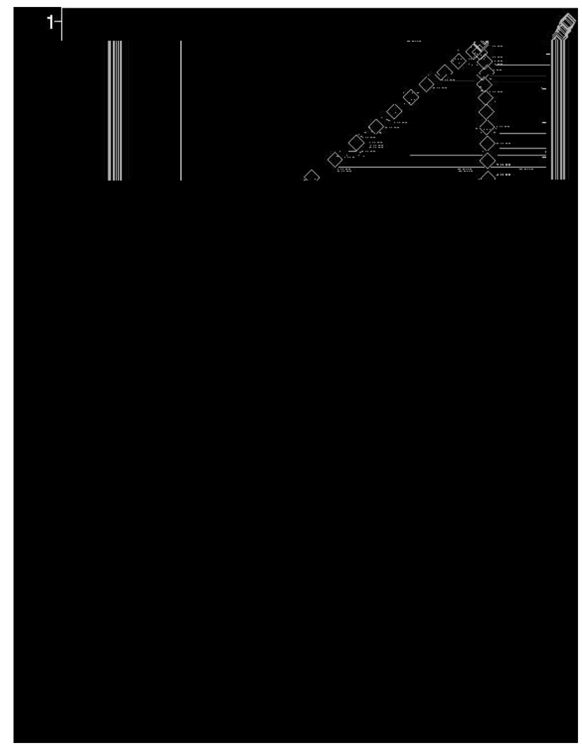
•Dissipations: need to be non-positive definite with respect to the SBP scalar product. Mattsson's solution: a prescription for all norms.

Theory: SBP

Second order case. Maximum = 1.414

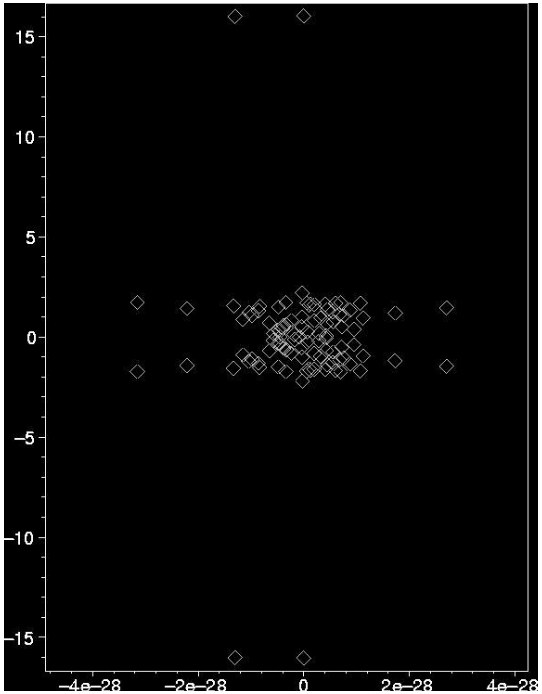


Periodic

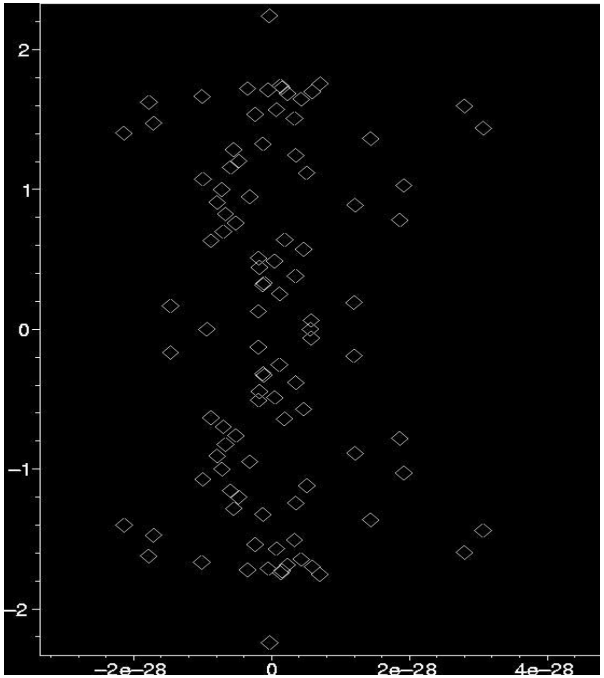


Boundary

Theory: SBP case D8-4

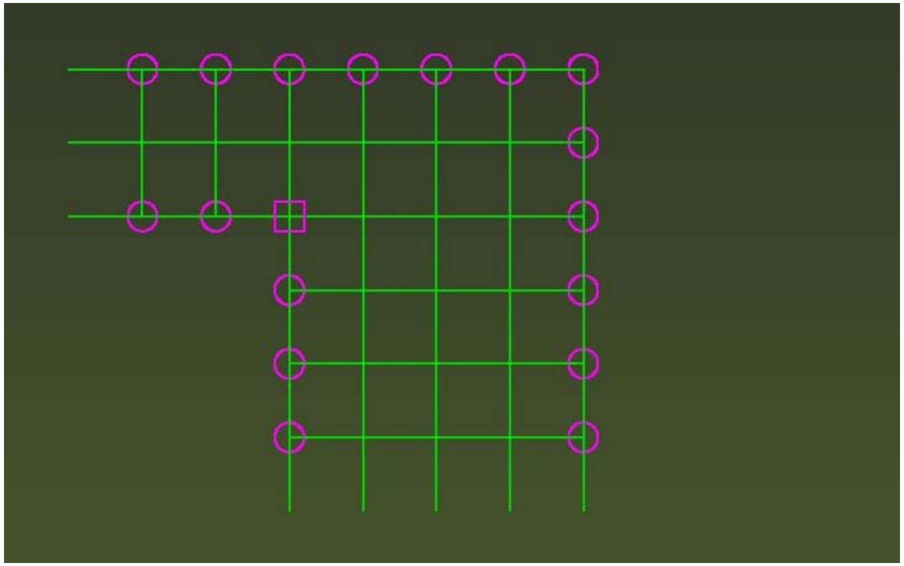


Minimum bandwith operator:Maximum 16.04!

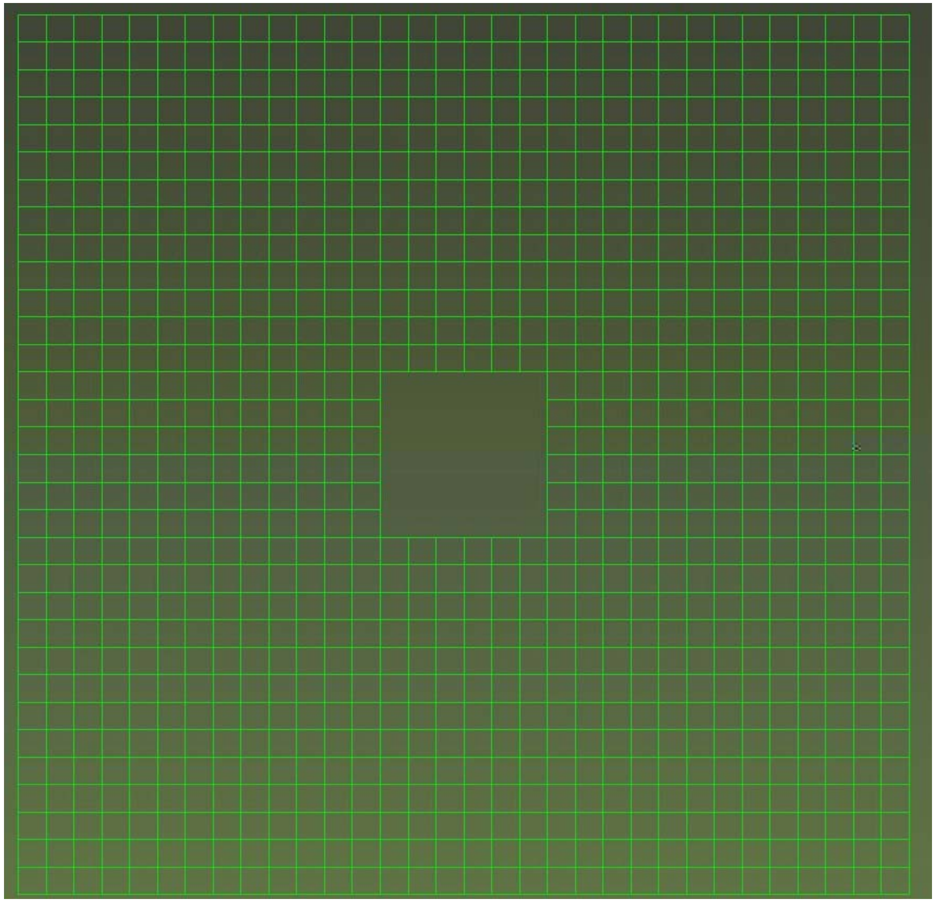


Optimized operator: Maximum 2.242

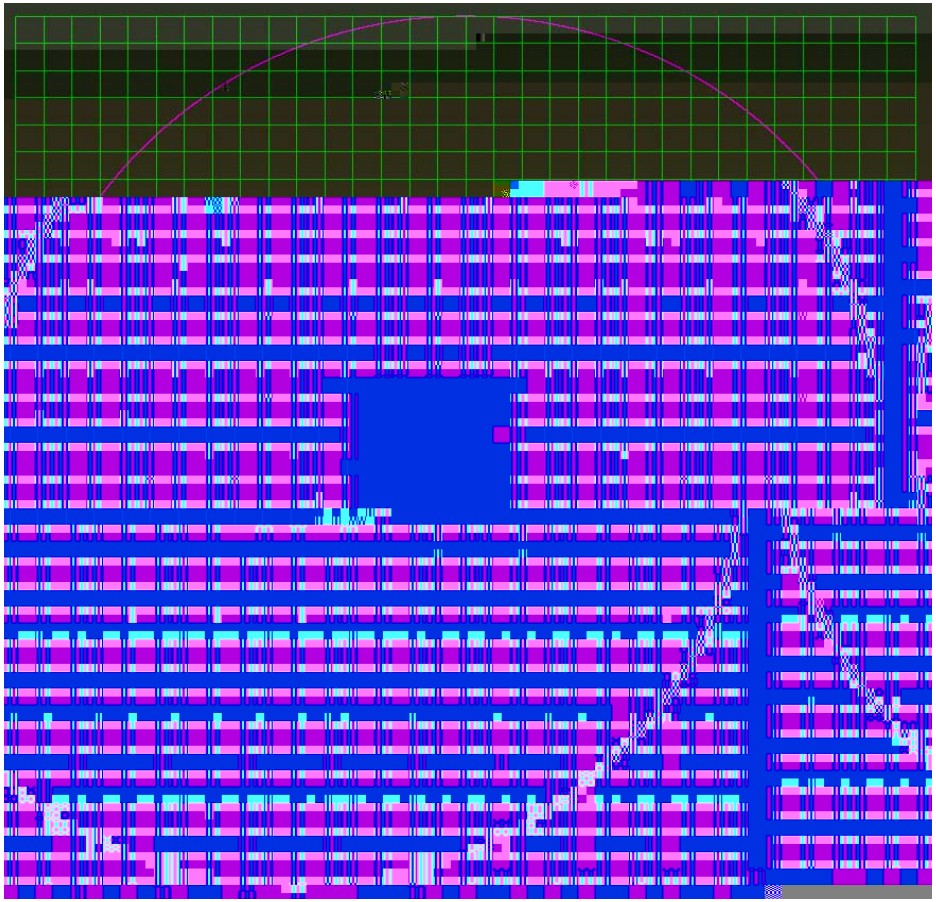
Implementation: SBP



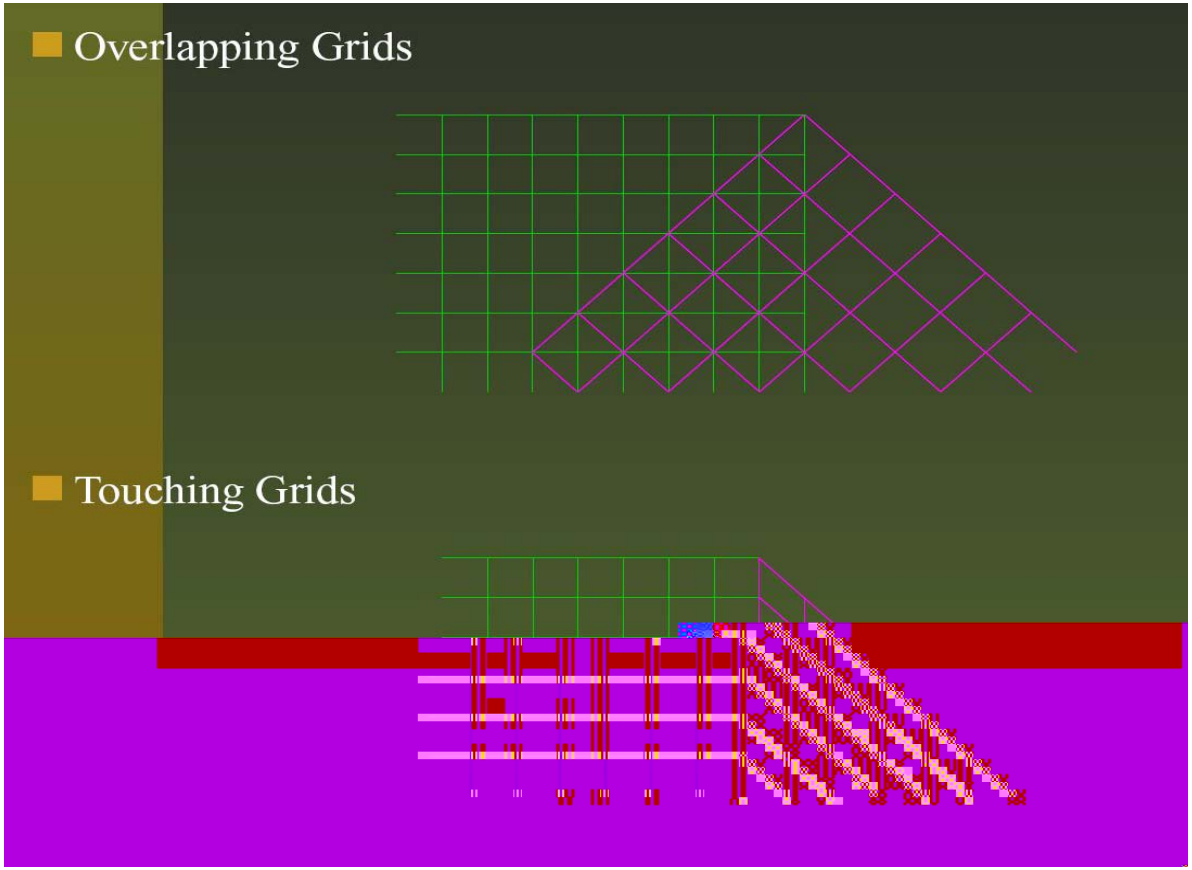
Implementation: SBP



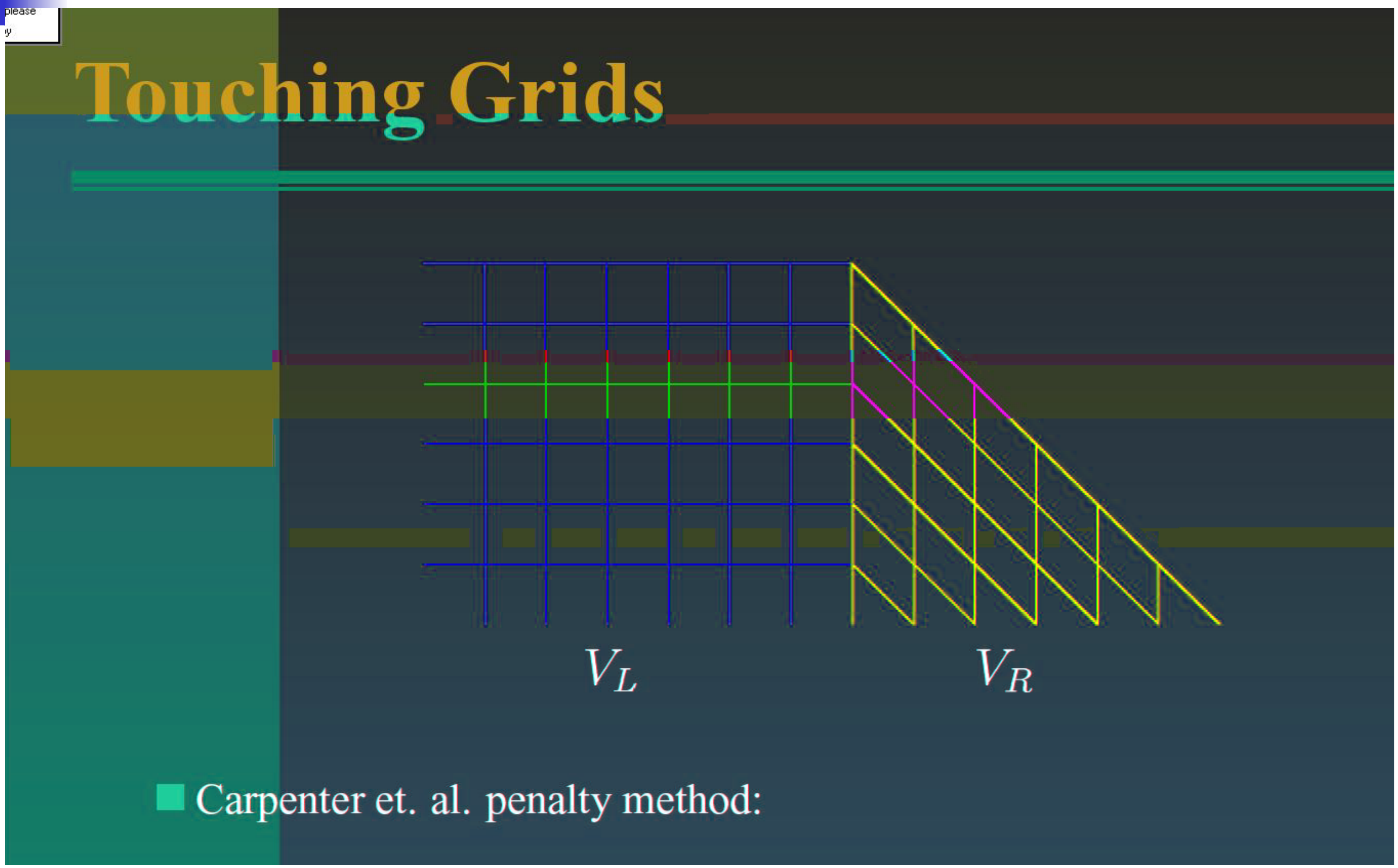
Implementation: SBP



Theory: Non-trivial domains



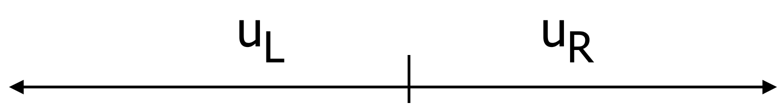
Theory: Non-trivial domains



Theory: Non-Trivial domains

[Carpenter, Nordstrom and Gottlieb '98]

- Say you want to discretize the advection equation $u_t = cu_x$, in two domains. The Left one covers (…,0], and the Right one [0,…)



- We use two fields to describe u, u_L and u_R. At x=0 the two fields are defined, and the solution is multivalued.

- Now discretize using penalty terms:

$$\frac{d}{dt}u_j^L = cDu_j^L - \frac{S^L}{h\sigma_{00}}(u_j^L - u_j^R)$$

$$\frac{d}{dt}u_j^R = cDu_j^R - \frac{S^R}{h\sigma_{00}}(u_j^R - u_j^L)$$

- And use <u>any</u> operator D satisfying the SBP property.

Theory: Non-Trivial domains

- Define the energy

$$E = (u,u)_\Sigma^L + (u,u)_\Sigma^R$$

$$(u,v)_\Sigma^L = h\sum_{-\infty}^{0}\sigma_{ij}u_j v_j \qquad (u,v)_\Sigma^R = h\sum_{0}^{\infty}\sigma_{ij}u_j v_j$$

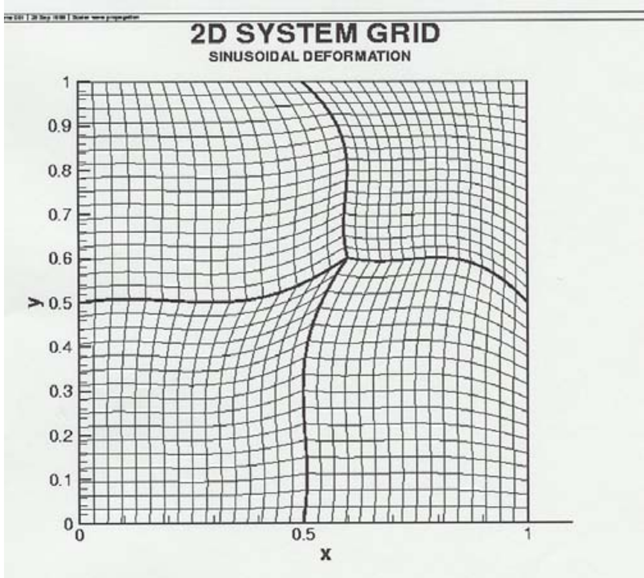
- Take its time derivative and use the SBP property to get

$$\frac{d}{dt}E = (c-2S^L)(u_0^L)^2 - (c+2S^R)(u_0^R)^2 + 2(S^L+S^R)u_0^L u_0^L$$

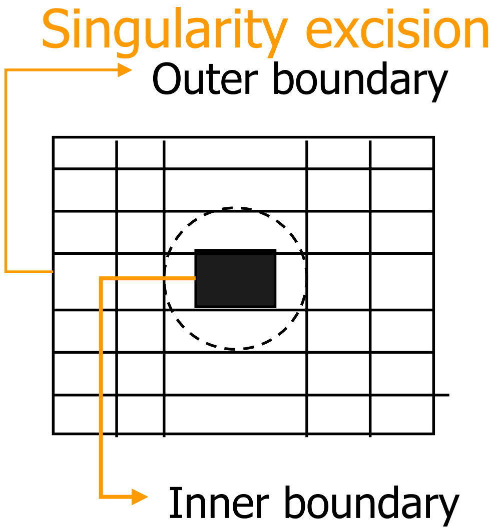
- If c>0, choosing $S^L = c+\delta$, $S^R = \delta$ gives

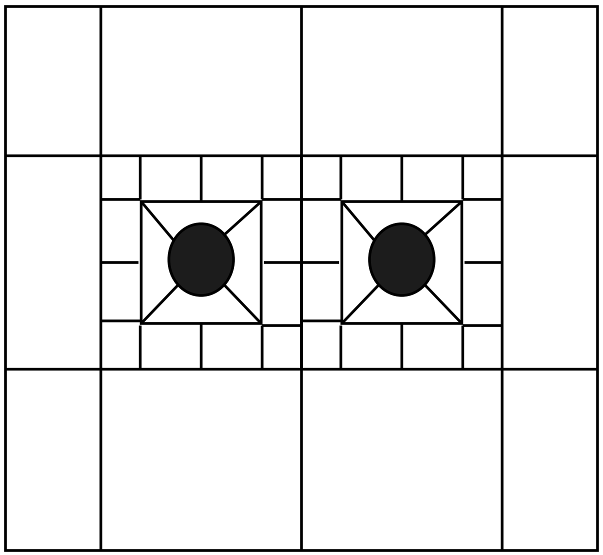
$$\frac{d}{dt}E = -(u_0^L - u_0^R)^2(c+2\delta)$$

- And the energy estimate follows if $\delta >= -\lambda/2$



Implementation: N.T.Domains

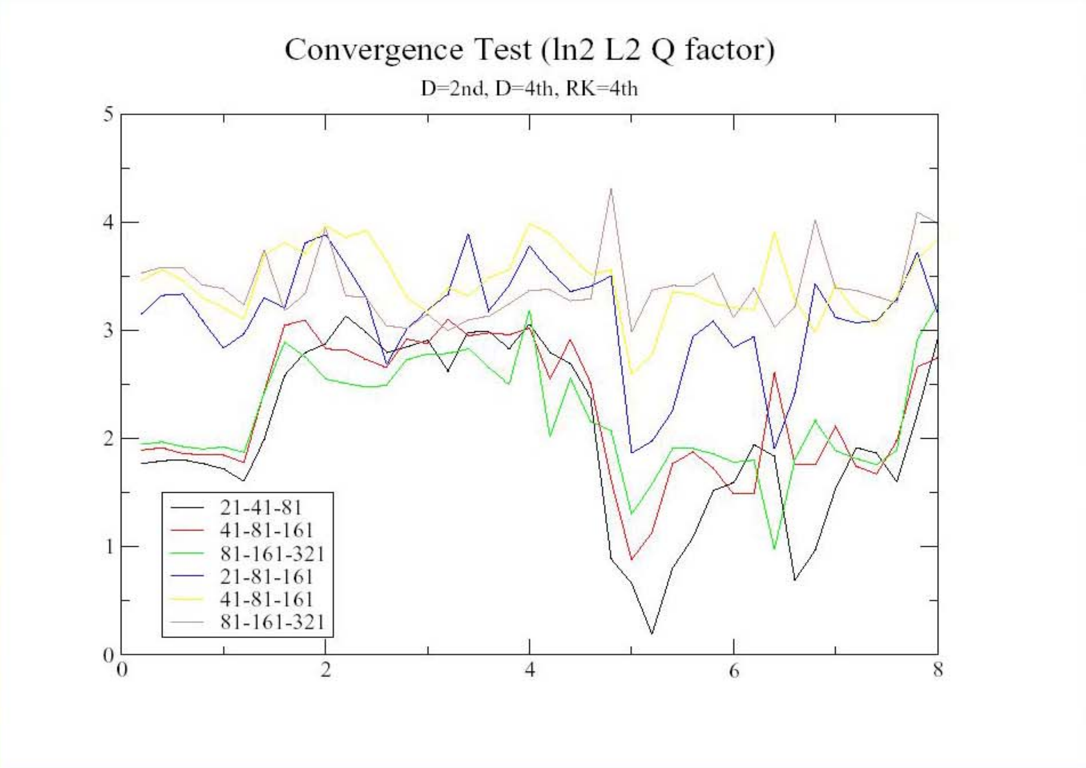




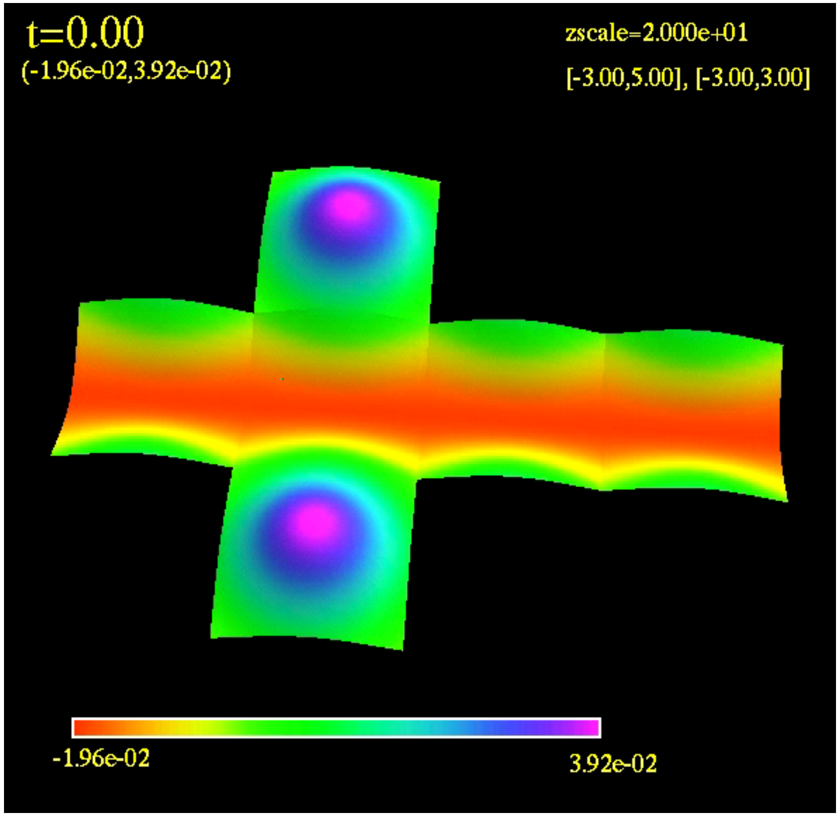
- L. Lindblom et al
- J. Thornburg et al

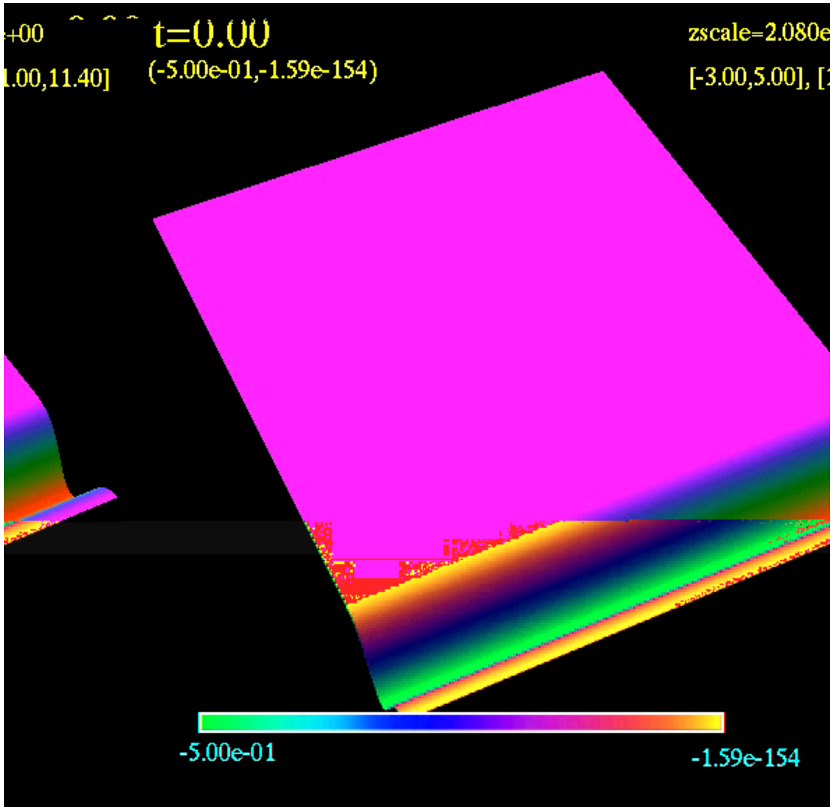
Implementation: N.T.Domains

- Wave equation on a sphere
  - 6 touching grids
  - no-dissipation
  - strict stability
  - penalty method



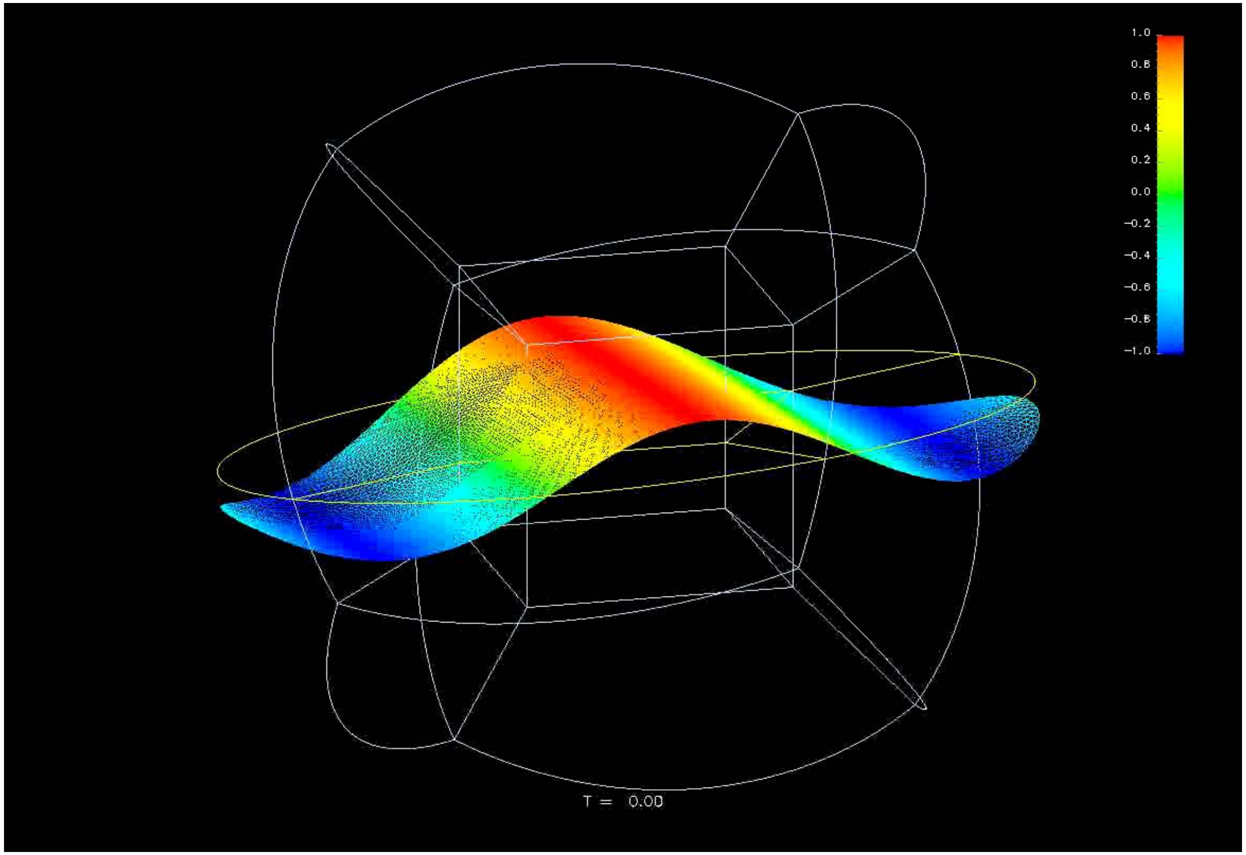
Implementation: N.T.Domains



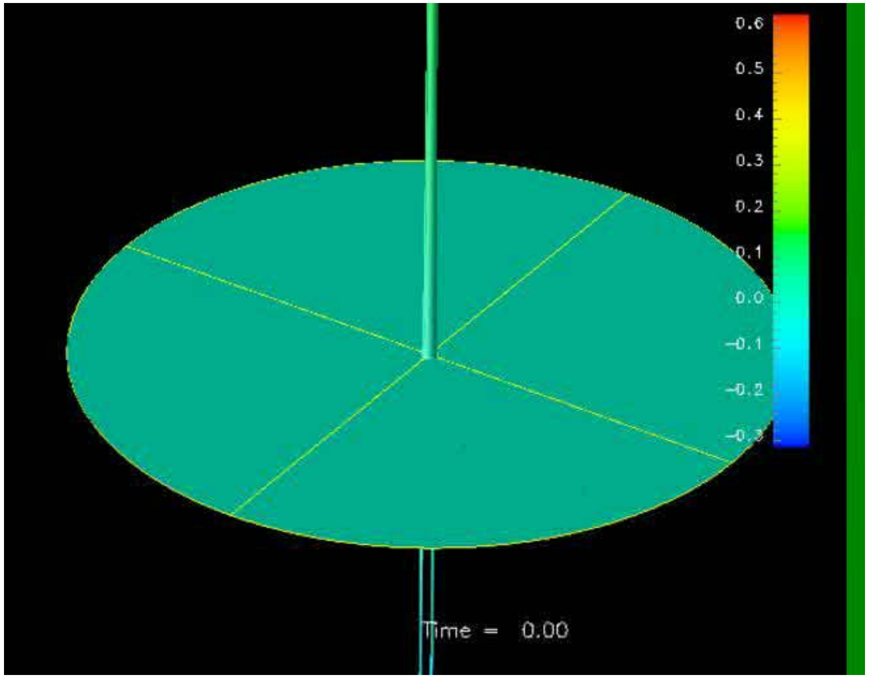


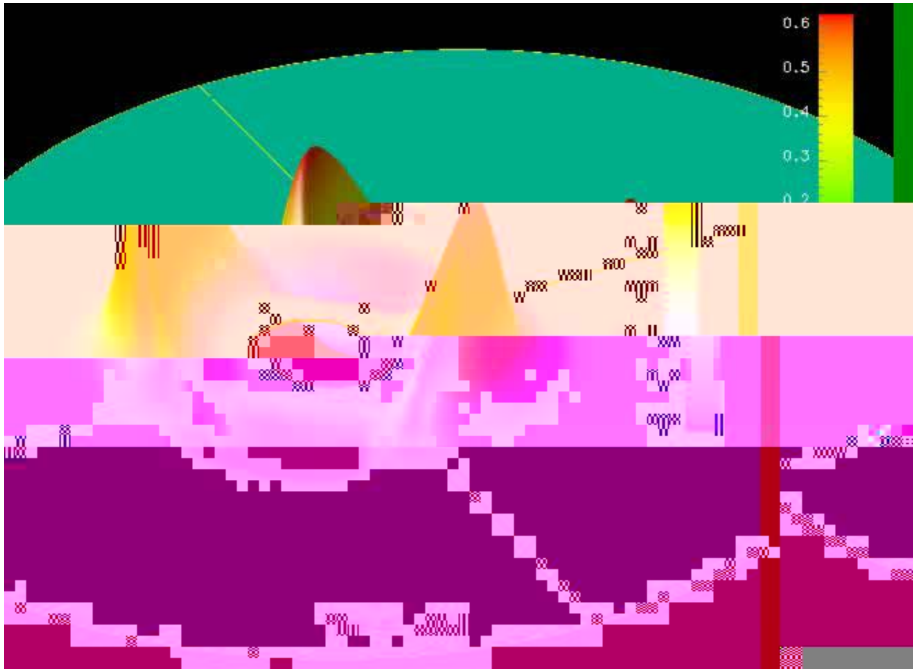
Implementation: N.T.Domains

Carpet/Cactus



Implementation: N.T.Domains





Implementation: N.T.Domains

- Other topologies: S^3 (8 cubes)
- Shallow Water Equations on Sphere.