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1 Overview

1.1 Notation and terminology

In these lecture notes I use **boldface** to highlight technical terms when they are being defined, either informally by using them, or in a formal definition. These terms also appear in the index at the end of these notes. I use *italics* to highlight something that is important. In equations, either \( a := b \) or \( b := a \) means that \( a \) is being defined in terms of \( b \). Three dots \( \ldots \) in a mathematical expression signify something that I do not write out in full because it repeats something earlier – what, should be clear from the context.

1.1.1 Partial derivatives and index notation

In these notes, I often abbreviate partial derivatives by commas, as in

\[
u_x := \frac{\partial u}{\partial x}, \quad u_{xy} := \frac{\partial^2 u}{\partial x \partial y}
\]  

(1)

Obviously, \( u_{xy} = u_{yx} \), as partial derivatives commute. It is helpful to stick to some preferred order (say \( x \) first). To take an example of this notation, the most general linear second-order PDE in one dependent variable \( u \) and two independent variables (or coordinates) \( x \) and \( y \) can be written as

\[
a u_{xx} + 2b u_{xy} + c u_{yy} + pu_x + qu_y = f.
\]  

(2)

**Remark 1.1.** In the PDE literature (for example in Renardy and Rogers), it is more common to write \( u_x, u_{xy} \) etc., without the comma, but I find it helpful to distinguish partial derivatives from other indices.

To write a PDE even more concisely when discussing general theory, we may number the independent variables, for example

\[
x_1 := x, \quad x_2 := y.
\]  

(3)

and do the same for the partial derivatives, obviously in the same numbering:

\[
u_{,1} := u_x, \quad u_{,2} := u_y.
\]  

(4)

We do the same for the coefficients

\[
a^{11} := a, \quad a^{12} := b, \quad a^{22} := c, \quad b^1 := p, \quad b^2 := q
\]  

(5)

and then we can write our example PDE (2) as

\[
\sum_{i=1}^{2} \sum_{j=1}^{2} a^{ij} u_{,ij} + \sum_{i=1}^{2} b^i u_{,i} = f.
\]  

(6)

or in more relaxed notation

\[
\sum_{i,j} a^{ij} u_{,ij} + \sum_{i} b^i u_{,i} = f.
\]  

(7)

Make sure you understand where the factor 2 in (2) has gone when it is written in the form (6). The form (6) has the advantage that it can be easily generalised to an arbitrary number of independent variables. We will use this in definitions and theorems later.

1.1.2 Systems of PDEs

Consider now a **PDE system** of \( N \geq 2 \) PDEs for \( N \) dependent variables, in \( n \) independent variables. A simple example with \( N = 2 \) and \( n = 2 \) would be the Cauchy Riemann equations (206,207) below.

To make general statements, we may number the dependent variables as \( u^\alpha, \alpha = 1, \ldots, N \) and the equations by \( \beta = 1, \ldots, N \). Then a general system of \( N \) second-order linear PDEs for \( N \)
dependent variables in \( n \) independent variables can be written as (for simplicity we assume there are no first derivatives)

\[
\sum_{\beta=1}^{N} \sum_{i=1}^{n} \sum_{j=1}^{n} a^{ij\alpha}_{\beta} u_{ij}^{\beta} = f^{\alpha}.
\]  

We do not sum over \( \alpha \), which appears on both sides. Hence (8) represents \( N \) equations, one for each value of \( \alpha = 1, \ldots, N \). Throughout these notes, \( n \) is the number of independent variables, while for systems of PDEs \( N \) is the number of dependent variables.

To make contact with something you already know, consider for a moment the system of algebraic linear equations

\[
\sum_{\beta=1}^{N} a^{\alpha\beta} u^{\beta} = f^{\alpha} \quad \text{for the \( N \) unknowns } u^{\alpha}.
\]

You will be familiar with the matrix notation \( A u = f \) for this, where \( A \) is an \( N \times N \) matrix with components \( a^{\alpha\beta} \) and \( u \) and \( f \) are column vectors in \( \mathbb{R}^{N} \) with components \( u^{\alpha} \) and \( f^{\alpha} \).

We can use similar vector and matrix notation for systems of linear PDEs. For any fixed \( i \) and \( j \) we define \( A_{ij} \) as the \( N \times N \) matrix with components \( a^{ij\alpha}_{\beta} \). With this notation, (8) can be written as

\[
\sum_{i=1}^{n} \sum_{j=1}^{n} A_{ij} u_{ij} = f.
\]

\( (\text{It is important to understand that we treat } \alpha, \beta \text{ as matrix indices, but not } i, j. \) We sometimes refer to a single PDE such as (7) as a scalar PDE to distinguish it from a PDE system such as (9). One sometimes refers to \( u \) as a vector in state space to distinguish it from a vector \( x \) in physical space.

1.1.3 PDE problems

A PDE is not usually solved in isolation. We must specify a domain in space, or in space and time, on which we want the PDE to hold. If the domain is bounded, then we will need to impose one or several boundary conditions on each boundary. For a time-dependent problem, boundary conditions at the initial time \( t = 0 \) are also called initial conditions. It is important to understand that the initial and boundary conditions are independent of the PDE in the interior.

If the domain is unbounded, for example all of \( \mathbb{R}^{n} \), then we still need boundary conditions, but they may be less obvious. For example, we may want the solution to fall off (go to zero) sufficiently fast as we approach infinity.

A PDE problem is a PDE, together with its domain and all necessary boundary and/or initial conditions.

1.1.4 Revision: Linear PDEs

Probably all the PDEs you have seen until now were linear PDEs. This means that they are linear expressions (containing the first power or the zeroth power) in the unknown. Our example PDE (2) is linear if the coefficients \( a, b, c, p, q, f \) are known functions of \( x \) and \( y \) (they may be constants), but not of \( u \). All terms in it are first order in the unknown \( u \), except for \( f \), which is zeroth order. If there is a zeroth-order term, also called the inhomogeneous term or source term, the PDE is called inhomogeneous. If this is absent (\( f = 0 \) in our example), the linear PDE is homogeneous.

Sometimes it is useful to write a linear PDE abstractly as

\[
Lu = f,
\]

where \( L \) is a homogeneous linear differential operator (say the Laplace operator) and \( f \) is the inhomogeneous term. For a PDE system, we could write \( Lu = f \).

The general solution (without imposing any boundary conditions) of any inhomogeneous linear PDE is of the form \( u = u_{\text{CF}} + u_{\text{PI}} \). Here the complementary function \( u_{\text{CF}} \) is the general solution of the corresponding linear PDE \( Lu = 0 \), and the particular integral \( u_{\text{PI}} \) is any one solution of the original inhomogeneous PDE (10). This is just the same as for linear ODEs. It works because

\[
Lu = L(u_{\text{CF}} + u_{\text{PI}}) = Lu_{\text{CF}} + Lu_{\text{PI}} = 0 + f = f.
\]
We have used the linearity of $L$ in the second equality. Recall that $L$ is linear if $L(f + g) = L(f) + L(g)$ and $L(cf) = cL(f)$, where $c$ is a constant. For example $L(f) = df/dx$ or $L(f) = h(x)f$ are linear, but $L(f) = f^2$ is not.

Similarly, in linear PDE problems we distinguish **homogeneous boundary conditions** and **inhomogeneous boundary conditions**. $u = 0$ at the boundary is an example of a homogenous boundary condition, while $u = g$, with $g(x)$ a given function, would be inhomogeneous. This $g(x)$ is an example of what is called **boundary data**. With inhomogeneous boundary conditions and a source term, we can then again write the general solution as the sum of a particular integral that solves the PDE problem with the source term and with inhomogeneous boundary conditions, and a complementary function that is the general solution of the corresponding PDE with zero source term and homogeneous boundary conditions.

### 1.2 Revision: Vector calculus

#### 1.2.1 Notation

In (9), we have written the dependent variables as a vector $u$, but we have used index notation for the independent variables. In other contexts, it may be useful to write the independent variables as a vector $x$, especially if they represent a position vector in three-dimensional Euclidean space.

A dependent variable may itself be a direction vector in Euclidean space, for example the velocity vector $v(x)$ in fluid dynamics. We then write

$$ x := (x_1, x_2, x_3) := (x, y, z), \quad (12) $$

$$ v(x) := (v^1, v^2, v^3) := (v^x, v^y, v^z), \quad (13) $$

$$ \nabla := \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right). \quad (14) $$

We can use $\nabla$ (pronounced **nabla**) to define the **gradient** of the scalar function $u$ as

$$ \nabla u := (u_{,x}, u_{,y}, u_{,z}) \quad (15) $$

and the **divergence** of the vector-valued function $v$ as

$$ \nabla \cdot v := v^x_{,x} + v^y_{,y} + v^z_{,z} = \sum_i v^i_{,i}. \quad (16) $$

Note that

$$ \nabla \cdot x = \sum_{i=1}^n x^i_{,i} = n, \quad (17) $$

the number of space dimensions.

**Remark 1.2.** In this course, I use the notation of denoting a vector by a boldface letter both for vectors in three (or two) space dimensions such as $x$, short for $x_i$ with $i = 1, \ldots, n$, and for vectors of variables such as $u$, short for $u^\alpha$ with $\alpha = 1, \ldots, N$. I use $\nabla$ only to denote derivatives with respect to space (the independent variables). (By contrast, Renardy and Rogers use it also for derivatives with respect to the dependent variables.)

#### 1.2.2 The Laplace operator

The second-order derivative operator $\nabla \cdot \nabla$ is called the Laplace operator and is usually denoted by either $\nabla^2$ or $\Delta$. In three-dimensional **Cartesian coordinates** $x = (x_1, x_2, x_3)$ the Laplace operator is

$$ \Delta(3) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}. \quad (18) $$

In two or one dimensions we have

$$ \Delta(2) = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}, \quad \Delta(1) = \frac{\partial^2}{\partial x^2}. \quad (19) $$
Without proof, we state a few other formulas. In cylindrical polar coordinates \((r, \theta, z)\), where
\[
x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \quad x_3 = z, \tag{20}
\]
we have
\[
\Delta^{(3)} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}, \tag{21}
\]
\[
\Delta^{(2)} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \tag{22}
\]
By just removing the coordinate \(z\) we obtain polar coordinates in the plane
\[
x_1 = r \cos \theta, \quad x_2 = r \sin \theta, \tag{23}
\]
where
\[
\Delta^{(2)} = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}. \tag{24}
\]
In spherical polar coordinates \((r, \theta, \varphi)\) where
\[
x_1 = r \sin \theta \cos \varphi, \quad x_2 = r \sin \theta \sin \varphi, \quad x_3 = r \cos \theta, \tag{25}
\]
we have
\[
\Delta^{(3)} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}, \tag{26}
\]
\[
\Delta^{(2)} = \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial}{\partial \theta} \left( \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2}. \tag{27}
\]
In the following, \(\Delta\) will be used to denote any of \(\Delta^{(3)}, \Delta^{(2)}\) or \(\Delta^{(1)}\). Note that in curvilinear coordinates, such as the polar coordinates we have used here, the Laplace operator generally contains first, as well as second, derivatives.

All of these formulas can be derived from the chain rule of partial derivatives
\[
\frac{\partial}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial x^j}{\partial x^i} \frac{\partial}{\partial x^j}, \tag{28}
\]
where \(x^j\) is one system of coordinates and \(x^i\) the other. As a simple example, where \(n = 2\), take the transformation from coordinates \((x, y)\) to \((\xi, \eta)\), given by
\[
\frac{\partial}{\partial \xi} = \frac{\partial x}{\partial \xi} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \xi} \frac{\partial}{\partial y}, \tag{29}
\]
\[
\frac{\partial}{\partial \eta} = \frac{\partial x}{\partial \eta} \frac{\partial}{\partial x} + \frac{\partial y}{\partial \eta} \frac{\partial}{\partial y}. \tag{30}
\]

### 1.2.3 The divergence theorem

There is more to vector calculus than concise notation. One result we will need is

**Theorem 1.3 (Divergence theorem).** Let \(V\) be a volume and \(S := \partial V\) its surface. Let \(dV\) be the volume element and \(dS\) the surface element, and let \(\mathbf{n}(x)\) be the outward-pointing unit normal vector at each point of \(S\). Then
\[
\int \int \int_V \nabla \cdot \mathbf{f} \, dV = \int \int_S \mathbf{n} \cdot \mathbf{f} \, dS, \tag{31}
\]
It is easy to prove the theorem in the simple case in three dimensions where \( V \) is the rectangular box \( x_0 \leq x \leq x_1, y_0 \leq y \leq y_1, z_0 \leq z \leq z_1 \), and hence \( S \) is the union of six rectangles. Using the fundamental theorem of calculus,

\[
\int_{x_0}^{x_1} \frac{df}{dx} \, dx = f(x_1) - f(x_0),
\]

separately in the \( x, y \) and \( z \) directions, one can show that

\[
\begin{align*}
&\int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy \int_{z_0}^{z_1} dz \left( f_x^x + f_y^y + f_z^z \right) \\
&= \int_{y_0}^{y_1} dy \int_{z_0}^{z_1} dz \left[ f_x^x(x_1, y, z) - f_x^x(x_0, y, z) \right] \\
&+ \int_{x_0}^{x_1} dx \int_{z_0}^{z_1} dz \left[ f_y^y(x, y_1, z) - f_y^y(x, y_0, z) \right] \\
&+ \int_{x_0}^{x_1} dx \int_{y_0}^{y_1} dy \left[ f_z^z(x, y, z_1) - f_z^z(x, y, z_0) \right]
\end{align*}
\]

In vector calculus notation this can be written as (31), where \( \mathbf{n} = (0, -1, 0) \) for all points in the part of \( S \) given by \( y = y_0, x_0 \leq x \leq x_1, z_0 \leq z \leq z_1 \), and so on for the other five faces of the box.

The actual divergence theorem above generalises our simple result in two ways. First, one can prove that (31) holds for any volume \( V \) with boundary \( S \), not only a rectangular box. Secondly, \( \nabla \cdot \mathbf{f}, dV, dS \) and \( \mathbf{n} \) are geometric objects, in the sense that the two integrals on either side of (31) can be defined and evaluated not only in Cartesian coordinates, but in arbitrary coordinates.

You have previously learned to evaluate (31) in some simple situations, but for this course we mostly need the abstract version.

### 1.3 Revision: Separation of variables

In MATH2038, MATH2047, MATH2048 or MATH2015 you have learned to solve PDEs using separation of variables. Keep in mind that this works only under the following conditions:

1. The PDE is homogeneously linear and separable. This means that an ansatz like

\[
u(x, y) = X(x) \ Y(y)
\]

 turns the linear PDE

\[
Lu(x, y) = 0,
\]

where \( L \) is a homogeneous linear differential operator into

\[
L_x X(x) = L_y Y(y),
\]

where \( L_x \) and \( L_y \) are homogeneous linear differential operators. Hence both sides must be equal to some constant \( K \), and so we have the ODEs

\[
L_x X_K(x) = K, \quad L_y Y_K(y) = K.
\]

These ODEs are then solved for arbitrary separation constant \( K \). Typically, homogeneous boundary conditions allow only discrete values of \( K \). The solution \( u(x, y) \) is obtained as the sum

\[
u(x, y) = \sum_K c_K X_K(x) Y_K(y).
\]

The constants \( c_K \) are typically determined by initial or inhomogeneous boundary conditions.

2. The domain of the problem is a coordinate rectangle. In other words, the domain has the form \( a < x < b \) and \( c < y < d \). The important consequence of this is that each part of the boundary affects only one of the ODEs above. So a boundary condition at \( x = 0 \) becomes a boundary condition for \( X(0) \) but does not affect \( Y(y) \). This in turn means that the ODEs for \( X \) and \( Y \) can be solved separately.
A coordinate rectangle does not have to be a rectangle in physical space. For example, a sphere in spherical coordinates is given by

\[ 0 \leq r \leq 1, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi, \]  

which is again a coordinate rectangle, but now in spherical polar coordinates.

We do not review the basics of separation of variables here, but it is worth reminding you of a few more advanced aspects below.

More than two independent variables: Separate in one variable first and introduce a first separation constant. You now have an ODE, and a PDE in one fewer independent variable. Repeat.

Inhomogenous boundary conditions: Say we want to solve the Laplace equation \( u_{xx} + u_{yy} = 0 \) on a coordinate rectangle, with inhomogenous (non-zero) boundary conditions on each side. We do this by writing the solution \( u \) as the sum of four solutions, each of which obeys homogeneous boundary conditions on three sides and the given inhomogeneous one on one side.

Another method works more generally. Here we turn inhomogeneous boundary conditions into a source term. Let \( u_1 \) be any function that obeys the inhomogenous boundary condition, but is not actually a solution of the PDE, i.e. \( Lu_1 \neq f \). Then write \( u = u_1 + u_2 \) where \( u_2 \) obeys the corresponding homogeneous boundary condition but has a non-vanishing source term, given by

\[ Lu_2 = -Lu_1 + f. \]  

You should convince yourself that this works.

1.4 Revision: Complex Fourier series and Fourier transforms

In this course, we will need the complex Fourier transform for two purposes: formally, in the classification of PDEs, and in solving PDEs by separation of variables where the domain in, say, \( x \) is not an interval \( a \leq x \leq b \) but the real line \( -\infty < x < \infty \).

In separation of variables on an interval, we typically have a boundary condition on each end of the interval. As a simple example, assume we want to solve some PDE for \( u(x,y,z) \) with boundary conditions \( u(0,y,z) = u(2\pi,y,z) = 0 \) on \( u \). After separation of variables with the usual ansatz \( u(x,y,z) = X(x)Y(y)Z(z) \), this gives the boundary conditions \( X(0) = X(2\pi) = 0 \) on \( X \). Then we would make each \( X_n(x) \) obey these boundary condition, by trying \( X_n(x) = \sin nx, n = 1, 2, \ldots \). Note that this choice of \( X_n(x) \) also implies the simple relation \( X''_n = -n^2 X_n \).

If instead of an interval we have the real line, we typically have a fall-off condition, \( u(x,y,z) \to 0 \) as \( |x| \to \infty \). In principle it would be possible to make each \( X_k(x) \) vanish separately at infinity. But then we would lose the useful property that \( X_k' \) is related to \( X_k \) in a simple way. Instead we use \( X_k(x) = e^{ikx} \). This is periodic (with period \( 2\pi/k \)), and so does not fall off at infinity. (It also obeys \( X''_k = -k^2 X_k \).) But by superimposing a continuum of such functions we can make a function that falls off at infinity. This leads us to the concept of a Fourier transform.

You have already covered the complex Fourier series in the prequisite course for this one. If \( f(x) \) is a periodic function with period \( L \), that is \( f(x + L) = f(x) \), then we can write it as

\[ f(x) = \sum_{n=-\infty}^{\infty} c_n e^{ik_n x}, \quad k_n := \frac{2\pi n}{L}, \]  

where the complex Fourier coefficients \( c_n \) are given by

\[ c_n = \frac{1}{L} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-ik_n x} \, dx. \]  

(Note the integral could be taken over any other full period, for example from 0 to \( L \).) If \( f(x) \) is real, then \( c_n = c_{-n}^* \), where the star denotes the complex conjugate.

[You can verify (42) directly by changing \( n \) to \( m \) and then substituting the expression (41) for \( f(x) \). After integrating, you get \( c_n = c_n \), as required.]

From the complex Fourier series we can obtain the Fourier transform as a limit. Define

\[ f(k_n) := \frac{c_n L}{\sqrt{2\pi}}, \quad \Delta k := k_{n+1} - k_n = \frac{2\pi}{L}. \]
With this notation, (41) and (42) become
\[
\begin{align*}
  f(x) &= \frac{1}{\sqrt{2\pi}} \sum_{n=-\infty}^{\infty} \hat{f}(k_n) e^{ik_n x} \Delta k, \\
  \hat{f}(k_n) &= \frac{1}{\sqrt{2\pi}} \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-ik_n x} \, dx.
\end{align*}
\] (44) (45)

So far, the function \( \hat{f}(k) \) is defined only at the discrete points \( k_n \). Now as we take the limit \( L \to \infty \), \( \Delta k \to 0 \) and we can think of it as \( dk \) under an integral, that is \( \sum \Delta k \to \int dk \). At the same time, \( \hat{f}(k) \) is then defined on the continuous real line. Hence
\[
\begin{align*}
  f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikx} \, dx, \\
  \hat{f}(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} \, dk.
\end{align*}
\] (46) (47)

Note the pleasing symmetry. When \( f(x) \) is real, then \( \hat{f}(k) = \hat{f}(-k)^* \).

\( f(x) \) admits a Fourier transform if and only if
\[
\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty. \tag{48}
\]

But this means that \( f(x) \) must vanish at infinity sufficiently fast for the integral to exist. Hence if we impose (46) as an ansatz for \( f(x) \), for example in the process of separation of variables, we automatically impose the fall-off condition \( f(x) \to 0 \) as \( |x| \to \infty \).

### 1.5 Common linear PDEs

#### 1.5.1 Laplace, Poisson and Helmholtz equations

We start with PDEs that do not have time derivatives. The **Laplace equation** is
\[
\Delta u = 0. \tag{49}
\]

The **Poisson equation** is just the Laplace equation with a (given) source term, or
\[
\Delta u = f(x). \tag{50}
\]

The (inhomogeneous) **Helmholtz equation**, which we will derive below, is
\[
(\Delta + k^2)u = f(x), \tag{51}
\]

Let us assume that we want to solve one of these linear PDEs on a bounded domain \( V \) with boundary \( S \), with unit normal vector \( n \). At each point on the boundary we can impose precisely one boundary condition. The standard classes of conditions that can be imposed are

- **Dirichlet boundary conditions**: Specify the value of \( u \) on \( S \).
- **Neumann boundary conditions**: Specify the value of the normal derivative
  \[
  u_n := n \cdot \nabla u = \sum_i n^i u_{,i}
  \]
  on \( S \), where \( n \) is the outward-pointing unit normal vector.
- **Robin boundary conditions**: Specify the value of a linear combination of \( u \) and \( u_n \) on \( S \).
For example, the Poisson equation with inhomogeneous Robin boundary conditions specifies the PDE problem

\[ \Delta u = f(x) \text{ in } V, \]
\[ \beta u_{,n} + \alpha u = g(x) \text{ on } S. \]

Here the special value \( \beta = 0 \) gives Dirichlet boundary conditions and \( \alpha = 0 \) gives Neumann boundary conditions. Here, and in similar examples, \( g(x) \neq 0 \) specifies an inhomogeneous boundary condition and \( g(x) = 0 \) the corresponding homogeneous boundary condition.

**Remark 1.4.** For the Poisson equation the solution with Neumann conditions is only unique up to an arbitrary constant, for if \( u_1 \) satisfies the problem so too does \( u_2 = u_1 + C \) where \( C \) is constant. This is so because the derivative of a constant is zero, and so both \( \Delta C = 0 \) and \( C_{,n} = 0 \).

**Remark 1.5.** For the Poisson or Laplace equation with Neumann boundary conditions,

\[ \Delta u = f(x) \text{ in } V, \]
\[ u_{,n} = g(x) \text{ on } S, \]

we must have

\[ \int_V f \, dV = \int_V \Delta u \, dV = \int_V \nabla \cdot (\nabla u) \, dV = \int_S (\nabla u) \cdot n \, dS = \int_S u_{,n} \, dS = \int_S g \, dS \]

for the source term \( f \) and boundary data \( g \) to be compatible. Otherwise, this problem has no solution. (We will come back to this in Sec. 8.4.3.)

**Remark 1.6.** Dirichlet, Neumann or Robin boundary conditions, homogeneous or inhomogeneous, all apply only at a boundary at finite distance. If the domain of the PDE is infinite, then most likely we will want to impose a fall-off condition \( u(x) \to 0 \) as \( |x| \to \infty \). Note that there is only one type of fall-off condition, and in particular we should think of it as a (homogeneous) boundary condition at infinity. This type of boundary condition is also called a free space boundary condition. It turns out that if we can impose either Dirichlet, Neumann or Robin boundary condition on a finite volume, as a fourth alternative we can also impose a free space boundary condition on an infinite (unbounded) volume. This is often the simplest boundary condition to work with. In particular, we will use it a lot when we find Green’s functions for PDEs.

The Laplace, Poisson and Helmholtz equations, in any number of space dimensions, are examples of a class of PDEs that are called elliptic equations. They are, in fact, the most important examples of linear second-order elliptic equations. A formal definition of ellipticity will be given in Sec. 3 below.

**Remark 1.7.** A key feature of elliptic PDEs is that they smooth out their data: the interior solution is more often differentiable than the boundary data.

**Remark 1.8.** The Helmholtz equation is derived from the wave equation by separating the time variable (assuming a periodic time dependence) as follows. Consider the wave equation with a time-periodic source term. For simplicity, we assume the source has only a single (angular) frequency \( \omega \), and we use complex notation:

\[ -\frac{1}{c^2} \psi_{,tt} + \Delta \psi = e^{-i\omega t} f(x) \]

If we look for a time-periodic solution of this, called a standing wave,

\[ \psi(x, t) = e^{-i\omega t} u(x), \]

we find that \( u(x) \) satisfies (51), where

\[ k = \frac{\omega}{c} = \frac{2\pi \nu}{c} \]

is the wave number. Here \( c \) is the wave speed, \( \nu \) the frequency and \( \omega := 2\pi \nu \) the angular frequency. The wavelength of a plane wave of frequency \( \nu \) is

\[ \lambda = \frac{c}{\nu} = \frac{2\pi}{k}. \]
1.5.2 Heat equation

For a time-dependent problem on a bounded domain, with a PDE that contains one or more time derivatives, we must specify both initial data throughout the domain (volume) $V$ at some time $t = 0$, and boundary data on the boundary of the domain (surface) $S$ for $t \geq 0$; see Fig. 1. The problem is then solved for $t > 0$ inside the volume $V$. This is called the Cauchy problem.

We begin by looking at the heat equation, also called the diffusion equation,

$$ u_t = \kappa \Delta u. \quad (62) $$

Here $\kappa > 0$ is a constant that must have dimension length$^2$/time, the diffusion constant.

Because the heat equation gives us $u_t(x,0)$ if we know $u(x,0)$, it is intuitively clear that we need to specify precisely $u(x,0)$ as initial data. A particular class of solutions of the heat equation are those that are independent of time: they then obey the Laplace equation. This suggests that the heat equation requires the same boundary conditions as the Laplace equation. Thus, the Cauchy problem for the heat equation is

$$ u_t = \kappa \Delta u, \quad t > 0 \quad (63) $$

$$ u(x,0) = f(x) \quad \text{for} \quad x \in V \quad (64) $$

$$ \beta u_x + \alpha u = g(x,t) \quad \text{for} \quad x \in S. \quad (65) $$

(Either $\alpha$ or $\beta$ can be zero, to give Neumann or Dirichlet boundary conditions.)

The heat equation, in any number of space dimensions, is an example of a class of PDEs that are called parabolic. We give a formal definition of parabolic second-order scalar PDEs in Sec. 3, but if $Lu(x) = 0$ is a linear elliptic equation for $u(x)$, then $u_t = Lu(x,t)$ is a parabolic equation for $u(x,t)$. Going from the Laplace equation (elliptic) to the heat equation (parabolic) is one example of this. One can prove that a parabolic equation with Dirichlet, Neumann or Robin boundary conditions and initial data for $u$ has a unique solution.

**Remark 1.9.** Parabolic PDEs have these key features:

- a) They smoothe out their initial data: the solution is more often differentiable than the initial data.
- b) They have infinite propagation speed. Even if the initial data at $t = 0$ is non-zero only in a finite region of space, at any $t > 0$ the solution is typically non-zero in the entire domain: in this sense, the solution has spread from the initial data at infinite speed.
- c) They cannot be run backwards. If the time evolution problem with initial data at $t = 0$ is well-posed for $t > 0$, it is ill-posed for $t < 0$.

1.5.3 Wave equation

The wave equation, in any number of space dimensions, is

$$ u_{tt} = c^2 \Delta u, \quad (66) $$
where \( c > 0 \) is the wave speed. For the wave equation to have consistent dimensions, \( c \) must have dimension length/time.

The wave equation contains two time derivatives, and so we need initial data for both \( u(x,0) \) and \( u_t(x,0) \) at \( t = 0 \). To understand this intuitively, consider the wave equation that describes the motion of a string, that is the wave dimension in one space dimension

\[
  u_{tt} = c^2 u_{xx}. \tag{67}
\]

(Here \( x \) is the position along the string, and \( u \) the transversal displacement of the string from its rest position, assumed small.) We need to specify the initial position \( u(x,0, \) ) and velocity \( u_t(x,0, \) ) of each part of the string. Newton's force law and Hooke's law of elasticity then give us the acceleration and we can evolve in time. (By contrast, heat has no inertia, so it is sufficient to specify the initial temperature.) If one considers a time-periodic solution to the wave equation, it reduces to the Helmholtz equation. This suggests that the wave equation needs Dirichlet, Neumann or Robin boundary conditions, like the Helmholtz equation.

All these conditions together constitute the Cauchy problem for the wave equation (on a volume \( V \) with boundary \( S \), and assuming Robin boundary conditions):

\[
\begin{align*}
  u_{tt} & = c^2 \Delta u \quad \text{for} \quad x \in V, \quad t \geq 0, \tag{68} \\
  u(x,0) & = f(x) \quad \text{for} \quad x \in V, \tag{69} \\
  u_t(x,0) & = g(x) \quad \text{for} \quad x \in V, \tag{70} \\
  \beta u_{tt} + \alpha u & = h(x,t) \quad \text{for} \quad x \in S. \tag{71}
\end{align*}
\]

The wave equation is an example of a class of PDEs called linear second-order hyperbolic equations. We give a formal definition below. However, if \( Lu = 0 \) is an elliptic equation for some differential operator \( L \), then \( c^2u_{tt} + bu_t = Lu \) is a hyperbolic second-order PDE, for any constant \( b \) and any \( c > 0 \). The wave equation is obviously in this class, but we shall see that the class of hyperbolic PDEs is much larger.

**Remark 1.10.** Hyperbolic PDEs have these key features:

a) Any features in the initial data move about in space at one or several specific velocities.

b) There is no smoothing: the solution is typically as often differentiable as the initial data and boundary data.

c) They can be run backwards: the same initial data at \( t = 0 \) can in general be evolved to \( t > 0 \) or \( t < 0 \).

**Remark 1.11.** The one dimensional wave equation (67) is unusual among wave equations in that, on the infinite domain \( -\infty < x < \infty \), it can be solved in closed form. Its general solution, known as the d'Alembert solution, is

\[
  u(x,t) = F(x-ct) + E(x+ct). \tag{72}
\]

This can be easily verified by using the chain rule.

D'Alembert's solution does not help us for more general hyperbolic problems, but it provides a nice explicit illustration of all points of Remark 1.10: \( F(x-ct) \) represents a wave of shape \( F(x) \) (at time \( t = 0 \)) moving in the direction of increasing \( x \) at constant speed \( c \). That is, \( F(x-ct) \) represents the same shape as \( F(x) \), but with the origin moved to \( x = ct \), i.e. the same shape as \( F(x) \) but translated a distance \( ct \) in the positive \( x \) direction. (If you are confused about the signs, imagine the example where \( F \) has its maximum when its argument is zero, say.) Conversely, \( E(x+ct) \) represents a wave of shape \( E(x) \) going in the negative \( x \) direction at constant speed \( c \). It is also obvious that the solution \( u(x,t) \) is just as often differentiable as the initial data \( u(x,0) \) and once more than \( u_t(x,0) \).

### 1.6 Cauchy data and the Cauchy-Kowalewski solution

The initial data for the Cauchy problem consist of specifying \( u \) at \( t = 0 \) for a PDE that contains only a first time derivative, \( u \) and \( u_t \) for a PDE that contains up to the second time derivative, and so on: one fewer time derivative in the initial data than in the PDE. These initial data are called **Cauchy data.** The following two remarks are meant to motivate why Cauchy data can be expected to give rise to well-posed initial-value problems.
Remark 1.12. If we solve the heat equation numerically, we start with data \( u(x, 0) \) at \( t = 0 \). The PDE then gives us \( u_t(x, 0) \). We use this to obtain an approximation for \( u(x, h) \), where \( h \) is some short time interval. In the simplest case, called forward Euler, this is just the zeroth and first term of a Taylor series,

\[
    u(x, h) \simeq u(x, 0) + h u_t(x, 0) = u(x, 0) + h \kappa \Delta u(x, 0).
\]  

Then consider this as new initial data and repeat the process to get \( u(x, 2h) \), then \( u(x, 3h) \) and so on. This is the basic idea behind all numerical methods for solving time-dependent problems.

Remark 1.13. A related theoretical way of looking at Cauchy data is to try and write the solution for small \( t > 0 \) by expanding \( u(x, t) \) into its Taylor series about \( t = 0 \), that is

\[
    u(x, t) = u(x, 0) + u_t(x, 0) t + \frac{1}{2!} u_{tt}(x, 0) t^2 + O(t^3).  
\]  

To do this, we need to find all \( t \)-derivatives of \( u \) at \( t = 0 \). In principle, this can be done by replacing successive time derivatives by spatial derivatives and using the fact that partial derivatives, and in particular space and time derivatives, commute. In the example of the heat equation, we have

\[
    u_t = \kappa \Delta u \quad \Rightarrow \quad u_{tt} = (\kappa \Delta u)_t = \kappa \Delta u_{tt} = \kappa \Delta (\kappa \Delta u) = \kappa^2 \Delta^2 u,  
\]  

and so on for higher time derivatives. If we evaluate this at \( t = 0 \) and substitute it into (74), we obtain the Cauchy-Kowalewski solution: an infinite series in powers of \( t \), with coefficients that contain arbitrarily many spatial derivatives of the initial data. For the heat equation this is

\[
    u(x, t) = u(x, 0) + \kappa \Delta u(x, 0) t + \frac{1}{2!} \kappa^2 \Delta^2 u(x, 0) t^2 + O(t^3).  
\]

However, the Cauchy-Kowalewski solution is just a formal solution. In the first place, it only exists if \( u \) is infinitely often differentiable in \( t \) and \( x \). (The technical term for infinitely often differentiable is smooth.) More importantly, in almost any interesting situation, the infinite series does not converge for any finite \( t > 0 \), and so this solution does not make sense. We have introduced it here to show why a \( k \)-th order evolution equation requires the first \( k - 1 \) derivatives as Cauchy data.

1.7 Weak solutions

The d’Alembert solution of the wave equation gives us a glimpse of a general fact: hyperbolic PDEs have discontinuous solutions. To derive (72) we have assumed that \( u(x, t) \) is twice differentiable, but intuitively the d’Alembert solution makes sense for non-differentiable and even discontinuous functions \( F \) and \( E \). For example, one could consider initial data that correspond to a wave moving right and then make these initial data more and more square until they become a step function, which is discontinuous.

Remark 1.14. It is clear that a discontinuity in the solution arises when either \( F \) or \( E \) is a discontinuous function. Say \( F(z) \) jumps at \( z_0 \), then \( F(x - ct) \) jumps at \( x - ct = z_0 \). Similarly, a jump in \( E(z) \) at \( z_0 \) becomes a jump at \( x + ct = z_0 \). These lines are the characteristic curves of (67). This is a simple example of a general fact: the solution of a linear hyperbolic equation can be discontinuous along its characteristic curves.

In what sense does such a discontinuous solution still obey the wave equation? Consider initially a solution \( u(x, t) \) of the wave equation (67) that is at least twice differentiable (and so really can be said to obey the wave equation), and an arbitrary function \( \phi(x, t) \) that is smooth and vanishes at infinity. Then \( u_{tt} = c^2 u_{xx} \) for all \( x \) and \( t \) implies that

\[
    \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-u_{tt} + c^2 u_{xx}) \phi \, dx \, dt = 0  
\]  

for any such function \( \phi \), simply because the round bracket is zero. We can imagine in particular that \( \phi \), while smooth, is sharply peaked about the point \((x, t)\). It then probes if \( u \) obeys \( u_{tt} = c^2 u_{xx} \) at that point.
Integrating the first term by parts in $t$ twice, and the second part twice in $x$ gives
\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-\phi_{tt} + \epsilon^2 \phi_{xx}) u \, dx \, dt = 0.
\] (78)

The boundary terms vanish by the assumption that $\phi$ vanishes at infinity. But $\phi$ was an arbitrary function, and this integral no longer requires $u$ to be differentiable in order to be defined. This observation motivates

**Definition 1.15.** $u(x, t)$ is a weak solution of the wave equation (67) if it obeys (78) for all functions $\phi(x, t)$ that are twice differentiable and vanish at infinity.

**Definition 1.16.** A strong solution of (67) is a twice differentiable function $u(x, t)$ for which (67) holds for every $x$ and $t$.

**Remark 1.17.** Any strong solution is also a weak solution, as we have just shown by integrating by parts and considering all possible $\phi$. The reverse is not true [because we may not be able to differentiate $u$, and hence may not be able to go back from (78) to (77).]

Weak solutions are important for hyperbolic (wave equation-like) problems for two reasons. First, the solution of a linear hyperbolic equation with discontinuous initial data often makes physical sense. Secondly, the solution of a nonlinear hyperbolic equation typically becomes discontinuous in finite time even if the initial data are everywhere smooth. We will see examples of this when we discuss conservation laws.

### 1.8 Exercises

1. **Revision problem 1:** a) Solve the following PDE problem (Laplace equation in two dimensions on a square, with homogeneous Dirichlet boundary conditions, on three sides of the square, and inhomogeneous Dirichlet boundary conditions on one side):

   \[
   u_{xx} + u_{yy} = 0, \quad 0 \leq x \leq \pi, \quad 0 \leq y \leq \pi, \quad (79)
   \]
   \[
   u(0, y) = 0, \quad 0 \leq y \leq \pi, \quad (80)
   \]
   \[
   u(\pi, y) = 0, \quad 0 \leq y \leq \pi, \quad (81)
   \]
   \[
   u(x, 0) = 0, \quad 0 \leq x \leq \pi, \quad (82)
   \]
   \[
   u(x, \pi) = f(x), \quad 0 \leq x \leq \pi. \quad (83)
   \]

   b) Hence show that the solution for $f(x) = 1$ is

   \[
   u(x, y) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi \sinh[(2m+1)\pi]} \sin[(2m+1)x] \sinh[(2m+1)y] \quad (84)
   \]

   c) Use Maple, Matlab, Mathematica, or some other software, to plot the solution series with the first 1, 2, 10, 100 terms. Observe that the solution is differentiable in the interior even though the boundary data jumps (between 0 and 1) at the two corners $(x, y) = (0, \pi)$ and $(\pi, \pi)$.

2. **Revision problem 2:** a) Solve the following PDE problem (one-dimensional heat equation on an interval, with homogeneous Dirichlet boundary conditions at both ends of the interval):

   \[
   u_{xx} - u_y = 0, \quad 0 \leq x \leq \pi, \quad y \geq 0 \quad (85)
   \]
   \[
   u(0, y) = 0, \quad y \geq 0, \quad (86)
   \]
   \[
   u(\pi, y) = 0, \quad y \geq 0, \quad (87)
   \]
   \[
   u(x, 0) = f(x), \quad 0 \leq x \leq \pi. \quad (88)
   \]

   b) Hence show that the solution for $f(x) = 1$ is

   \[
   u(x, y) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin[(2m+1)x] e^{-(2m+1)^2 y} \quad (89)
   \]

   16
c) Use Maple, Matlab, Mathematica, or some other software, to plot the solution series with the first 1, 2, 10, 100 terms for $0 \leq y \leq \pi$. Observe that the solution is differentiable in the interior even though the there is a jump from 0 to 1 between the initial data and the boundary data at the corners $(x, y) = (0, 0)$ and $(\pi, 0)$. (Or put differently, the initial data do not obey the boundary condition.)

3. Revision problem 3: a) Solve the following PDE problem (one-dimensional wave equation on an interval, with homogeneous Dirichlet boundary conditions at both ends of the interval):

\begin{align}
  u_{,xx} - u_{,yy} &= 0, & 0 \leq x \leq \pi, & y \geq 0 \quad (90) \\
  u(0, y) &= 0, & y \geq 0, \quad (91) \\
  u(\pi, y) &= 0, & y \geq 0, \quad (92) \\
  u(x, 0) &= f(x), & 0 \leq x \leq \pi, \quad (93) \\
  u_y(x, 0) &= g(x), & 0 \leq x \leq \pi. \quad (94)
\end{align}

b) Hence show that the solution for $f(x) = 1$, $g(x) = 0$ is

\begin{equation}
  u(x, y) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin[(2m+1)x] \cos[(2m+1)y] 
\end{equation}

(95)

c) Use Maple, Matlab, Mathematica, or some other software, to plot the solution series with the first 1, 2, 10, 100 terms for $0 \leq y \leq \pi$. Observe that the solution is discontinuous, and that these discontinuities propagate with speeds $\pm 1$.

4. Homework 1a: (Very short) Write out

\begin{equation}
  \nabla \cdot (u \nabla u) = \nabla u \cdot \nabla u + u \Delta u 
\end{equation}

in Cartesian coordinates $(x, y, z)$.

5. Homework 1b: (A bit more work) By applying the chain rule of partial derivatives, transform the 2-dimensional Laplacian in Cartesian coordinates to polar coordinates. In other words, show that

\begin{equation}
  u_{,xx} + u_{,yy} = u_{,rr} + \frac{1}{r} u_{,r} + \frac{1}{r^2} u_{,\theta \theta}. 
\end{equation}

(97)

[Hint: solve for $r(x, y)$ and $\theta(x, y)$ first.]

6. Homework 2: Find the solution of the wave equation $u_{,tt} = c^2 u_{,xx}$ on the line $-\infty < x < \infty$ in terms of the initial data $u(x, 0) = f(x)$, $u_t(x, 0) = g(x)$. [Hint: use the d’Alembert solution.]

7. Homework 3: a) Write down the Cauchy-Kowalewski solution for the one-dimensional wave equation $u_{,tt} = c^2 u_{,xx}$. Use this to find the solution with the Cauchy data $u(x, 0) = \sin x$, $u_t(x, 0) = 0$ as an infinite series. Show that the series sums to $u(x, t) = \cos(ct) \sin(x)$. b) Then write this also as a d’Alembert solution.

8. * Consider the answer to a previous problem,

\begin{equation}
  u(x, y) = \sum_{m=0}^{\infty} \frac{4}{(2m+1)\pi} \sin[(2m+1)x] \cos[(2m+1)y] 
\end{equation}

(98)

Write this as

\begin{equation}
  u(x, y) = E(x + y) + F(x - y) 
\end{equation}

(99)

for two functions $E(z)$ and $F(z)$. Give these functions both as Fourier series, and explicitly. [Hint: to do the latter, note first from the Fourier series that these are periodic functions. Then use part of the answer to Homework 3 to determine what the value of these functions is over the interval $0 \leq z \leq \pi$. Hence work out what they are for all $z$.]
9. * Using separation of variables, find the solution of
\[ -u_{tt} + u_{xx} = 0, \quad a \leq x \leq b, \quad t \geq 0, \]  
\[ \alpha_L u(a, t) - \beta_L u_x(a, t) = 0, \quad t \geq 0, \]  
\[ \alpha_R u(b, t) + \beta_R u_x(b, t) = 0, \quad t \geq 0, \]  
\[ u(x, 0) = f(x), \quad a \leq x \leq b, \]  
\[ u_t(x, 0) = g(x), \quad a \leq x \leq b \]  
(100)
(101)
(102)
(103)
(104)
(wave equation on an interval with Robin BCs). (I have written \(-\beta_L\) because \(u_{,n} = -u_{,x}\) at the left boundary, but that is just a convention.)

Hint: Your main challenge in this problem is to find the basis functions \(X_n(x)\). Sturm-Liouville-theory then tells you that the \(X_n(x)\) must obey the orthogonality conditions
\[
\int_a^b X_n(x) \bar{X}_m(x) \, dx = \begin{cases} 
0 & n \neq m, \\
N_n & n = m,
\end{cases}
\]  
(105)
where \(\bar{X}\) denotes the complex conjugate (if you have chosen to use complex notation for the \(X_n(x)\)) for some coefficients \(N_n > 0\). In your answer you can use \(N_n\) without finding them explicitly – we leave that for the next problem.

10. * Show explicitly by integration that (105) holds for the basis functions \(X_n(x)\) from your solution of Problem 38 and find the coefficients \(N_n\).

Hint: Use integration by parts twice, and the boundary condition.

11. * Using separation of variables, find the solution of
\[ -u_{tt} + u_{xx} = 0, \quad -\infty \leq x \leq \infty, \quad t \geq 0 \]  
\[ u(x, t) \to 0, \quad |x| \to \infty, \quad t \geq 0 \]  
\[ u(x, 0) = f(x), \quad -\infty \leq x \leq \infty, \]  
\[ u_t(x, 0) = g(x), \quad -\infty \leq x \leq \infty, \]  
(106)
(107)
(108)
(109)
(1-dimensional wave equation with free space boundary conditions).

Hint: Here of course you have a Fourier transform instead of a Fourier series, and this problem is much simpler than Problem 38. It is included here for comparison, and to reassure you that Fourier transforms are not difficult.

12. Let
\[ G = \frac{1}{r}, \]  
\[ \text{where } r = \sqrt{x_1^2 + x_2^2 + x_3^2}. \]  
Show that
\[ \Delta_{(3)} G = 0 \text{ if } r > 0. \]  
(110)
(111)

13. Give \(E\) and \(F\) in (72) in terms of the Cauchy data \(u(x, 0) = f(x), \ u_t(x, 0) = g(x)\) for the one-dimensional wave equation \(u_{tt} = c^2 u_{xx}\) on the half-line \(0 \leq x < \infty\) with the Dirichlet boundary condition \(u(0, t) = 0\) at \(x = 0\). [Hint: First find a d’Alembert solution on all of \(-\infty < x < \infty\) that obeys \(u(0, t) = 0\). This will require some relation between the functions \(E\) and \(F\).]
2 Well-posedness

2.1 Main definition

Some boundary conditions are unsuitable for certain types of PDE in that they can lead to unphysical behaviour. For example, Cauchy type conditions are unsuitable for the Laplace equation and Dirichlet conditions are unsuitable for the wave equation. This leads to the notion of a well-posed problem. Problems which arise in practical applications are usually well-posed boundary value problems (for PDEs in space only) or well-posed Cauchy problems (for PDEs in time and space). It is always a PDE problem that is well-posed (or not), not a PDE on its own.

**Definition 2.1.** A PDE problem is well-posed if and only if

1. a solution exists (*existence*);
2. for given data there is only one solution (*uniqueness*);
3. a small change in the data (boundary data, initial data, source terms) produces only a small change in the solution (*continuous dependence on the data*).

Existence and uniqueness are clear, and seem only common sense, but in practice they may be hard to prove, and typically need to be proved separately. Only for very simple problems can one prove existence by explicitly deriving the solution. The third condition, continuity, will be less familiar to you. To give it a precise meaning requires some technical preparation. For motivation, we look at a few simple but illustrative examples of ill-posed problems first.

2.2 Examples of ill-posed problems

2.2.1 Wave equation with Dirichlet boundary conditions: many solutions

Consider the wave equation

\[ u_{tt} = u_{xx}, \quad 0 < x < \pi, \quad 0 < t < \pi, \]  

(112)

with the homogeneous Dirichlet boundary conditions in space

\[ u(0,t) = 0, \quad u(\pi,t) = 0, \]  

(113)

and the homogeneous initial and final Dirichlet boundary conditions in time,

\[ u(x,0) = 0, \quad u(x,\pi) = 0. \]  

(114)

Separation of variables leads us to look for solutions of the form \( u(x,t) = X(x)T(t) \), and it is easy to see that there are infinitely many such solutions which obey all boundary conditions, namely

\[ u(x,t) = A_n \sin nx \sin nt, \]  

(115)

where \( n \) is an integer and \( A_n \) a constant. Thus there are infinitely many solutions to this problem. It is ill-posed.

We have not considered inhomogeneous boundary conditions, or a source term, but we can immediately see that those problems are also ill-posed. Suppose they have a solution (if not, they are ill-posed). As this is a linear PDE, we can add any solution of the problem with homogeneous boundary conditions and zero source term, and get another solution. But, as we have just shown, there are infinitely many of these. So the problem with a source term and/or inhomogeneous boundary conditions is also ill-posed.
2.2.2 Wave equation with Dirichlet boundary conditions: no continuous dependence on the boundary data

Perhaps what was wrong with the previous example was that we imposed final conditions at the special time \( t = \pi \). So let us impose them at a general value \( t = a \) instead, with \( a \) not an integer multiple of \( \pi \). We now also allow for inhomogeneous final conditions. Hence consider

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad 0 < t < a, \tag{116}
\]

with homogeneous Dirichlet boundary conditions in space

\[
u(0, t) = 0, \quad u(\pi, t) = 0, \tag{117}\]

as before, and the homogeneous initial and inhomogeneous final Dirichlet boundary conditions in time,

\[
u(x, 0) = 0, \quad u(x, a) = f(x). \tag{118}\]

It is easy to see that

\[
u(x, t) = \sum_{n=1}^{\infty} A_n \sin nx \sin nt \tag{119}\]

obeys the wave equation and all three homogeneous boundary conditions. Substituting this into the inhomogeneous boundary condition gives

\[
\sum_{n=1}^{\infty} A_n \sin nx \sin na = f(x), \tag{120}\]

and we can solve this using by determining the constants \( \sin na A_n \) as the coefficients of a sine series:

\[
A_n = \frac{1}{\sin na \pi} \int_0^{\pi} \sin nx f(x) \, dx. \tag{121}\]

Hence we have a unique solution for any given function \( f(x) \). For \( f(x) = 0 \) this unique solution is just \( \nu(x, t) = 0 \). So we avoid the problem of the previous example, as we suspected.

But now there is another problem instead. Focus on the factor \( 1/\sin na \). As we have assumed that \( a \) is not an integer multiple of \( \pi \), \( \sin na \) is not zero for any \( n \). But how small can it be? One can prove that it can become arbitrarily small, and hence \( A_n \) can become arbitrarily large, in the following technical sense. Fix any \( \epsilon > 0 \), as small as you like. Then there exists some positive \( n \) (which depends on \( \epsilon \)), such that \( |\sin na| < \epsilon \). Now consider \( f(x) = \sin nx \). The absolute value of these data is bounded by \( 1 \). But the solution is bounded only by \( 1/\epsilon \), which we can make arbitrarily large. In this sense, the solution does not depend continuously on the data. The problem is ill-posed.

Note that it was sufficient to show ill-posedness to look at one (suitably chosen) particular family of data, here \( f(x) = \sin nx \).

2.2.3 Cauchy problem for the Laplace equation: no continuous dependence on the boundary data

We now look at another problem that is ill-posed because its solution does not depend continuously on the data. Consider the solution of the Laplace equation

\[
u_{xx} + \nu_{yy} = 0, \quad 0 < x < \pi, \quad y > 0, \tag{122}\]

with the homogeneous Dirichlet boundary conditions

\[
u(0, y) = 0, \quad \nu(\pi, y) = 0, \tag{123}\]

and the Cauchy data

\[
u(x, 0) = 0, \quad \nu_y(x, 0) = \sin nx. \tag{124}\]
where $n > 0$ is an integer. Separation of variables suggests that we look for a solution of the form $u(x, y) = X(x)Y(y)$. Doing this, and taking into account all boundary conditions, we find the unique solution

$$u(x, y) = \frac{1}{n} \sin nx \sinh ny. \quad (125)$$

Now, as $n \to \infty$, the Cauchy data remain finite and bounded between 1 and $-1$, but the solution for $y > 0$ diverges at almost every $x$, because for any $y > 0$,

$$\lim_{n \to \infty} \frac{\sinh ny}{n} = \infty. \quad (126)$$

[We say at almost every $x$, because obviously $u(m\pi/n, y) = 0$, for any integer $m$.] Hence once again, we have given an example of a family of data (with parameter $n$) for which the solution does not depend continuously on the data, and so the problem is ill-posed.

### 2.3 Continuous dependence on the data

We now introduce some concepts that allows us to define the intuitive notion in which the solution did not depend continuously on the data in the previous example.

**Definition 2.2.** A real vector space $V$ is a set whose elements (called vectors) can be added and multiplied by real numbers (called scalars in contrast to vectors). Hence if $X, Y \in V$ and $c \in \mathbb{R}$ then $cX \in V$ and $X + Y \in V$.

**Example 2.3.** The prototypical vector space that you know already is $\mathbb{R}^n$. However, when using this example keep in mind that a norm or an inner product is not part of the definition of a vector space.

**Example 2.4.** Functions from $\mathbb{R}$ to $\mathbb{R}$ also form a vector space, with $f + g$ defined by $(f + g)(x) := f(x) + g(x)$ and $cf$ defined by $(cf)(x) = cf(x)$. More generally, we could consider the vector space of functions from $\mathbb{R}^n$ to $\mathbb{R}^N$, or in other words, vector-valued functions on $\mathbb{R}^n$. Clearly a vector space of functions is infinite-dimensional in the sense that it is not spanned by a finite number of basis vectors, but we can still do many of the same things that we can do with finite-dimensional vector spaces such as $\mathbb{R}^n$.

**Definition 2.5.** A norm on a real vector space $V$ is a function from $V$ to the non-negative real numbers with the properties

1. $\|X\| = 0$ if and only if $X = 0$;
2. $\|cX\| = |c| \|X\|$ for all $X \in V$ and $c \in \mathbb{R}$;
3. $\|X + Y\| \leq \|X\| + \|Y\|$ for all $X, Y \in V$ (the triangle inequality).

**Example 2.6.** The Euclidean norm, also called the $l^2$ norm, on $\mathbb{R}^n$ is given by

$$|x| := \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \quad (127)$$

I use $|x|$ instead of $\|x\|$ for this norm because that is the established notation for this particular norm. Also, for $n = 1$, this norm is just the absolute value of real numbers. Convince yourself that the square root is necessary for property 2 above to hold.

This is the most important norm on finite-dimensional vector spaces, and we mention two more only to illustrate the concept of norm.

**Example 2.7.** The maximum norm on $\mathbb{R}^n$ is given by

$$\|x\|_{\text{max}} := \max(|x_1|, |x_2|, \ldots, |x_n|), \quad (128)$$

where of course $|x_1|$ stands for the absolute value.
Example 2.8. The $l^1$ norm on $\mathbb{R}^n$ is given by

$$
\|x\|_1 := |x_1| + |x_2| + \cdots + |x_n|.
$$

(129)

Convince yourself that properties 1 and 2 above hold for these norms. (Property 3 is a little harder to check.)

We will only introduce one norm on a function space:

Example 2.9. The $L^2$ norm on functions from $\mathbb{R}^n$ to $\mathbb{R}^N$ defined by

$$
\|f(\cdot)\|_{L^2} := \sqrt{\int_{\mathbb{R}^n} |f(x)|^2 \, dx},
$$

(130)

where $|f(x)|$ denotes the Euclidean norm on $\mathbb{R}^N$. The notation $f(\cdot)$ indicates that the function norm depends on the value of $f(x)$ for all $x$. Functions from $\mathbb{R}^n$ to $\mathbb{R}$ with finite $L^2$ norm are said to be in the vector space $L^2(\mathbb{R}^n)$, but there is no space in this course to give a formal definition.

Definition 2.10. The function $f$ from the vector space $V$ with norm $\|\cdot\|_V$ to the vector space $W$ with norm $\|\cdot\|_W$ is Lipshitz continuous with Lipshitz constant $K > 0$ if

$$
\|f(X_1) - f(X_2)\|_W \leq K\|X_1 - X_2\|_V
$$

(131)

for all pairs of vectors $X_1, X_2 \in V$.

Remark 2.11. You will get the right intuitive idea if you simply take both $V$ and $W$ to be $\mathbb{R}$ with the usual absolute value of real numbers $|x|$ as the norm: the change in $f$ cannot be bigger than $K$ times the change in $x$.

We are now ready to formally define continuous dependence on the initial data. For simplicity, we consider an evolution problem that is first order in time and has homogeneous BCs, so that $u(x,0)$ are the only data. Consider two solutions $u_1(x,t)$ and $u_2(x,t)$ with initial data $u_1(x,0)$ and $u_2(x,0)$, respectively.

Definition 2.12. The solution $u(x,t)$ of a first-order in time evolution problem depends continuously on the initial data $u(x,0)$ in the function norm $\|\cdot\|$ if and only if there exists a function $K(t) > 0$ such that

$$
\|u_1(\cdot,t) - u_2(\cdot,t)\| \leq K(t)\|u_1(\cdot,0) - u_2(\cdot,0)\|
$$

(132)

for all pairs of initial data $u_1(x,0), u_2(x,0)$, and with $K(t)$ independent of these data.

Note that continuity, and hence well-posedness, is defined only in some particular function norm. An inequality such as (132) in PDE theory is called an estimate. The notion of continuity we have used here is Lipshitz continuity, where the Lipshitz constant is allowed to depend on $t$. The choice of function norm is an art, and depends on the problem in hand. (You will not be asked to find estimates in this course, just to check and use them.)

Remark 2.13. I have kept the definition as simple as possible by restricting it to a first-order in time evolution problem, and by using the same norm for the solution as for the initial data. In practice, it may be necessary to use different function norms on the initial data and the solution in order to prove an estimate at all, or in order to get the most useful one. For second-order in time evolution problems, we need a norm on $u$ and $u_t$, for the initial data, and if there are boundary data, we need a norm for those as well, and the solution must be bounded by the sum of all these norms. For a system of PDEs, the norm combines a norm over state space with a function norm, as in the example (130).

Proposition 2.14. Continuous dependence on all the data implies uniqueness.

Proof. We again restrict to a problem with only initial data for simplicity. If there are two solutions $u_1$ and $u_2$ that have the same data, then the right-hand side of (132) is zero. As the left-hand side cannot be negative, it must also be zero, so $u_1 - u_2$ has zero norm for all $t \geq 0$. But the only function that has zero norm is the zero function, and so $u_1(x,t) - u_2(x,t)$ vanishes for all $t$. Hence $u_1 = u_2$ for all $t$. \qed
Remark 2.15. If the PDE or PDE system is linear, then the difference $u = u_1 - u_2$ of any two solutions $u_1$ and $u_2$ is itself a solution of the corresponding homogeneous problem (zero inhomogeneous term) with homogeneous BCs (zero boundary data). Hence we can reformulate the continuity condition for linear problems as
\[
\|u(\cdot, t)\| \leq K(t)\|u(\cdot, 0)\|
\] (133)
for all initial data $u(\cdot, 0)$, with $K(t)$ independent of these data.

Remark 2.16. The condition (133) can also be viewed as a form of stability with respect to the initial data, where by stability we mean that the solution cannot grow arbitrarily rapidly. It is clear that often solutions will grow in time, so we cannot demand that $K(t)$ is a constant. But it is essential that it is independent of the initial data $u(x, 0)$, or else the condition would be trivial. In fact, a typical way in which an evolution problem fails to be well-posed is that initial data that vary more rapidly in space lead to solutions that grow more quickly in time. We have seen an example of this in Sec. 2.2.3.

Remark 2.17. In fact, (132) is often too strong for nonlinear problems. Instead one may want the problem to be well-posed only for solutions $u$ close to a given reference solution $u_0$. The relevant criterion is then
\[
\|\delta u(\cdot, t)\| \leq K(t)\|\delta u(\cdot, 0)\|,
\] (134)
where $\delta u := u - u_0$ denotes any sufficiently small perturbation about a reference solution $u_0$. One then says that the problem is well-posed in a neighbourhood of $u_0$.

2.4 Examples of well-posedness results

2.4.1 An energy estimate for the heat equation

Consider now the 1-dimensional heat equation
\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\] (135)
on all of space. (For simplicity we have set $\kappa = 1$.) Define the “energy”
\[
E(t) := \frac{1}{2} \int_{-\infty}^{\infty} u^2 \, dx.
\] (136)
and again consider initial data where $E(0)$ is finite. We then have
\[
\frac{dE}{dt} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u^2 \, dx
= \int_{-\infty}^{\infty} uu_t \, dx
= \int_{-\infty}^{\infty} uu_{xx} \, dx
= -\int_{-\infty}^{\infty} u_x^2 \, dx \leq 0.
\] (137)
In the first equality we have differentiated under the integral, and in the second we have used the product rule. We have used (135) in the third equality, and in the fourth equality we have integrated by parts and used that finite $E$ implies that $u$ vanishes $x = \pm \infty$. Hence we have obtained the estimate
\[
\|u(\cdot, t)\|_E \leq \|u(\cdot, 0)\|_E
\] (138)
in the norm defined by
\[
\|u(\cdot, t)\|_E := \sqrt{E(t)}.
\] (139)
The square root is required here to obtain the property $\|cu\| = |c| \|u\|$ of a norm. In this example we have $K(t) = 1$, and we use the same norm for the initial data and the solution, but neither is typical.

With this estimate, we have shown continuous dependence of the solution on the initial data in the energy norm, for (differentiable) solutions. From Prop. 2.14 we also have uniqueness for (differentiable) solutions.

### 2.4.2 An energy estimate for the wave equation

Consider the 1-dimensional wave equation

$$u_{tt} = u_{xx}$$

(140)
on all of space. (For simplicity we have set $c = 1$.) Define the energy

$$E(t) := \frac{1}{2} \int_{-\infty}^{\infty} (u_{t}^2 + u_{x}^2) \, dx,$$

(141)
and consider initial data where $E(0)$ is defined and is finite. We then have

$$\frac{dE}{dt} = \frac{1}{2} \int_{-\infty}^{\infty} (u_{t} u_{tt} + u_{x} u_{xt}) \, dx$$

(142)

In the first equality we have differentiated under the integral, and in the second we have used the product rule. We have used (140) in the third equality, and in the fourth equality we have used the fact that finite $E$ implies that $u_{t}$ and $u_{x}$ must vanish at $x = \pm \infty$. This gives us the estimate

$$\|u(\cdot, t)\|_{E} = \|u(\cdot, 0)\|_{E}$$

(143)
in the energy norm defined by

$$\|u(\cdot, t)\|_{E} := \sqrt{E(t)}.$$  

(144)

As for the heat equation, we have $K(t) = 1$, and we use the same norm for the initial data and the solution. The inequality (133) is actually a strict equality here. Finally, we note that $E$ is really the physical energy (potential energy plus kinetic energy) carried by the wave. This has lent the name “energy norm” to this type of function norm.

As for the heat equation, we have shown continuous dependence on the initial data, and hence uniqueness, for strong (differentiable) solutions.

### 2.4.3 Uniqueness of solutions of the boundary value problem for the Poisson equation

**Theorem 2.18.** The solution of the Poisson equation (53) with either Dirichlet boundary conditions, or Robin boundary conditions (54) with $\alpha \beta > 0$, is unique. With Neumann boundary conditions, the solution is unique up to addition of a constant.

**Proof.** Suppose we have two solutions $u_1$ and $u_2$ that have the same boundary conditions and source term but differ in $V$. We define their difference

$$u := u_1 - u_2.$$  

(145)

Convince yourself that if $u_1$ and $u_2$ are solutions of the Poisson equation with some inhomogeneous boundary condition, $u$ satisfies Laplace’s equation with the homogeneous boundary condition of the same type (say Dirichlet) as in the original problem. Hence we have $\Delta u = 0$, so that

$$\int_{V} u \Delta u \, dV = 0.$$  

(146)
Now $\nabla \cdot (u \nabla u) = u \Delta u + \nabla u \cdot \nabla u$, so that

$$\int_V \left[ \nabla \cdot (u \nabla u) - |\nabla u|^2 \right] dV = 0$$

(147)

for any volume $V$. The divergence theorem gives us

$$\int_V \nabla \cdot (u \nabla u) dV = \int_S uu_n dS,$$

(148)

and hence

$$\int_V |\nabla u|^2 dV = \int_S uu_n dS$$

(149)

for any volume $V$ with surface $S$. Now if our original boundary conditions were inhomogeneous Dirichlet boundary conditions then $u = 0$ on the boundary, and if they were Neumann, then $u_n = 0$ on the boundary. Either way, the right-hand side of (149) vanishes. Finally, if $\alpha \beta > 0$, then $uu_n = -(\alpha/\beta)u^2 \leq 0$. But then the left-hand side of (149) is non-negative and the right-hand side is non-positive, so both must be zero. Hence $u$ must be constant. But for either Dirichlet or Robin boundary conditions, this constant must be zero. Hence $u_1 = u_2$, and the solution is unique. For Neumann boundary conditions, $u = u_1 - u_2$ can be constant, and so the solution of the Neumann problem is unique only up to addition of a constant.

Remark 2.19. Here we have assumed that the solutions are twice differentiable, so we have proved uniqueness only in the space of strong solutions.

2.5 The importance of well-posedness

Why is well-posedness important? Obviously we want existence. What about uniqueness?

Consider, for example, an elastic pole loaded with a weight on top, and look for equilibrium solutions. There is one solution where it is straight. This is an equilibrium but it is unstable to buckling. The buckling can happen into any direction. So here multiple solutions of a problem that is formally ill-posed make some sense. The mathematical problem is ill-posed because we have idealised the physical problem, and some physical information is missing. (A real pole buckles in a particular direction because it is not completely round, or perhaps there is a wind pushing it, or something else breaks the symmetry.)

If the solution exists and is unique but if arbitrarily small changes in data result in large changes in the solution, then it is more likely that the solution is useless for engineering and science because, in practice all data come from measurements which have small errors in them. Similarly, any numerical solution of the PDE is only an approximation and so will have small errors in it. Consider some initial data at $t_0$ and evolve them to $t_1$. The final data at $t_1$ then have some numerical error in them, and so we have a small random change of initial data when we evolve further from $t_1$ to $t_2$. If this gives rise to a very large change in the solution later on, the numerical solution becomes nonsense.

There is a large mathematical literature on the well-posedness or ill-posedness of PDEs. You need to understand the basic concept of well-posedness so that you know when to worry about possible ill-posedness and consult this literature. There are several reasons why an ill-posed PDE problem may not be recognised as such:

- The mathematical problems that you encounter in this course are both standard problems, and well-posed, but in modelling an engineering system, a system of ODEs and PDEs may well arise that has never been investigated mathematically before and that may or may not be well-posed.
- Many PDEs or systems of PDEs that one can write down are neither elliptic, hyperbolic or parabolic, and then the question of well-posedness becomes tricky.
- A physical or engineering problem may be “physically well posed” or may “clearly have a unique solution” but the mathematical problem one solves may be ill-posed because it does not completely reflect the physical situation one had in mind. As a (further) example, the
solutions of the Navier-Stokes equations in the limit of small viscosity are completely different, mathematically and in fact (they have boundary layers), from solutions of the Euler equations (where the viscosity is set to zero), but the only difference is a tiny amount of dissipation that one might naively consider irrelevant.

- Trying to solve an ill-posed problem numerically, there is a risk that you get an answer that looks reasonable, but is actually nonsense. However, there is one important clue: a numerical solution to a well-posed problem will converge with increasing resolution (finer numerical mesh). A numerical simulation to an ill-posed problem will actually get worse, by the mechanism of Sec. 2.2.3: a finer mesh allows for sin nx with larger n to be resolved.

2.6 Exercises

14. Homework 4: Find the solution of the Cauchy problem (sic!) for the Laplace equation on the half-plane
\[ u_{xx} + u_{yy} = 0, \quad -\infty < x < \infty, \quad y \geq 0, \quad (150) \]
\[ u(x, 0) = f(x), \quad u_y(x, 0) = g(x), \quad -\infty < x < \infty, \quad (151) \]
\[ u(x, y) \to 0 \quad \text{as} \quad x \to \pm \infty. \quad (152) \]

Show that it is ill-posed (because the solution can grow arbitrarily rapidly compared to the initial data).

15. Homework 5: Show that the Cauchy problem for the backwards in time heat equation
\[ u_t = -\kappa u_{xx} \quad \text{with} \quad \kappa > 0 \]
is ill-posed, because the solution cannot depend continuously on the initial data in any possible function norm over x. [Hint: to focus on the essence of the argument, neglect boundary conditions.]

16. Homework 6: Derive the values of the constants \( \alpha_L, \beta_L, \alpha_R \) and \( \beta_L \) for which the 1-dimensional heat equation on the interval \([a, b]\) with Robin boundary conditions
\[ u_t = u_{xx}, \quad a \leq x \leq b, \quad t \geq 0, \quad (153) \]
\[ \alpha_L u(a, t) + \beta_L u_x(a, t) = 0, \quad t \geq 0, \quad (154) \]
\[ \alpha_R u(b, t) + \beta_R u_x(b, t) = 0, \quad t \geq 0, \quad (155) \]
is well-posed in the energy norm
\[ \|u(\cdot, t)\|_{\tilde{E}} := \sqrt{\tilde{E}(t)}, \quad \text{where} \quad \tilde{E}(t) := \int_a^b \frac{1}{2} u^2 \, dx. \quad (156) \]

17. Homework 7: (Straightforward) State and prove an energy estimate for the wave equation on \( \mathbb{R}^2 \),
\[ u_{tt} = u_{xx} + u_{yy}, \quad -\infty < x < \infty, \quad -\infty < y < \infty \quad (157) \]
where \( u \) and its derivative vanish as \( x \to \pm \infty \) or \( y \to \pm \infty \).

18. * (Hard) In a previous problem we showed that the Cauchy problem for the backwards in time heat equation \( u_t = -\kappa u_{xx} \) with \( \kappa > 0 \) is ill-posed, because the solution does not depend continuously on the initial data. In the model answer we neglected boundary conditions. Extend that answer assuming that the domain is \( 0 \leq x \leq \pi, \quad t \geq 0 \), with initial data and inhomogeneous Dirichlet boundary data
\[ u(x, 0) = f(x), \quad u(0, t) = g(t), \quad u(\pi, t) = h(t). \quad (158) \]

[Hint: write \( u = u_1 + u_2 \), where
\[ u_1(x, 0) = f(x), \quad u(0, t) = 0, \quad u(\pi, t) = 0 \quad (159) \]
and
\[ u_2(x, 0) = 0, \quad u_2(0, t) = g(t), \quad u_2(\pi, t) = h(t). \quad (160) \]
so that \( u_1 \) takes care of the initial data and \( u_2 \) of any inhomogeneous boundary conditions. You can use the triangle inequality

\[
||X + Y|| \geq ||X|| - ||Y||
\]

(161)

(the right-hand side denotes the absolute value of the difference of the two norms).]

19. How would you define the \( l^p \) norm, where \( p \) is any positive integer, such that \( l^1 \) and \( l^2 \) give the special cases we have already seen?

20. With the answer to the previous question, can you show that the limit as \( p \to \infty \) of the \( l^p \) norm is the maximum norm? (It is in fact also called the \( l^\infty \) norm).

21. For a function \( f \) from an interval \( I \) to \( \mathbb{R} \), show that

\[
differentiable in I \Rightarrow \text{Lipschitz continuous in } I \Rightarrow \text{continuous in } I
\]

(162)

and find \( K \) for the case where \( f \) is differentiable.

22. Show that the energy norm for the wave equation can be written as

\[
E = \sqrt{||u(\cdot,t)||^2 + ||u_x(\cdot,t)||},
\]

(163)

where \( || \cdot || \) denotes the \( L^2 \) norm. Assuming that both the \( L^2 \) norm of functions and the \( l^2 \) norm of vectors in \( \mathbb{R}^2 \) (that is \( |X| = \sqrt{X_1^2 + X_2^2} \)) obey the three conditions for a norm, show this for the energy norm.
3 Classification of PDEs from their symbol

3.1 Introduction

In Sec. 1.5 we highlighted some typical features of what we called, provisionally, elliptic, parabolic and hyperbolic PDEs. In Sec. 2.2 we saw examples of boundary conditions that make these PDEs well-posed or ill-posed. Both are closely related to the coefficients of the highest derivatives in the PDE.

We begin by looking at three examples of linear scalar PDEs with constant coefficients, namely
\[ u_{xx} + u_{yy} = 0, \]
\[ u_{xx} - u_{yy} = 0 \] and
\[ u_{xx} - u_y = 0 \] (elliptic, hyperbolic, and parabolic). These examples are supposed to give you an intuitive feeling for the relation between the coefficients of the highest derivatives and the qualitative behaviour of solutions, and how this determines which boundary conditions are appropriate.

From Sec. 3.2 on we then look at the formal theory, initially again for scalar linear PDEs with constant coefficients (as in the examples). Our key tool for this will be the complex Fourier transformation. We then generalise from scalar PDEs to systems.

Next (with some handwaving), we generalise from linear PDEs (or PDE systems) with constant coefficients to general linear PDEs (or systems). From there (with a lot of handwaving), we generalise to nonlinear PDEs (or systems).

We do not provide any well-posedness theorems for the class of PDEs that we define, as stating them rigorously goes beyond this course.

3.1.1 A model elliptic PDE

Consider the Laplace equation in two dimensions,
\[ u_{xx} + u_{yy} = 0, \] (164)
initially without worrying about the domain or boundary conditions. Using separation of variables, we find the elementary solutions
\[ u(x, y) = \sin kx \sinh ky, \] (165)
for any real constant \( k \). We could also replace \( \sin x \) by \( \cos x \), and/or \( \sinh y \) by \( \cosh y \), giving us four possibilities for each value of \( k \). We can write this in complex notation as
\[ u(x, y) = e^{\pm ikx \pm ky}, \] (166)
giving again four possible combinations. (We could also interchange \( x \) and \( y \), but that would confuse the following argument.)

Focus now on the behaviour of \( \sin kx \sinh ky \). This solution remains finite and oscillates in the \( x \)-direction, with periodic zeros, and it increases in the \( y \)-direction, with only one zero. We can make the solution vanish at two values of \( x \) and at one value of \( y \), but not at two values of \( y \).

This example solution is typical in that there is no solution that vanishes on all four sides of a rectangle in the \( xy \)-plane. But this proves uniqueness of the Dirichlet problem on a rectangle for this PDE: if there were two different solutions that obeyed the same Dirichlet boundary conditions, their difference would be a non-zero solution that vanished everywhere on the boundary. Uniqueness is actually true for the Laplace equation on a domain \( V \) of arbitrary shape, and with other types of boundary condition, as we have already shown in Sec. 2.4.3.

Now consider the Cauchy problem for the same PDE, considering \( y \) as the time variable. Because the PDE is second order, we should be able to specify both \( u \) and \( u_y \) on \( y = 0 \). But now we run into the problem we have encountered already in Sec. 2.2.3: There is a unique solution for these Cauchy data, but it grows without bound as \( k \to \infty \) because \( \sinh ky \to \infty \) for any \( y > 0 \). Hence there is no continuous dependence on the initial data.

It is very important to understand that the problem is not that the solution grows exponentially with increasing \( y \). The problem is that it grows without bound with \( k \) at fixed \( y \). As \( u \) can grow as \( \sinh ky \), for any \( k \), the same is true for \( \|u(\cdot, y)\| \). This means that there can be no norms \( \|u\|_{\text{data}} \) and \( \|u\|_{\text{solution}} \) such that
\[ \|u(\cdot, y)\|_{\text{solution}} \leq K(y) \|u(\cdot, 0)\|_{\text{data}}, \] (167)
where $K(y)$ does not depend on the data and so does not depend on $k$ in this example.

But how do we know that there is not some norm for which (167) does hold? From the property of a norm that $\|cu\| = |c|\|u\|$ for any constant $c$ it follows that any norm over $x$ of $\sin kx \sinh ky$ must be proportional to $\sinh ky$. So we would have the problem of unlimited growth in any norm.

The key property of elliptic PDEs is that every solution must grow in some direction. This gives us uniqueness of the Dirichlet problem but also destroys the continuous dependence on the data of the Cauchy problem: the well-posed problem for this PDE is a boundary value-problem, not a Cauchy problem.

### 3.1.2 A model parabolic PDE

Consider now the diffusion equation in one dimension

$$u_{xx} - u_{yy} = 0. \tag{168}$$

An elementary solution is

$$u(x, y) = \sin kxe^{-k^2y}, \tag{169}$$

for any real constant $k$. We could also replace $\sin$ by $\cos$. In complex notation,

$$u(x, y) = e^{\pm ikx - k^2y}. \tag{170}$$

This solution decays with $y$ for all $k$. For larger $k$ (more rapidly changing initial data), it just decays faster. Essentially for this reason, the Cauchy problem is in fact well-posed for $y > 0$, as we have seen in Sec. 2.4.1. But the Cauchy problem would be ill-posed for $y < 0$ (the backwards heat equation), because then the solution grows with $k$ without bound at fixed $y$.

The Dirichlet problem on a rectangle in the $xy$-plane is also ill-posed, simply because the solution is already determined by data on, for example, $y = 0$, $x = 0$ and $x = \pi$: there is no freedom to specify boundary data at any $y > 0$.

The key property of a parabolic equation determining all this is that every solution decays in one and the same direction (time).

### 3.1.3 A model hyperbolic PDE

Consider now the wave equation in one space dimension,

$$u_{xx} - u_{yy} = 0. \tag{171}$$

An elementary solution is

$$u(x, y) = \sin kx \sin ky, \tag{172}$$

where again we can replace one or both sines by cosines, or in complex notation

$$u(x, y) = e^{\pm ikx \pm iky}. \tag{173}$$

We saw in Sec. 2.2.1 that the Dirichlet problem on the rectangle $0 \leq x \leq \pi$, $0 \leq y \leq \pi$ is ill-posed because it does not have unique solutions. This is because $u(x, y) = \sin kx \sin ky$ vanishes on the four sides of this rectangle for $k = 1, 2, \ldots$. Intuitively, this is possible because this solution oscillates, with repeating zeros, in both the $x$ and $y$-direction. We saw that if we choose a square, rather than a generic rectangle, there is no zero solution, but the solution can still become arbitrarily large compared to the boundary data. So then the problem is ill-posed because we lack continuous dependence on the initial data. Either way, the Dirichlet problem is ill-posed.

We also saw from the d’Alembert solution that any Cauchy data $u(x, 0) = f(x)$ and $u_y(x, 0) = g(x)$ give rise to a unique solution. Basically, we can reassemble $g(x)$ and $g(x)$ into a right-moving wave $F(x - y)$ and a left-moving wave $G(x + y)$. We get continuous dependence on initial data for the Cauchy problem because the solution oscillates but does not grow. We showed this more formally in Sec. 2.4.2.

The key property of a (strongly or strictly hyperbolic) PDE is that there are enough solutions oscillating in all directions (space and time) to split Cauchy data into “waves”.

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3.2 The symbol of a linear PDE

3.2.1 Definition of the symbol

Definition 3.1. We can write any PDE system in the abstract form

\[ F(x, \nabla, u) = 0, \quad (174) \]

meaning some (vector-valued) function \( F \) of \( x, u \) and the partial derivatives of \( u \) with respect to \( x \) vanishes. If the PDE system is linear, we stress this by writing

\[ L(x, \nabla)u = f, \quad (175) \]

where \( L \) is a linear derivative operator acting to the right on \( u(x) \). If we have a system, then the coefficients of \( L \) are matrix-valued in state space and \( Lu \) is a vector in state space, so \( Lu = 0 \) represents \( N \) equations for \( N \) unknowns \( u \). \( L^p \) denotes the principal part of \( L \), the highest derivatives.

Definition 3.2. The symbol of the linear differential operator \( L(x, \nabla) \) is the algebraic expression \( L(x, i\kappa) \). (For a system of PDEs, the symbol is matrix-valued). The principal symbol \( L^p(x, i\kappa) \) is just the symbol of the principal part.

Example 3.3. The homogeneous second-order linear PDE

\[ \sum_{ij} a^{ij}(x) u_{ij} + \sum_i b^i(x) u_i + c(x) u = 0 \quad (176) \]

can be written in operator form as \( Lu = 0 \), where

\[ L(x, \nabla) = \sum_{ij} a^{ij}(x) \frac{\partial}{\partial x^j} + \sum_i b^i(x) \frac{\partial}{\partial x^i} + c(x). \quad (177) \]

(To avoid ambiguity about what the partial derivatives act on, we write all coefficients to the left of all partial derivatives, so that the partial derivatives act only on \( u \).) Its symbol is therefore

\[ L(x, ik) = - \sum_{ij} a^{ij}(x) k_i k_j + i \sum_i b^i(x) k_i + c(x), \quad (178) \]

where \( k_i \) are the components of the (co)vector \( k \). This is simply a quadratic expression in \( k \). The principal symbol is

\[ L^p(x, ik) = - \sum_{ij} a^{ij}(x) k_i k_j, \quad (179) \]

that is, a homogeneous quadratic expression in \( k \).

3.2.2 The Fourier transform in \( \mathbb{R}^n \)

The motivation for introducing the symbol of a PDE, and in particular for including the factor \( i \) in \( ik \), is this. For any (vector-valued) function \( u(x) \) defined on all of \( \mathbb{R}^n \), we define its Fourier transform

\[ \hat{u}(k) := (2\pi)^{-n/2} \int_{\mathbb{R}^n} u(x) e^{-ik \cdot x} d^n x, \quad (180) \]

where

\[ k \cdot x := \sum_{i=1}^n k_i x^i \quad (181) \]

is the standard inner product on \( \mathbb{R}^n \). \( k \) is called the wave number (even though for \( n > 1 \) is not a number but a (co)vector). If the Fourier transform exists, one can show that it has the unique inverse

\[ u(x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{u}(k) e^{ik \cdot x} d^n k. \quad (182) \]
In the case \( n = 1 \), the formulas (180) and (182) reduce to the formulas (47) and (46) above.

Now assume that the coefficients of the PDE (system) are constant (independent of \( x \)). Hence we can write \( L(x, \nabla) = L(\nabla) \). If the Fourier transform of \( Lu \) exists, then it is given by

\[
(\hat{L(\nabla)}u)(k) = L(ik) \hat{u}(k)
\]  

(183)

In words, one obtains the Fourier transform of \( Lu \) by simply multiplying \( \hat{u} \) by the symbol of \( L \). This is easy to see by differentiating under the integral in the Fourier transform.

Clearly, (183) still holds if instead of the Fourier transform on an infinite domain we have a complex Fourier series on a finite domain. The only difference is that \( k \) can then only take discrete real values.

3.3 Classification of linear PDEs

It turns out that we can formulate conditions for a scalar PDE or PDE system to admit well-posed PDE problems in terms of algebraic conditions on its principal symbol. Two of these conditions are ellipticity and strict hyperbolicity. To formulate complete conditions for the well-posedness of a PDE problem, we also need to take into account the domain and the boundary conditions, of course.

In our first definition of ellipticity, Def. 3.4 we will consider a linear PDE with constant coefficients. We then relax the assumption of constant coefficients in the more general definition of ellipticity, Def. 3.9, and also in all following definitions in this Section, but we keep the assumption of linearity. Finally, in Sec. 3.4 we also relax the assumption of linearity.

3.3.1 Ellipticity

Definition 3.4. (Provisional definition) The linear PDE \( Lu = 0 \) in \( n \) independent variables with constant coefficients is called elliptic if and only if

\[
L^p(ik) \neq 0 \quad \forall k \in \mathbb{R}^n \neq 0.
\]

(184)

Remark 3.5. Look for plane wave solutions of the form

\[
u(x) = e^{ik \cdot x},
\]

(185)

with \( k \) real, that is, solutions that oscillate in every direction. Substituting this into the PDE \( Lu = 0 \) gives

\[
L(\nabla)u(x) = 0 \iff L(ik) = 0.
\]

(186)

Hence for a PDE (system) with constant coefficients, the absence of real \( k \) that solve \( L(ik) = 0 \) means that there are no plane wave solutions that oscillate in every direction – any plane wave solution must be growing exponentially in at least one direction (the direction in which \( k \) has an imaginary part). But the general solution is made from a superposition of plane wave solutions, via a Fourier transform or Fourier series. We saw in Sec. 3.1.1 how this growth in at least one direction gives both uniqueness of the boundary problem and ill-posedness of the Cauchy problem.

Remark 3.6. Just in passing we also mention the maximum principle: roughly speaking this states that the solution of an elliptic problem on a bounded region takes its maximum value on the boundary. You can see intuitively why this might be so: a typical solution like \( e^{ikx}e^{iky} \) is largest when \( y \) is largest, which will be somewhere on the boundary.

Remark 3.7. Consider now a linear PDE system, still with constant coefficients, and look for plane wave solutions of the form

\[
u(x) = e^{ik \cdot x} r,
\]

(187)

where \( r \) is a constant vector in state space. (But \( x \) and \( k \) are vectors in physical space!) Then

\[
L(\nabla)u = 0 \iff L(ik)r = 0 \implies \det L(ik) = 0.
\]

(188)

The middle equality states that \( r \) is an eigenvector of the matrix \( L(ik) \) with eigenvalue 0, and the last equality is a necessary condition for this. Hence if there is no real \( k \) such that \( \det L(ik) = 0 \), there can be no oscillating plane wave solutions. This motivates the following definition.
**Definition 3.8.** (Provisional definition) The linear PDE system with constant coefficients $Lu = 0$ for $N$ dependent variables in $n$ independent variables is called **elliptic** if and only if
\[ \det L^p(i\mathbf{k}) \neq 0 \quad \forall \mathbf{k} \in \mathbb{R}^n \neq 0. \] (189)

#### 3.3.2 The high-frequency approximation

What about linear PDEs with coefficients that do depend on $x$? If a PDE fails to be well-posed, it is often because solutions are badly behaved when they vary very rapidly in space (or space and time). For example, we saw in Sec. 3.1.1 that continuous dependence of the solution on the data typically breaks down because the solution becomes arbitrarily large relative to its data if we consider arbitrarily large wave numbers $k$.

Denote by $\ell_{\text{coeff}}$ the typical distance on which the coefficients of the PDE vary. Consider solutions $u(x)$ which vary on a typical length scale $\ell_{\text{soln}}$, with $\ell_{\text{soln}} \ll \ell_{\text{coeff}}$. (190)

We shall refer to this as the **high-frequency approximation**. We can then consider the coefficients of the PDE as approximately constant. The resulting Fourier transform is then $L(i\mathbf{k})\hat{u}$. Furthermore, because the solution is rapidly varying, this is dominated by $L^p(i\mathbf{k})\hat{u}$, because when $|\mathbf{k}|$ is large, a higher power of it is much larger. This motivates a new definition, where a PDE is elliptic at a point $x$ if it is elliptic taking into account only the highest derivatives, and “freezing” the coefficients of the PDE at their values at $x$. Hence we have the following two definitions.

**Definition 3.9.** (Generalises and replaces Def. 3.4) The linear PDE (now with variable coefficients) $L(x, \nabla)u = 0$ is called **elliptic at the point** $x$ if and only if
\[ L^p(x, i\mathbf{k}) \neq 0 \quad \forall \mathbf{k} \in \mathbb{R}^n \neq 0. \] (192)

**Definition 3.10.** (Generalises and replaces Def. 3.8.) The linear PDE system $L(x, \nabla)u = 0$ is called **elliptic at the point** $x$ if and only if
\[ \det L^p(x, i\mathbf{k}) \neq 0 \quad \forall \mathbf{k} \in \mathbb{R}^n \neq 0. \] (191)

To save repetition, in the following subsections we will define hyperbolicity and parabolicity (at a point) immediately for PDEs and PDE systems with variable coefficients, and not bother with constant coefficients.

#### 3.3.3 Strict hyperbolicity

**Definition 3.11.** The linear PDE $Lu = 0$ in $n$ independent variables is called **strictly hyperbolic** in the time direction $\mathbf{n}$ at the point $x$ if and only if
\[ L^p(x, i\mathbf{n}) \neq 0, \] (192)

and all roots $\omega$ of the polynomial equation
\[ L^p(x, i\mathbf{k} + i\omega \mathbf{n}) = 0 \] (193)

are real and distinct for all $\mathbf{k} \in \mathbb{R}^n$ that are not zero or a multiple of $\mathbf{n}$.

This is a complicated definition, and we look at it piece by piece.

**Remark 3.12.** By the same argument as before, for a PDE with constant coefficients, $u(x) = e^{(i\mathbf{k} + i\omega \mathbf{n}) \cdot x}$ is a solution of $L^p u = 0$ if and only if (193) holds. Moreover, it is an approximate solution of $Lu = 0$ in the high-frequency approximation.
Remark 3.13. It is useful to change coordinates so that \( x = (y, t) \), \( n = (0, -1) \) (the minus sign is just for convenience) and \( k = (\eta, 0) \). Here \( \eta \in \mathbb{R}^{n-1} \) and \( y \in \mathbb{R}^{n-1} \). For this coordinate transformation to exist we need (192), and that \( k \) is neither zero nor a multiple of \( n \). We can then think of \( t \) as time and \( y \) as space and of \( n \) as the time direction in the original coordinates \( x \). With this change of variables, (194) becomes

\[
u(y, t) = e^{i\eta \cdot y - i\omega t} = e^{i|\eta|(\hat{\eta} \cdot y - \hat{\eta} |\eta|)},
\]

where

\[
\eta \cdot y := \sum_{\alpha=1}^{n-1} \eta_\alpha y^\alpha, \quad \hat{\eta} := \frac{\eta}{|\eta|}.
\]

We see that this corresponds to a plane wave travelling with velocity \( \omega/|\eta| \) in the spatial direction \( \hat{\eta} \).

Remark 3.14. Why the condition that all eigenvalues \( \omega \) be real? As the coefficients of \( L^p \) are real, if \( \omega \) is a solution of (193), so is its complex conjugate \( \omega^* \). Hence unless \( \omega \) is real, one of \( \omega \) and \( \omega^* \) has positive imaginary part, and so (195) grows exponentially with time for one of them. Furthermore, if \( L \) is a differential operator of order \( m \), \( L^p(ik) \) is a homogeneous polynomial of order \( m \), or \( L^p(\lambda k) = \lambda^m L^p(ik) \). Hence if the pair \( (k, \omega) \) is a solution of (193), so is any multiple \((\lambda k, \lambda \omega)\). Hence if there are complex \( \omega \), \( Lu = 0 \) has solutions (195) that grow the more rapidly the faster they oscillate in space. Hence there cannot be an estimate of the form (132) because no \( K(t) \) would grow fast enough, and so the time evolution problem cannot be well-posed. (This is precisely what happened in our example of trying to solve the Cauchy problem for the Laplace equation, Sec. 2.2.3.)

Remark 3.15. Why the condition that all eigenvalues \( \omega \) be distinct? Consider a Fourier transform in \( y \), but not in \( t \), that is \( \hat{u}(\eta, t) \). The condition that all eigenvalues \( \omega \) be distinct then guarantees that we can obtain \( \hat{u}(\eta, t) \) (the Fourier transform of the solution) from the the Fourier transform of the Cauchy data \( \hat{u}(\eta, 0), \hat{u}_t(\eta, 0), \hat{u}_{tt}(\eta, 0) \), and so on, up to \( k - 1 \) time derivatives for a \( k \)-th order in time PDE.

For a system of PDEs, we again look for plane waves of the form

\[
u(x) = e^{(ik + i\omega n) \cdot x},
\]

where \( r \) is a constant vector in state space that obeys the matrix equation

\[
L(ik + i\omega n)r = 0.
\]

A necessary condition for such an \( r \neq 0 \) to exist is that the determinant of this matrix vanishes. We will not spell out all the details, but arguments similar to the ones we gave in the scalar case then motivate the following definition.

Definition 3.16. The linear PDE system \( Lu = 0 \) in \( n \) independent variables is called **strictly hyperbolic** in the time direction \( n \) at the point \( x \) if

\[
\det L^p(x, in) \neq 0,
\]

and all roots \( \omega \) of the polynomial equation

\[
\det L^p(x, ik + i\omega n) = 0
\]

are real and distinct for all \( k \in \mathbb{R}^n \) that are not zero or a multiple of \( n \).

3.3.4 Strong hyperbolicity

Many interesting PDE systems are first-order, for example the Euler equations of fluid mechanics, or the Maxwell equations of electrodynamics. For such system, a weaker condition than strict hyperbolicity is sufficient to establish well-posedness results for the Cauchy problem. (The Euler equations are strictly hyperbolic, but the equations of elasticity and the Maxwell equations are only strongly hyperbolic.)
Definition 3.17. The linear first-order PDE system \( L(x, \nabla)u = 0 \) is called strongly hyperbolic in the direction \( n \) at the point \( x \) if
\[
det L^p(x, i n) \neq 0, \tag{201}\]
and, for all \( k \in \mathbb{R}^n \) that are not zero or a multiple of \( n \), all eigenvalues \( \omega \) of the eigenvalue problem
\[
L^p(x, ik + i\omega n)r = 0 \tag{202}\]
arare real, there is a complete set of right eigenvectors \( r \) (in state space) for each eigenvalue \( \omega \), and the eigenvalues \( \omega \) and eigenvectors \( r \) depend continuously on \( k \).

Remark 3.18. Because for each eigenvalue of a matrix there exists at least one eigenvector, strict hyperbolicity implies strong hyperbolicity (for a first-order system). The reverse is not true!

Remark 3.19. The fact that there is a complete set of eigenvectors \( r \) means that Fourier-transformed Cauchy data \( \hat{u}(\eta, 0) \) can be decomposed uniquely into multiples of all the eigenvectors \( r \), evolved as plane waves, and put back together again to make the (Fourier-transformed) solution \( \hat{u}(\eta, t) \) for \( t > 0 \). As above, let \( x = (y, t) \), \( n = (0, -1) \) and \( k = (\eta, 0) \). Let \( \hat{u}(\eta, t) \) represent the partial Fourier transform with respect to \( y \) only. For simplicity, consider a PDE system with constant coefficients. Let \( R \) be the square matrix consisting of all right eigenvectors \( r \). (We consider \( u \) and \( r \) as column vectors in state space). Let \( \Omega \) be the diagonal matrix of the corresponding eigenvalues \( \omega \), and let \( \exp -i\Omega t \) be the diagonal matrix with \( \exp -i\omega t \) in the diagonal. Then
\[
\hat{u}(\eta, t) = R e^{-i\Omega t} R^{-1} \hat{u}(\eta, 0). \tag{203}\]
We see that the eigenvectors \( r \) must form a complete set in order for \( R \) to have an inverse.

3.3.5 Examples

Example 3.20. The second-order scalar PDE
\[
u_{,xx} + u_{,yy} = 0 \tag{204}\]
(Laplace equation) is elliptic. Proof: Write \( k := (\xi, \eta) \). Then
\[
L(ik) = -((\xi^2 + \eta^2) = -|k|^2. \tag{205}\]
But this is positive definite in \( k \), so the only real solution of \( |k|^2 = 0 \) is \( k = 0 \).

Example 3.21. The first-order PDE system
\[
u_y + v_x = 0, \quad u_x - v_y = 0 \tag{206, 207}\]
(the Cauchy-Riemann equations) is elliptic. Proof: This system can be written as
\[
\sum_{i=1,2} A^i u_{,i} = 0, \text{ where } u := \begin{pmatrix} u \\ v \end{pmatrix}, \quad A^x := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad A^y := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{208}\]
Hence
\[
L(ik) = i(A^x \xi + A^y \eta) = i \begin{pmatrix} \eta & \xi \\ -\xi & -\eta \end{pmatrix} \Rightarrow \det L(ik) = \xi^2 + \eta^2. \tag{209}\]
Note that by taking a derivative this first-order system implies \( u_{,xx} + u_{,yy} = 0 \) and \( v_{,xx} + v_{,yy} = 0 \), both of which are also elliptic.

Example 3.22. The second-order scalar PDE
\[-u_{,xx} + u_{,yy} + u_{,zz} = 0 \tag{210}\]
is strictly hyperbolic in all directions
\[
n = (1, v) := (1, v_y, v_z) \quad \text{with} \quad |v|^2 := v_y^2 + v_z^2 < 1. \tag{211}\]
Beginning of proof: With \( k := (\alpha, \beta, \gamma) := (\alpha, \beta) \), we have det \( L(ik) = \alpha^2 - |\beta|^2 \). Also, \( \omega n + k = (\omega + \alpha, \omega \beta + \beta) \). Hence

\[
L(i\omega n + ik) = (\omega + \alpha)^2 - |\omega \beta + \beta|^2 = (1 - v^2)\omega^2 + 2\omega(\alpha - \beta \cdot \nu) + (\alpha^2 - |\beta|^2) =: A\omega^2 + 2B\omega + C. \quad (212)
\]

Setting this to zero has two distinct real solutions if and only if \( \Delta := B^2 - AC > 0 \). To finish the proof, we need to show that in fact \( \Delta > 0 \) for all \( \alpha \) if and only if \( |v|^2 < 1 \). (We do this first for \( \alpha = 0 \), and then for \( \alpha \neq 0 \).)

Example 3.23. The first-order PDE system

\[
\begin{align*}
u, x &= v, y + w, z, \\ v, x &= u, y, \\ w, x &= u, z
\end{align*}
\]

is a) strictly hyperbolic for the same directions \( n \) as in Example 3.22, and b) actually equivalent to (210).

Example 3.24. The first-order PDE system

\[
\begin{align*}
u, x + \mu u, y &= 0, \\ v, x + \nu v, y + \lambda u, y &= 0,
\end{align*}
\]

where \( \mu, \nu \) and \( \lambda \) are real constants, is strictly hyperbolic in the time direction \( n = (-1, 0) \) (so \( x \) is the time coordinate) if and only if \( \mu \neq \nu \). For \( \mu = \nu \) it is not strictly hyperbolic but still strongly hyperbolic if and only if \( \lambda = 0 \). Otherwise it is not hyperbolic (and the Cauchy problem is actually ill-posed). Beginning of proof: Let \( k := (\alpha, \beta) \). Then

\[
L(i\omega n + ik) = i \begin{pmatrix}
-\omega + \alpha + \mu \beta & 0 \\
\lambda \beta & -\omega + \alpha + \nu \beta
\end{pmatrix}.
\]

(218)

The result follows by explicitly calculating the eigenvalues and eigenvectors of this matrix. (The way in which this system fails to be strongly hyperbolic is in fact typical.)

3.3.6 First order scalar PDEs

Remark 3.25. Any scalar first-order linear PDE in \( n \) independent variables

\[
\sum_{i=1}^{n} a^i(x) u, i = f(u) \quad (219)
\]

is strictly hyperbolic in the sense of Def. 3.11 for all \( n \) that are not normal to \( a \). Hence all such PDEs are of the same type.

Remark 3.26. This is not so for first-order systems, which can be hyperbolic, elliptic, or neither.

Remark 3.27. Solving a scalar, first-order PDE is equivalent to first solving a system of first-order ODEs. This gives the solution in an implicit form, and an algebraic equation needs to be solved to give \( u(x) \) in explicit form. This solution method is called the method of characteristics. Either step may be difficult in practice, but because of this formal equivalence to ODEs, scalar first-order PDEs are not a good example of PDEs in general, and the method of characteristics can be a misleading way of building up intuition for PDEs. We do not have time to present the method of characteristics in general, but intuitively the principal part of (219) can be thought of as just a directional derivative along the vector field \( a^i(x) \). \( u \) can then be obtained from initial data by integrating along this vector field.
3.3.7 Second order scalar PDEs

For scalar, second-order linear PDEs in \( n \) independent variables, there is a complete classification into four types, as follows.

**Definition 3.28.** Consider the scalar linear second-order PDE for the unknown \( u \) in \( n \) independent variables \( x \),

\[
L(x, \nabla) u = \sum_{i,j=1}^{n} a^{ij}(x) u_{,ij} + \sum_{i=1}^{n} b^i(x) u_{,i} + c(x) = 0. \tag{220}
\]

Hence we have the principal part

\[
L^p(x, \nabla) u = \sum_{i,j=1}^{n} a^{ij}(x) u_{,ij} \tag{221}
\]

and principal symbol

\[
L^p(x, ik) = -\sum_{i,j=1}^{n} a^{ij}(x) k_i k_j. \tag{222}
\]

Now consider \( a^{ij} \) as a symmetric \( n \times n \) matrix. Then we call the PDE parabolic at \( x \) if \( a^{ij} \) is singular (so one or more eigenvalues are zero). Conversely, if all eigenvalues are nonzero, we call the PDE elliptic at \( x \) if all eigenvalues of \( a^{ij} \) have the same sign. We call it hyperbolic at \( x \) if one eigenvalue has the opposite sign from the others. The remaining case, where there is more than one eigenvalue of each sign (but none zero), is called ultrahyperbolic at \( x \). The PDE is simply called parabolic if it is parabolic at every \( x \), and so on.

**Proposition 3.29.** This definition of ellipticity agrees with Def. 3.9, and this definition of hyperbolicity agrees with strict hyperbolicity for at least one choice of \( n \) defined in Def. 3.11.

**Proof.** If all eigenvalues of the matrix \( a^{ij} \) have the same sign, then without loss of generality we can assume they are all positive. Hence the matrix is positive definite which implies that

\[
\sum_{i,j} a^{ij} k_i k_j \geq 0 \quad \forall k \in \mathbb{R}^n, \quad \text{with} \quad \sum_{i,j} a^{ij} k_i k_j = 0 \iff k = 0. \tag{223}
\]

Hence we have ellipticity.

Now assume that one eigenvalue is negative and all others are positive. Let the corresponding eigenvector be \( n \), so that

\[
\sum_{i,j} a^{ij} n_i n_j < 0. \tag{224}
\]

We can uniquely decompose any other vector \( k \) as a part in the direction of \( n \) and a part “orthogonal” to it, in the sense that

\[
k = \alpha n + \beta, \quad \text{with} \quad \sum_{i,j} a^{ij} n_i \beta_j = 0. \tag{225}
\]

Moreover,

\[
\sum_{i,j} a^{ij} \beta_i \beta_j \geq 0 \quad \text{with} \quad \sum_{i,j} a^{ij} \beta_i \beta_j = 0 \iff \beta = 0. \tag{226}
\]

Therefore the equation

\[
\sum_{i,j} a^{ij}(\omega n_i + k_i)(\omega n_j + k_j) = (\omega + \alpha)^2 \sum_{i,j} a^{ij} n_i n_j + \sum_{i,j} a^{ij} \beta_i \beta_j = 0 \tag{227}
\]

has two distinct real solutions \( \omega \), for any \( k \) that is not a multiple of \( n \). \qed

**Remark 3.30.** The only type of parabolic second-order PDE we will consider in this course are those with precisely one zero eigenvalue. The corresponding eigenvector \( n \) is the direction of time.

We will not say anything about ultrahyperbolic PDEs, which arise less naturally in physics and engineering than elliptic, parabolic and hyperbolic ones.
3.4 Nonlinear PDEs and systems

Some PDEs and systems in engineering and physics are linear at a deep physical level, for example the Poisson equation for the gravitational field, the Maxwell equations, or the Schrödinger equation.

Other linear PDEs arise as an approximation when we consider small perturbations of an equilibrium solution. For example, the equations of acoustics are valid for small pressure and velocity perturbations of the Euler equations. For larger perturbations, for example in an explosion, the full nonlinear Euler equations are needed, and interesting things such as shock formation will happen. In another example, the heat equation is linear because the heat flux is approximately proportional to the temperature gradient, but this is only an approximation.

Definition 3.31. A system of PDEs is called \textit{quasilinear} if derivatives of the principal order occur only linearly. (Their coefficients may depend nonlinearly on the lower derivatives and the independent coordinates).

As an example, the PDE (2) is quasilinear if we allow \(a, b, c\) and \(f\) to depend only on \(x, y, u, u_x, u_y\) (but not on \(u_{xx}, u_{yy}\) and \(u_{xy}\), or any higher derivatives).

The systems of conservation laws we look at later (for example the Euler equation) are quasi-linear. So are many or most other PDEs of interest in physics and engineering.

Also, every nonlinear PDE (or system) can be turned into a quasilinear PDE (or system) that is one order higher by taking one more derivative. A simple example is

\[
\begin{align*}
  u_{tt} &= (u_x)^2 \Rightarrow u_{tt} = 2u_x u_{xt},
\end{align*}
\]

The PDE on the left is first-order and is quadratic in \(u_x\), so it is not quasilinear. The equivalent PDE on the right is second-order, and is linear in all the highest derivatives (\(u_{tt}\) and \(u_{xt}\)), so it is quasilinear.

Definition 3.32. The \textit{linearisation} of a system of PDEs of the form

\[
  F(x, u, \nabla) = 0
\]

about a solution \(u_0(x)\) is

\[
  L(x, u_0, \nabla) \delta u = 0,
\]

where the linear differential operator \(L\) is defined by

\[
  L(x, u_0, \nabla) \delta u := \frac{d}{d\epsilon} \bigg|_{\epsilon=0} F(x, u_0 + \epsilon \delta u, \nabla)
\]

Example 3.33. Consider again the nonlinear PDE (scalar with constant coefficients) (228), that is

\[
  F(u, \nabla) = u_t - (u_x)^2 = 0.
\]

We linearise about a solution \(u_0(x, t)\) by computing

\[
  L(u_0, \nabla) \delta u = \frac{d}{d\epsilon} \bigg|_{\epsilon=0} [(u_{0,t} + \epsilon \delta u_t) - (u_{0,x} + \epsilon \delta u_x)^2] = \delta u_t - 2u_{0,x} \delta u_x,
\]

so

\[
  L(u_0, \nabla) = \frac{\partial}{\partial t} - 2u_{0,x} \frac{\partial}{\partial x}.
\]

Example 3.34. The equation of motion of a pendulum is

\[
  \frac{d^2 u}{dt^2} + \omega^2 \sin u = 0,
\]

where \(u\) is angle away from the vertical. (If the pendulum has a rod rather than a string this holds for any value of \(u\) and would, for example, apply to a pendulum going over the top.) If we linearise about the solution \(u_0(t) = 0\), we obtain

\[
  \frac{d^2 \delta u}{dt^2} + \omega^2 \delta u = 0,
\]

the harmonic oscillator equation. Intuitively, \(\sin \delta u \simeq \delta u\) for \(|\delta u| \ll 1\), and the formal calculation is \(d/d\epsilon|_{\epsilon=0} \sin(0 + \epsilon \delta u) = \cos(0 + \epsilon \delta u)|_{\epsilon=0} \delta u = \delta u\).
**Definition 3.35.** A nonlinear PDE or PDE system is called **hyperbolic**, **elliptic** or **parabolic** about a solution $u_0(x)$ if its linearisation about $u_0(x)$ has the appropriate property. In other words, we look at algebraic properties of the principal symbol of the **linearised** system, $L^p(u_0, x, ik)$, where $L(u_0, x, \nabla)$ is defined by (231).

**Remark 3.36.** A quasilinear system and its linearisation have the same principal part (which is, of course, linear). Hence a quasilinear system is strongly hyperbolic, elliptic, and so on, in the sense that its linearisation is, if and only if the principal symbol obeys the conditions defined above. (That means we can save the trouble of linearising the non-principal part).

**Example 3.37.** Consider the quasilinear first-order PDE

$$u_t + f'(u) u_x = 0 \iff \left( \frac{\partial}{\partial t} + f'(u) \frac{\partial}{\partial x} \right) u = 0. \quad (237)$$

Its linearisation about a solution $u_0(x, t)$ is

$$\delta u_t + f'(u_0) \delta u_x + f''(u_0) \delta u u_0_x = 0. \quad (238)$$

Hence

$$L = \frac{\partial}{\partial t} + f'(u_0) \frac{\partial}{\partial x} + f''(u_0) u_0_x \implies L^p = \frac{\partial}{\partial t} + f'(u_0) \frac{\partial}{\partial x}. \quad (239)$$

Note the similarity between the last part of (237) and the last part of (239).

**Remark 3.38.** Well-posedness of the linearisation of a PDE problem is necessary for well-posedness of the full nonlinear system, but it is not sufficient. (And finding sufficient conditions is an art.)

### 3.5 Exercises

23. **Homework 8:** a) Using separation of variables, find the general solution the PDE

$$u_{xxxx} - u_{yyyy} = 0 \quad (240)$$

on the domain $0 \leq x \leq \pi$ with boundary conditions

$$u(0, y) = 0, \quad u(\pi, y) = 0 \quad (241)$$

for all $y$. [Hint: this is not a complete PDE problem yet. We have not specified the domain in $y$ and the related boundary conditions, so your solution should contain free constants.] Based on this solution, does it look as if the PDE (240) has any of the following properties:

1) every solution grows in at least one direction, 2) every solution oscillates in all directions, 3) every solution decays in one and the same direction?

b) Show that the initial value problem consisting of (240) with (241) on the domain $0 \leq x \leq \pi$, $y \geq 0$ with initial conditions

$$u(x, 0) = u_0(x), \quad u_x(x, 0) = u_1(x), \quad u_{xx}(x, 0) = u_2(x), \quad u_{xxx}(x, 0) = u_3(x) \quad (242)$$

has a unique solution. Show that this initial value problem is nevertheless ill-posed because the solution cannot depend continuously on the initial data in any function norm.

c) Using separation of variables, find the general solution the PDE

$$u_{xxxx} + u_{yyyy} = 0 \quad (243)$$

on the domain $0 \leq x \leq \pi$ with boundary conditions (241). Does it look as if (243) has any of the three properties in part a)?

24. **Homework 9:** (Quite short) For the linear PDE with constant coefficients

$$Lu = 0 \iff u_{xx} + 7u_{xy} + 3u_x = 0, \quad (244)$$

find $L(\nabla)$ and $L(ik)$. Show explicitly that $\hat{L}(\nabla)u = L(ik)\hat{u}(k)$. 38
25. **Homework 10:** Show that

\[ -u_{,xx} + u_{,yy} + u_{,zz} + \cdots = 0 \]  

(245)

is strictly hyperbolic in all directions

\[ n = (1, v^y, v^z, \ldots) := (1, v) \]  

(246)

with \( v < 1 \), where \( v := |v| := \sqrt{v \cdot v} \).

26. **Homework 11:** Show that the system of first-order PDEs

\[
\begin{align*}
  u_{,x} &= v_{,y} + w_{,z}, \\
  v_{,x} &= u_{,y}, \\
  w_{,x} &= u_{,z}
\end{align*}
\]

(247) (248) (249)

is (a) strictly hyperbolic for suitable \( n \) and (b) equivalent to the second-order wave equation.

27. **Homework 12:** Show that the compressible Euler equations in one space dimension,

\[
\begin{align*}
  \rho_t + (\rho v)_x &= 0, \\
  (\rho v)_t + (\rho v^2 + P)_x &= 0, \\
  \left( e + \frac{1}{2} \rho v^2 \right)_t + \left[ v \left( e + \frac{1}{2} \rho v^2 + P \right) \right]_x &= 0,
\end{align*}
\]

(250) (251) (252)

where \( P = P(\rho, e) \) is a given function with \( P_\rho > 0 \) and \( P_e > 0 \), is strictly hyperbolic in the \( t \)-direction. Here \( \rho > 0 \) is mass/volume, \( v \) is velocity, \( e > 0 \) is internal (heat) energy per volume and \( P > 0 \) is pressure. (And as we are in one space dimension, “volume” means length.) You can assume \( v = 0 \) at a sufficiently late stage in the discussion. Show that the characteristic speeds are \( v, v + c \) and \( v - c \), and give and expression for \( c \).

[This is a long and difficult but rewarding problem, so see how far you can get. Start by expanding \( (\rho v)_t = \rho v_t + \rho x v \) and similarly for the other derivatives. Use the chain rule to write \( P, x = P_\rho \rho, x + P_e e, x \). We now have a quasilinear first-order PDE system for \( u = (\rho, v, e) \). Write it as

\[ A^t u_t + A^z u_x = 0 \]  

(253)

and read off the \( 3 \times 3 \) matrices \( A^t \) and \( A^z \). We have seen that for classification purposes we can treat a quasilinear first-order system as if it is linear. So you can now check the conditions for strict hyperbolicity. We have to say that the system is hyperbolic at the point \( x \) because \( \rho, v \) and \( e \) in the background solution all depend on \( x \).]

28. **a** Show that the PDE

\[ -4 u_{,xy} + u_{,zz} = 0 \]  

(254)

for \( u(x, y, z) \) is hyperbolic, using the criterion for linear second-order PDEs. **b** Show that neither the \( x \), \( y \) or \( z \)-directions are good time directions for this PDE. In other words, show that the PDE is not strictly hyperbolic in the directions \( n = (1, 0, 0), (0, 1, 0) \) or \( (0, 0, 1) \). **c** Show that the direction \( n = (1, 1, 0) \) is a good time direction. **d** Change variables from \( (x, y, z) \) to \( (t, s, \zeta) \), where \( t = (x + y)/2, s = (x - y)/2 \) and \( \zeta = z \), and show that you get the standard form of the wave equation. [This is consistent with the fact that we have already shown that \( \nabla t = (1/2, 1/2, 0) \) is a good time direction.]

29. Show that the first-order PDE system

\[
\begin{align*}
  u_{,x} + \mu u_{,y} &= 0, \\
  v_{,x} + \nu v_{,y} + \lambda u_{,y} &= 0,
\end{align*}
\]

(255) (256)

where \( \mu, \nu \) and \( \lambda \) are real constants, is strictly hyperbolic in the time direction \( n = (-1, 0) \) (so \( x \) is the time coordinate) if and only if \( \mu \neq \nu \). For \( \mu = \nu \) it is not strictly hyperbolic but still strongly hyperbolic if and only if \( \lambda = 0 \).
30. Show that any scalar first-order linear PDE in \( n \) independent variables

\[
\sum_{i=1}^{n} a^i(x) u, i = f(u)
\]  

(257)

is strictly hyperbolic in the sense of Def. 3.11 for all \( \mathbf{n} \) that are not normal to \( \mathbf{a} \).
4 Conservation laws

4.1 Integral and differential form

4.1.1 One space dimension

To give the simplest example of a conservation law, consider the flow of mass through a pipe, so we have a problem in one space dimension \( x \) and time \( t \). Let \( u(x, t) \) be the density of mass, measured in units of mass/length, at position \( x \) and time \( t \).

Let \( f(x, t) \) be the mass flux through the pipe, measured in mass/time, again at position \( x \) and time \( t \). We count a flux going in the direction of increasing \( x \) as positive. Consider a segment of pipe \( a \leq x \leq b \). The mass in that segment at time \( t \) is

\[
m(t) = \int_a^b u(x, t) \, dx
\]

(258)

This mass changes with time because of the fluxes through the ends \( x = a \) and \( x = b \), as follows:

\[
\frac{dm}{dt} = f(a, t) - f(b, t).
\]

(259)

Note the signs: flux towards increasing \( x \) means into the segment at \( x = a \) but out of it at \( x = b \).

Combining the last two equations, we have

\[
\frac{d}{dt} \int_a^b u(x, t) \, dx = f(a, t) - f(b, t).
\]

(260)

We now turn (260) into a PDE. Assuming \( u(x, t) \) is once differentiable in \( t \), we can write the left-hand side as

\[
\frac{d}{dt} \int_a^b u(x, t) \, dx = \int_a^b \frac{\partial}{\partial t} u(x, t) \, dx
\]

(261)

Also, if \( f(x, t) \) is once differentiable in \( x \), we can write the right-hand side as

\[
f(a, t) - f(b, t) = -\int_a^b \frac{\partial}{\partial x} f(x, t) \, dx.
\]

(262)

Combining these two results and bringing both terms on the same side we obtain

\[
\int_a^b [u_{,t}(x, t) + f_{,x}(x, t)] \, dx = 0.
\]

(263)

If we want this to hold for any segment of pipe, \( a \) and \( b \) can take any values. Then the integral can vanish only if the integrand in square brackets vanishes for every \( x \), or

\[
u_{,t} + f_{,x} = 0.
\]

(264)

This first-order PDE is a conservation law in differential form or strong form. A solution of this is called a strong solution of the conservation law.

However, we are often interested in the case where \( u \) and \( f \) are not differentiable, and are in fact discontinuous. (This will lead us to "shocks"). Hence we want to go the other way from (260) and also remove the time derivative. For this, we integrate (260) over a time interval \( t_0 \leq t \leq t_1 \) to get

\[
\int_a^b u(x, t_1) \, dx - \int_a^b u(x, t_0) \, dx = \int_{t_0}^{t_1} f(a, t) \, dt - \int_{t_0}^{t_1} f(b, t) \, dt
\]

(265)

This consists of four integrals, one over each side of the rectangle \( (a \leq x \leq b, t_0 \leq t \leq t_1) \). Note that \( u \) and \( f \) now only need to be integrable, not differentiable.

**Definition 4.1.** A weak solution of the conservation law (264) is a function \( u(x, t) \) that obeys (265) for all \( a, b, t_0 \) and \( t_1 \).
In defining this, we use (264) only as of shorthand notation for (265). Be sure you get the signs right. This definition of a weak solution looks different from the weak solutions of the wave equation we defined in Sec. 1.7, but is closely related. (265) itself is called the integral form or weak form of the conservation law (264). We have just shown by construction that any strong solution is also a weak solution. The reverse is clearly not true, as a weak solution does not have to be once differentiable. In fact, interesting weak solutions are typically discontinuous.

Note that until we have given an expression for \( f \) in terms of \( u \), the problem has not been completely specified.

4.1.2 Higher space dimensions

It is straightforward to generalise the integral and differential form to any number of spatial dimensions. Consider the rectangle \((a \leq x \leq b, c \leq y \leq d)\) in two space dimensions. \( u \) is the density of mass, now measured in mass/(length)\(^2\). The mass in the rectangle at time \( t \) is

\[
m(t) = \int_a^b dx \int_c^d dy u(x, y, t).
\]

(266)

It changes with time because of the fluxes through four sides of the rectangle:

\[
\frac{dm}{dt} = \int_c^d \left[ f^x(a, y, t) - f^x(b, y, t) \right] dy + \int_a^b \left[ f^y(x, c, t) - f^y(x, d, t) \right] dx.
\]

(267)

Here \( f^x(x, y, t) \) is the mass flux in \( x \)-direction, and \( f^y(x, y, t) \) the mass flux in the \( y \)-direction. Both are measured in mass/(length-time). Once again, this is not the form we need. The differential form is

\[
u_{,t} + f^x_{,x} + f^y_{,y} = 0
\]

(268)

and the integral form is

\[
\int_a^b dx \int_c^d dy \left[ u(x, y, t_1) - u(x, y, t_0) \right] \\
+ \int_{t_0}^{t_1} dt \int_c^d dy \left[ f^x(b, y, t) - f^x(a, y, t) \right] \\
+ \int_{t_0}^{t_1} dt \int_a^b dx \left[ f^y(x, d, t) - f^y(x, c, t) \right] = 0
\]

(269)

The integration is along the six faces of the rectangular box \((t_0 \leq t \leq t_1, a \leq x \leq b, c \leq y \leq d)\). Each of these six faces is itself two-dimensional.

The three-dimensional case should now be clear. \( u \) is then measured in units of mass/(length)\(^3\) and the three fluxes in units of mass/(length\(^2\)-time). The integral form is

\[
u_{,t} + f^x_{,x} + f^y_{,y} + f^z_{,z} = 0
\]

(270)

and the integral form now has eight integrals, each over three of the four coordinates \((x, y, z, t)\).

In \( n \) space dimensions and time, this can be written as

\[
u_{,t} + \sum_{i=1}^n f^{i\,i} = 0
\]

(271)

In vector calculus notation, the same equation is

\[
u_{,t} + \nabla \cdot \mathbf{f} = 0.
\]

(272)

We can also write the mixed form of our conservation law as

\[
\frac{d}{dt} \int_V u dV + \int_S \mathbf{f} \cdot \mathbf{n} dS = 0,
\]

(273)
and the fully integral form as

\[
\int_V \left[ u(x, t_1) - u(x, t_0) \right] dV + \int_{t_0}^{t_1} dt \int_S f \cdot n \, dS = 0,
\]

(274)

where \( S \) is the boundary of \( V \). Because they use the divergence theorem, these now hold for any volume \( V \) with boundary \( S \) (not just rectangular boxes). In any conservation law of the form (274), \( u \) is called the conserved quantity and \( f \) the corresponding flux.

4.2 Scalar conservation laws in one space dimension

The general form of a scalar conservation law in one space dimension

\[
u_t + \left[ f(u, x, t) \right]_x = 0.
\]

(275)

As for other PDEs, “scalar” means that there is only one PDE for one dependent variable \( u \), as opposed to a system of conservation laws (such as the Euler equations). The appropriate initial data are

\[
u(x, 0) = g(x).
\]

(276)

Often, \( f(u, x, t) \) is actually independent of \( x \) and \( t \). (Any ODE or PDE whose coefficients are independent of all the independent variables is called autonomous.) In this autonomous case, (275) reduces to

\[
u_t + [f(u)]_x = 0, \quad u(x, 0) = g(x).
\]

(277)

The function \( f(u) \) is sometimes called the flux function or flux law. In the following we consider only the autonomous case, which comprises many physical applications.

Using the chain rule, we can also write (277) as

\[
u_t + f'(u) u_x = 0,
\]

(278)

where \( f'(u) := df/du \). This form is explicitly quasilinear, but no longer explicitly in conservation law form. The two forms are equivalent if and only if \( u(x, t) \) is at least once differentiable. By contrast (277), is often used as a shorthand for the integral form (265), which is defined for any solution \( u(x, t) \) that is integrable.

We define (in the autonomous case)

\[
v(u) := \frac{f(u)}{u}
\]

(279)

or

\[
f(u) = uv(u).
\]

(280)

If we interpret \( u \) as the density of something, say particles per length of pipe, then \( v \) is the velocity with which these particles move, the particle velocity.

4.2.1 The advection equation

The simplest case of a scalar conservation law is the one where \( v \) is just constant in space and time, \( v = v_0 \) and hence \( f(u) = v_0 u \). This is called the advection equation (here in one space dimension). It can be solved in closed form. It is easy to verify that the solution of

\[
u_t + (uv_0)_x = 0, \quad u(x, 0) = g(x),
\]

(281)

is

\[
u(x, t) = g(x - v_0 t).
\]

(282)

The advection equation is a linear first-order PDE that is strongly hyperbolic, so this solution is also unique.

The solution means that if \( g(x_0) = u_0 \), then \( u = u_0 \) all along the characteristic curve \( x(t) = x_0 + v_0 t \). (Compare this with the discussion of the d’Alembert solution in Remark 1.11.)
We see that the advection equation just translates the initial data along characteristic curves, or with velocity \( v_0 \).

It is natural to also admit weak solutions of (281), of the form (282) but where the initial data \( g(x) \) and hence the solution \( u(x, t) \) are discontinuous. Weak solutions of a conservation law are not everywhere differentiable, and hence do not obey the differential form of the conservation law, but they obey its corresponding integral form. Note that any discontinuities in \( g(x) \) also propagate along characteristics.

4.2.2 Method of characteristics

Now consider the generic autonomous scalar conservation law (275,276), where \( f(u) \) is some given function. We can derive a solution either in implicit form or graphically using the method of characteristics.

Assume \( u \) is constant on the curve \( x = \xi(x_0, t) \) that starts at \( x_0 \) at \( t = 0 \). In other words

\[
\frac{\partial}{\partial t} u[\xi(x_0, t), t] = u_{,t} + u_{,x} \frac{\partial \xi}{\partial t} = -f'(u)u_{,x} + u_{,x} \left( \frac{\partial \xi}{\partial t} - f'(u) \right)
\]

(284)

In the second equality we have used the chain rule, and in the third equality we have used (278).

Generically \( u_{,x} \neq 0 \), and hence we must have

\[
\frac{\partial}{\partial t} \xi(x_0, t) = f'[u(x_0, 0)] = f'[g(x_0)].
\]

(285)

We also have the initial condition

\[
\xi(x_0, 0) = x_0.
\]

(286)

Although \( \xi \) depends on \( x_0 \) and \( t \), this differential equation contains no derivative with respect to \( x_0 \), and so it is in effect an ODE in \( t \), for each value of the parameter \( x_0 \). Moreover, the right-hand side of (285) does not depend on \( t \), and so we can simply integrate both sides to obtain

\[
\xi(x_0, t) = x_0 + f'[g(x_0)] t,
\]

(287)

where the integration constant is fixed by the initial condition (286). \( u \) is constant on characteristics, and we have just derived that the characteristics are simply straight lines in the \( xt \)-plane. We now have the solution \( u(x, t) \) in implicit form. If we can solve the algebraic equation

\[
x = x_0 + f'[g(x_0)] t.
\]

(288)

for \( x_0(x, t) \) in closed form [if this is possible depends on \( f'(u) \)], we can also write the solution in explicit form as

\[
u(x, t) = g[x_0(x, t)].
\]

(289)

\( f'(u) \) is called the characteristic velocity. Note that this is different from the particle velocity \( v = f'(u_0)/u \), except for the advection equation, where both are equal to \( v_0 \).

4.2.3 Propagation of small disturbances

Now consider a solution of (277) which is a small perturbation around a constant solution \( u_0 \),

\[
u(x, t) = u_0 + \delta u(x, t)
\]

(290)

and that \( u(x, t) \) is at least once differentiable so that we can use the quasilinear form (278). Expanding \( f'(u) \) into a Taylor series as

\[
f'(u_0 + \delta u) = f'(u_0) + f''(u_0) \delta u + \frac{1}{2!} f'''(u_0) (\delta u)^2 + O((\delta u)^3),
\]

(291)
we have to leading order in $\delta u$
\[
 u_{,t} + f'(u)u_{,x} = \delta u_{,t} + f'(u_0)\delta u_{,x} + O((\delta u)^2). \tag{292}
\]

But this is just the advection equation for $\delta u$, so small perturbations travel at the characteristic velocity $f'(u_0)$.

Considering that sound can convey information, and sound is a (very) small perturbation of gas pressure and velocity, it is said (in a non-rigorous sense) that information in hyperbolic equations travels at the characteristic velocity or velocities.

### 4.3 Weak solutions

#### 4.3.1 Shock formation and Riemann problems

For the conservation law (277) with initial data (276), consider initial data $g(x)$ such that the function $f'[g(x)]$ is an increasing function of $x$. Then the characteristics fan out from $t = 0$ and never intersect. Hence the initial density profile is stretched out. The solution remains smooth for all $t > 0$.

Now consider initial data $g(x)$ such that the function $f'[g(x)]$ is a decreasing function of $x$. Then the characteristics converge from $t = 0$. Intuitively, “particles” at the back move faster than particles in front, and hence catch them up. The initial density profile is compressed and becomes steeper until the solution given by following characteristics becomes multivalued, and no longer make. At this point the physical solution has become discontinuous, and hence non-differentiable, and we do not know how to continue. A shock has formed. Shocks develop generically in nonlinear hyperbolic PDEs, of which nonlinear conservation laws are an example.

To understand what happens once a discontinuity has formed, or when a discontinuity is already present in the initial data, we consider the **Riemann problem**, which consists of a conservation law (here, a scalar conservation law) with piecewise constant initial data:
\[
 u_{,t} + [f(u)]_{,x} = 0, \quad u(x, 0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases}. \tag{293}
\]

#### 4.3.2 Propagating shock solutions

We have seen that the advection equation admits solutions with a travelling discontinuity. This motivates us to look for a solution to the Riemann problem with a discontinuity that propagates with constant velocity:
\[
 u(x, t) = \begin{cases} u_L, & x < st \\ u_R, & x > st \end{cases}, \tag{294}
\]

for $t \geq 0$, where $s$ is a constant shock velocity. The shock location is $x = st$. (294) cannot be a solution of the differential form $u_{,t} + [f(u)]_{,x} = 0$ of the conservation law because it is not differentiable. Instead, consider the equivalent integral form
\[
 \int_a^b [u(x, t_1) - u(x, t_0)] \, dx + \int_{t_0}^{t_1} (f[u(b, t)] - f[u(a, t)]) \, dt = 0, \tag{295}
\]

which must hold for all rectangles ($t_0 \leq t \leq t_1, a \leq x \leq b$). On a rectangle where $u$ is simply constant, this is trivial. Instead consider a rectangle that is cut diagonally into two triangles by the propagating shock, for example the rectangle ($0 \leq t \leq \Delta t, 0 \leq x \leq \Delta x$), where $\Delta x := s\Delta t$ (assuming here that $s > 0$). Then (295), after dividing by $\Delta t$, immediately gives
\[
 s(u_L - u_R) = f(u_L) - f(u_R), \tag{296}
\]

the **Rankine-Hugoniot condition**. This is often written as
\[
 s[u] = [f(u)], \tag{297}
\]

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where the square brackets denote the jump across the shock. It is also called the **jump condition** at the shock.

By making the rectangle under consideration arbitrarily small, it is easy to show that the jump condition still holds when the solution is once differentiable but not necessarily constant on either side of the shock. Intuitively, if we zoom in on an isolated discontinuity, the smooth derivative becomes less and less relevant, and the discontinuity looks like a step function. Hence, what happens in the neighbourhood of the shock should depend only on the value just to the left and right. The shock still moves with velocity \( s \) given by \( (296) \), but that velocity will in general depend on time as \( u_L \) and \( u_R \) change.

For a scalar conservation law, we can obtain a shock solution for arbitrary values of \( u_L \) and \( u_R \), and simply read off the shock speed as

\[
s = \frac{f(u_L) - f(u_R)}{u_L - u_R} \tag{298}
\]

If we now let \( u_L = u + \Delta u \) and \( u_R = u \) in \( (298) \) and take the limit \( \Delta u \to 0 \), we see that the right-hand side is just the formal definition of a derivative as a limit, so we find that \( s \to f'(u) \). Hence weak shocks (shocks with an infinitesimally small jump) propagate with the characteristic velocity.

### 4.3.3 Rarefaction waves

The initial data in a Riemann problem do not single out any particular length scale, and neither does the solution \( (294) \), and so the solution must be **scale-invariant**. Dimensional analysis shows that there is no length scale that can be formed from the conservation law itself and the initial data of the Riemann problem. The solution must be a **similarity solution**, which here means that it must be a function of \( x/t \) only. Looking back, we see that this does hold for \( (294) \).

There is in fact another type of similarity solution of the Riemann problem that is continuous, although still only a weak solution. We look for a solution of the form

\[
u(x,t) = \phi(z), \quad z := \frac{x}{t}, \quad t > 0. \tag{299}\]

Substituting this into the differential form of the conservation law we find

\[
u_t + f'(u)\nu_x = \phi'(z) \left( -\frac{x}{t^2} \right) + f'\phi(z)\phi'(z) \left( \frac{1}{t} \right) = 0, \tag{300}\]

and after multiplying by \( t \)

\[
f'\phi(z)\phi'(z) = z\phi'(z) \tag{301}\]

We can assume \( \phi'(z) \neq 0 \) (or else the solution we are constructing would be constant) and divide by it to obtain

\[
f'\phi(z) = z. \tag{302}\]

As \( f(u) \) and hence \( f'(u) \) is a known function, this is just an algebraic equation that in principle can be solved for \( \phi(z) = (f')^{-1}(z) \). Assume there exist \( z_L \) and \( z_R \) with \( z_L < z_R \) such that \( \phi(z_L) = u_L \) and \( \phi(z_R) = u_R \). Then a solution to the Riemann problem is given by

\[
u(x,t) = \begin{cases} 
  u_L, & x < zLt, \\
  \phi(x/t), & zLt < x < zRt \\
  u_R, & x > zRt,
\end{cases} \tag{303}\]

where \( \phi(z_{L,R}) = u_{L,R} \) and \( (302) \) give

\[
z_{L,R} = f'(u_{L,R}). \tag{304}\]

So this solution can only exist when

\[
f'(u_L) < f'(u_R). \tag{305}\]

This solution is called a **rarefaction wave**. Clearly, it is continuous but not differentiable at \( x = z_{L,R}t \). One can show that it is a weak solution of the conservation law.
4.3.4 The Lax condition

A Riemann problem with given left and right states \( u_L \) and \( u_R \) may well admit more than one shock or rarefaction solution. However, there are often additional criteria to pick out a unique solution for each initial data. One is that the problem should be well-posed. If we add a small \( x \)-dependent perturbation to the left and right states then the resulting solution should only differ from the Riemann solution by a small perturbation.

Another consideration may be that the hyperbolic conservation law is actually only an approximation to a PDE that has a little bit of dissipation, for example

\[
 u_t + [f(u)]_x = \kappa u_{xx}
\]

for some small \( \kappa > 0 \). This PDE is actually parabolic, not hyperbolic. Its solutions will behave like solutions to the conservation law, but will gradually become smoother. In particular, as \( \kappa \to 0 \), the term \( u_{xx} \) will be important only where the gradient of the solution is very large, that is in place where a shock is trying to form. The shock solution for \( \kappa = 0 \) should then coincide with the limit \( \kappa \to 0 \) of the equation with dissipation.

It turns out that, from either argument, shocks are the correct solution to the Riemann problem only if characteristics run into them. This is called the Lax shock condition. Using our expressions for the shock velocitiess and the characteristic velocities to the left and right, this is

\[
 f'(u_L) > \frac{f(u_L) - f(u_R)}{u_L - u_R} > f'(u_R).
\]

Of course, this is only possible for

\[
 f'(u_L) > f'(u_R).
\]

In the other case, the rarefaction wave is the correct solution to the Riemann problem.

Recall that for a one-dimensional scalar conservation law \( u_t + [f(u)]_x = 0 \) with initial data \( u(x,0) = g(x) \), a shock will form at some \( t > 0 \) precisely if \( f'[g(x)] \) is a decreasing function. We see that the Lax criterion (308) is then automatically obeyed once the shock has formed.

4.3.5 A few words on systems and higher dimensions

For a system of \( N \) conservation laws in one space dimension, the Rankine-Hugoniot condition can be derived in the same way, and is

\[
 s(u_L - u_R) = f(u_L) - f(u_R).
\]

This poses \( N-1 \) constraints between the \( 2N \) components of \( u_L \) and \( u_R \), as well as one equation for the shock speed \( s \), so we cannot choose \( u_L \) and \( u_R \) freely. A similar statement holds for rarefaction waves in a system.

Hence constructing weak solutions for systems is more complicated. Roughly speaking, if we look for the solution of a Riemann problem with arbitrarily given left and right states \( u_L \) and \( u_R \) in a system of \( N \) conservation laws, the solution will consist of \( N \) “waves” sandwiched between the left and right state and \( N-1 \) intermediate states. Here a “wave” means a shock, rarefaction wave, or a third kind of similarity solution called a contact discontinuity.

The statement that a weak shock propagates at a characteristic speed is still true for a system, except that a system of \( N \) conservation laws now has \( N \) characteristic speeds (the eigenvalues of the matrix \( \partial f/\partial u \)).

In more space dimensions, say three, an isolated shock between constant states is planar, so we can simply orient our coordinate system so that the shock propagates in the \( x \)-direction, and nothing depends on \( y \) and \( z \). A similar statement holds for rarefaction waves. So there is nothing fundamentally different from one space dimension, but of course solutions can become extremely complicated in practice.
4.3.6 Inequivalence of different weak forms of a conservation law

As an example, consider Burgers’ equation

\[ u_t + uu_x = 0. \]  
\[ (310) \]

(It is a toy model for the Euler equation with constant pressure, and the simplest conservation law after the advection equation.) If we think of \( u \) as the conserved quantity, then the corresponding flux is \( f(u) = u^2/2 \), and the conservation law form is

\[ u_t + \left( \frac{u^2}{2} \right)_x = 0, \]
\[ (311) \]

with corresponding integral form

\[ \int_a^b u(x, t_1) \, dx - \int_a^b u(x, t_0) + \int_{t_0}^{t_1} \frac{u(b, t)^2}{2} \, dt - \int_{t_0}^{t_1} \frac{u(a, t)^2}{2} \, dt = 0. \]
\[ (312) \]

The shock velocity is then

\[ s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = \frac{u_L^2 - u_R^2}{2(u_L - u_R)} = \frac{u_L + u_R}{2}. \]
\[ (313) \]

But if we multiply (310) by \( 2u \), we are led to the conservation law form

\[ (u^2)_t + \left( \frac{2}{3} u^3 \right)_x = 0, \]
\[ (314) \]

or, with \( U := u^2 \),

\[ U_t + \left( \frac{2}{3} U^{3/2} \right)_x = 0, \]
\[ (315) \]

and a different integral form. That the two integral forms are actually inequivalent is easy to see by considering the shock velocity

\[ s = \frac{f(U_L) - f(U_R)}{U_L - U_R} = \frac{2(u_L^3 - u_R^3)}{3(u_L^2 - u_R^2)}, \]
\[ (316) \]

which is different from (313).

But for differentiable solutions, both integral forms are equivalent to the differential form (310) and hence to each other. So the integral form contains additional information about non-differentiable solutions. In practice, it is usually clear what the correct integral form is because of what is physically conserved (mass, momentum, energy, molecules, cars on a road, electric charge, etc., but not, say, mass^2).

4.4 Example: Traffic flow

Look again at (280) but interpret it as the conservation of cars on a road, where \( u \) is now measured in cars/length, and \( v \) is the velocity of the traffic flow, measured of course in length/time. In this context, (280) is called the traffic flow equation.

Now let us look a particular velocity law for traffic flow. There is maximum density \( u_{\text{max}} \) when cars are bumper to bumper, and there is a maximum velocity \( v_{\text{max}} \) given by the speed limit. On an open road, cars will go at the speed limit, but in dense traffic they will slow down until they reach the maximum density at zero speed – a traffic jam. For simplicity, we assume \( v(u) \) to be the linear function defined by these two points, namely

\[ v(u) = v_{\text{max}} \left( 1 - \frac{u}{u_{\text{max}}} \right), \]
\[ (317) \]
and hence
\[ f(u) = uv_{\text{max}} \left( 1 - \frac{u}{u_{\text{max}}} \right). \tag{318} \]

The characteristic velocities are therefore
\[ f'(u) = v_{\text{max}} \left( 1 - \frac{2u}{u_{\text{max}}} \right). \tag{319} \]

Hence \( f'(u) \) is a decreasing function. Finally, the shock velocity is given by
\[ s = \frac{f(u_L) - f(u_R)}{u_L - u_R} = v_{\text{max}} \frac{u_L \left( 1 - \frac{u_L}{u_{\text{max}}} \right) - u_R \left( 1 - \frac{u_R}{u_{\text{max}}} \right)}{u_L - u_R} \]
\[ = v_{\text{max}} \frac{(u_L - u_R) - (u_L^2 - u_R^2)}{u_L - u_R} = v_{\text{max}} \left( 1 - \frac{u_L + u_R}{u_{\text{max}}} \right). \tag{320} \]

Consider now the evolution of two kinds of initial data:

1) Assume that \( g(x) \) is a decreasing function. From (319) we see that \( f'(u) \) is a decreasing function of \( u \). Hence \( f'(g(x)) \) is an increasing function of \( x \). In other words, the characteristic velocity in the initial data increase with \( x \). Hence the characteristics fan out from \( t = 0 \) and never intersect. Physically, the cars in front are in less dense traffic and hence move faster. Hence the initial density profile is stretched out. The solution remains smooth for all \( t > 0 \).

2) \( g(x) \) is an increasing function. Then the characteristics converge from \( t = 0 \). Physically, the cars in front are in denser traffic and hence move more slowly, allowing the cars behind to catch up. Hence the initial density profile is compressed and becomes steeper, until a moving shock forms in the traffic flow at which each driver suddenly hits the brakes.

### 4.5 Exercises

31. **Homework 13:** (Very short) What is the dimension of \( u \) and \( f_i \) in \( n \) space dimensions? Check that \( u_t + \sum f_i, i = 0 \) is dimensionally consistent.

32. **Homework 14:** Use separation of variables and Fourier transforms to solve the Cauchy problem for the advection equation on the line,
\[ u_t + v_0 u_x = 0, \quad -\infty < x < \infty, \quad t \geq 0, \]
\[ u \rightarrow 0 \text{ as } x \rightarrow \pm \infty, \]
\[ u(x, 0) = g(x) \]
Then(!) show that this can be written as \( u(x, t) = g(x - v_0 t) \).

33. **Homework 15:** Use the method of characteristics to show that the solution of the Burgers equation with linear initial data,
\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \]
\[ u(x, 0) = ax, \]
is
\[ u(x, t) = \frac{ax}{1 + at} \]

34. **Homework 16:** Find the shock solution of the Riemann problem to the Burgers equation
\[ u_t + \left( \frac{u^2}{2} \right)_x = 0, \quad u(x, 0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases} \]

Find when the shock solution is physical.
35. **Homework 17:** Find the rarefaction wave solution to the Riemann problem for the Burgers equation, and state when it actually exists.

36. **Homework 18:** (Hard, but important) For the traffic flow Riemann problem

\[ u_t + \left( v_{\text{max}} \left( u - \frac{u^2}{2u_{\text{max}}} \right) \right)_x = 0, \quad u(x,0) = \begin{cases} u_L = au_{\text{max}}, & x < 0 \\ u_R = bu_{\text{max}}, & x > 0 \end{cases}, \tag{328} \]

find the shock location \( x_s(t) \), the characteristics \( \xi(x_0,t) \). Let \( v_{\text{max}} = 60 \text{mph} \), and consider the two cases where \( a = 0.3, b = 0.9 \), and \( a = 0.9, b = 0.3 \).

37. * (Also hard, for enthusiasts) For the solution of the traffic flow example, also find the car trajectories \( \tilde{\xi}(x_0,t) \).

38. * Consider the PDE problem

\[ u_t + [f(u)]_x = 0, \quad -\infty \leq x \leq \infty, \quad t \geq 0, \tag{329} \]
\[ u(x,0) = g(x) \tag{330} \]

Determine if these data will form a shock, and if so compute the time \( t_s \) when the shock first forms.

**Hint:** a) Recall the method of characteristics for solving this problem graphically, and recall that a shock forms when two characteristics cross. b) By drawing a picture, or otherwise, convince yourself that the first characteristics that cross will be two neighbouring characteristics. In other words, you cannot have characteristics starting at \( x_01 \) and \( x_02 \) crossing without some characteristics starting from intermediate values of \( x_0 \) crossing first, or at the same time. c) Now look at characteristics starting from \( x_0 \) and \( x_0 + h \), and find out where they cross, working to leading order in \( h \). You will find that to leading order \( t_s = t_s(x_0,h) \) does not depend on \( h \). d) Now find the smallest value of \( t_s(x_0) \) and you are done.

39. * Solve the PDE problem

\[ u_t + \left( \frac{1}{2} u^2 \right)_x = 0, \quad -\infty \leq x \leq \infty, \quad t \geq 0, \tag{331} \]
\[ u(x,0) = \begin{cases} u_L, & x < 0 \\ u_L + \frac{u_L - u_R}{L} x, & 0 < x < L, \\ u_R, & x > 0. \end{cases} \tag{332} \]

(Burgers’ equation with continuous, piecewise linear initial data.)

**Hint:** You will need to solve this problem separately for the two cases \( u_L < u_R \) and \( u_L > u_R \), as the solutions are qualitatively different. Sketch the initial data. Sketch some characteristics in the \( (t,r) \)-plane. Recall shocks and rarefaction waves. Recall the solution of another homework problem where \( u(x,0) = ax \) for \(-\infty \leq x \) (what is \( a \) here?). Try to glue together a solution from these ingredients in different regions of the \( (x,t) \)-plane.

40. Show that (303) cannot be a strong solution of the conservation law \( u_t + f_x = 0 \), because it is not differentiable at \( x = z_{L,R}t \). By considering suitable rectangles in the \( rt \)-plane whose diagonals are \( x = z_{L,R}t \), show that (303) is a weak solution in the sense of Def. 4.1.

41. Show that the first-order system of conservation laws in one space dimension

\[ v_t - w_x = 0 \tag{333} \]
\[ w_t - v_x = 0 \tag{334} \]

is equivalent to the wave equation in one dimension \( u_{tt} = u_{xx} \). State and solve the Riemann problem for arbitrary \( v_L, w_L, u_R, w_R \). [Hint: start from the d’Alembert solution (72), or try a solution that is piecewise constant in the \( xt \)-plane, as we did to obtain (294).]
5 Elementary generalised functions

5.1 Test functions

Definition 5.1. A function \( \phi(x) \) is a test function if it has the following properties:

- \( \phi(x) \) and all its derivatives exist and are continuous at all points \(-\infty < x < \infty \) (\( \phi \) is smooth);
- the integrals \( \int_{-\infty}^{\infty} \phi(x) \) and all its derivatives exist and are finite.

Note that since the integrals of a test function and all its derivatives exist, the test function and all its derivatives must vanish as \( x \to \pm \infty \).

A simple example of a test function is

\[
\phi(x) = e^{-x^2},
\]

which is analytic. Another example is the function

\[
f(x) = \begin{cases} 
0, & x \leq a \\
\frac{1}{\sqrt{\pi}} e^{-\frac{1}{x-a}(b-x)}, & a < x < b \\
0, & x \geq b
\end{cases}
\]

which is smooth but vanishes outside the interval \((a,b)\) (and is not analytic at \( x = a \) and \( x = b \)).

5.2 The \( \delta \)-function

Definition 5.2. A generalised function \( G \) (also called a distribution) is a linear map from test functions to real numbers. That is, it assigns to each test function \( \phi \) a number \( G[\phi] \).

This may seem a little abstract, but in practice we can think of generalised functions as “functions” that are only defined when integrated over a test function. We can then write

\[
G[\phi] := \int_{-\infty}^{\infty} G(x)\phi(x) \, dx.
\]

In the following, we only use this integral notation, and never \( G[x] \).

The most important generalised function is the \( \delta \)-function, which in spite of its name is not a function, but a generalised function. Intuitively, one can think of it as a “function” that is only defined under an integral.

Definition 5.3. The \( \delta \)-function is the generalised function defined by

\[
\int_{-\infty}^{\infty} \delta(x)\phi(x) \, dx := \phi(0)
\]

for every test function \( \phi(x) \).

Remark 5.4. This can be shown to be equivalent to

\[
\delta(x) = 0 \quad \text{if} \quad x \neq 0, \quad \int_{-\infty}^{\infty} \delta(x) \, dx = 1.
\]

Note that we do not, and cannot, assign a value to \( \delta(0) \).

Remark 5.5. We can also define the \( \delta \)-function as the limit of various sequences of regular functions, for example

\[
\delta(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2}.
\]

This agrees with the previous definition because

\[
\lim_{\epsilon \to 0} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2} = 0 \quad \text{if} \quad x \neq 0
\]
and
\[
\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2} \, dx = \frac{1}{\pi} \arctan \left( \frac{x}{\epsilon} \right) \bigg|_{-\infty}^{\infty} = 1
\]
for any \( \epsilon > 0 \), so that
\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2} \, dx = 1, \tag{343}
\]
see Fig. 2. This definition of the \( \delta \)-function as a limit has a physical interpretation in terms of a very large force acting over a very short time, while conveying a finite momentum. In applications the \( \delta \)-function can be used to represent an impulse, e.g. when a string is hit with a hammer.

**Remark 5.6.** More generally, if \( f(x) \) is any function such that \( \int_{-\infty}^{\infty} f(x) \, dx = 1 \) then
\[
\lim_{\epsilon \to 0} \frac{1}{\epsilon} f \left( \frac{x}{\epsilon} \right) = \delta(x). \tag{344}
\]
Say \( \phi(x) \) is a test function. Then, with the change of variable \( y = x/\epsilon \),
\[
\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} \frac{1}{\epsilon} f \left( \frac{x}{\epsilon} \right) \phi(x) \, dx = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f(y) \phi(\epsilon y) \, dy = \int_{-\infty}^{\infty} f(y) \phi(0) \, dy = \phi(0) \int_{-\infty}^{\infty} f(y) \, dy = \phi(0). \tag{345}
\]

### 5.3 The Heaviside and signum functions

**Definition 5.7.** The **Heaviside function** is the generalised function defined by
\[
\int_{-\infty}^{\infty} H(x) \phi(x) \, dx := \int_{0}^{\infty} \phi(x) \, dx \tag{346}
\]
for all test functions \( \phi(x) \).

**Remark 5.8.** It is easy to see that this is equivalent to
\[
H(x) = \begin{cases} 
1, & x > 0 \\
0, & x < 0 
\end{cases} \tag{347}
\]
but note that we do not need to assign a value to \( H(0) \).
Remark 5.9. We have
\[
\int_{-\infty}^{x} \delta(y) \, dy = H(x).
\] (348)

Because \(\delta(x) = 0\) for \(x \neq 0\), it only matters if \(y = 0\) is part of the integration domain \(-\infty < y < x\). Note also that the left-hand side of this equation is not defined for \(x = 0\), and hence for consistency we cannot give \(H(x)\) a value at \(x = 0\) either.

In applications the Heaviside function is often used as a “switch”. For example, it can be used to mathematically model an electrical circuit where the power is switched on at a specific moment in time.

Definition 5.10. The signum function \(\text{sgn}(x)\) is defined by
\[
\text{sgn}(x) = H(x) - H(-x).
\] (349)

Hence we have
\[
\text{sgn}(x) = \begin{cases} 
1, & x > 0 \\
-1, & x < 0 
\end{cases}.
\] (350)

Clearly \(\text{sgn}(x)\) is just the sign of the number \(x\), but \(\text{sgn}(0)\) is not defined. Considering \(\text{sgn}(x)\) as a generalised function allows us to show that
\[
\text{sgn}(x) = \frac{d|x|}{dx}
\] (351)

and that
\[
\frac{d}{dx} \text{sgn}x = 2\delta(x).
\] (352)

5.4 Generalised functions and derivatives

We are going to use generalised functions to solve inhomogeneous differential equations. In manipulations we will often need the derivative, \(G'(x)\), of a generalised function, \(G(x)\).

Definition 5.11. The derivative \(G'(x)\) of the generalised function \(G(x)\) is defined by
\[
\int_{-\infty}^{\infty} G'(x) \phi(x) \, dx := - \int_{-\infty}^{\infty} G(x) \phi'(x) \, dx
\] (353)

for all test functions \(\phi(x)\).
Remark 5.12. This definition is essentially integration by parts (since test functions vanish at \( \pm \infty \)). The right-hand-side of this equation is always known since that is how \( G(x) \) is defined, and if \( \phi \) is a test function so is \( \phi' \). In this sense we have
\[ H'(x) = \delta(x), \] (354)
because
\[ \int_{-\infty}^{\infty} H'(x) \phi(x) \, dx = - \int_{-\infty}^{\infty} H(x) \phi'(x) \, dx = - \int_{0}^{\infty} \phi'(x) \, dx = - \phi(x)|_{0}^{\infty} = \phi(0). \] (355)

Remark 5.13. The derivative of the \( \delta \)-function, \( \delta'(x) \), is defined by
\[ \int_{-\infty}^{\infty} \delta'(x) \phi(x) \, dx = - \int_{-\infty}^{\infty} \delta(x) \phi'(x) \, dx = - \phi'(0) \] (356)
for all test functions \( \phi(x) \).

We can also define \( \delta' \) as the limit of a sequence of functions such as
\[ \delta'(x) = \lim_{\epsilon \to 0} \frac{d}{dx} \frac{1}{\pi} \frac{\epsilon}{\epsilon^2 + x^2} = \lim_{\epsilon \to 0} -\frac{2}{\pi} \frac{\epsilon x}{(\epsilon^2 + x^2)^2} \] (357)
as shown in Fig. 4.

5.5 Properties of the \( \delta \)-function
All of the following properties are a consequence of the definition of the \( \delta \)-function and the standard properties of integrals:
1. \( \int_{-\infty}^{\infty} \phi(x) \delta(x) \, dx = \phi(0) \)
2. \( \int_{-\infty}^{\infty} \phi(x) \delta(x-a) \, dx = \phi(a) \)
3. \( \delta(-x) = \delta(x) \)
4. \( \delta(ax) = \delta(x)/|a| \)
5. $\delta(a^2 - x^2) = [\delta(x-a) + \delta(x+a)] / (2|a|)$

6. $x\delta(x) = 0$

7. $g(x)\delta(x) = g(0)\delta(x)$ provided $g(x)$ is continuous and $g(0)$ exists.

8. $H'(x) = \delta(x)$ and $H(x) = \int_{-\infty}^{x} \delta(x) \, dx$.

9. $\int_{-\infty}^{\infty} \delta'(x) \phi(x) \, dx = -\phi'(0)$

10. $(g(x)\delta(x))' = g(0)\delta'(x)$

**Example 5.14.** Let us take a closer look at (5). Consider $\delta(a^2 - x^2)$ as a generalised function and let it operate on a test function $\phi(x)$, i.e. evaluate

$$I = \int_{-\infty}^{\infty} \phi(x)\delta(a^2 - x^2) \, dx.$$ (558)

Change variables to $y = a^2 - x^2$ and use

$$x = \sqrt{a^2-y} \quad \text{for} \quad x > 0,$$

$$x = -\sqrt{a^2-y} \quad \text{for} \quad x < 0.$$ (559)

Then we get

$$I = \int_{x=0}^{\infty} \phi(x)\delta(y) \left[ -\frac{dy}{2\sqrt{a^2-y}} \right] + \int_{x=-\infty}^{x=0} \phi(x)\delta(y) \left[ -\frac{dy}{2\sqrt{a^2-y}} \right] =$$

$$= \int_{-\infty}^{a^2} \frac{\phi(\sqrt{a^2-y})\delta(y) \, dy}{2\sqrt{a^2-y}} + \int_{-\infty}^{a^2} \frac{\phi(-\sqrt{a^2-y})\delta(y) \, dy}{2\sqrt{a^2-y}} =$$

$$= \frac{1}{2|a|}\left[\phi(|a|) + \phi(-|a|)\right] = \frac{1}{2|a|} \int_{-\infty}^{\infty} \left[\delta(x-a) + \delta(x+a)\right] \phi(x) \, dx.$$ (561)

**5.6 Exercises**

42. **Homework 19:** Show that $\lim_{\epsilon \to 0} f_\epsilon(x) = \delta(x)$, where

$$f_\epsilon(x) := \frac{\epsilon}{\pi \epsilon^2 + x^2}.$$ (562)

43. * Prove that, in the sense of generalised functions, $\delta(-x) = \delta(x)$.

44. * Prove that

$$\int_{-\infty}^{\infty} \delta(x-y)f(x) \, dx = f(y).$$ (563)

45. Verify that as $\epsilon \to 0$ we have

$$\lim_{\epsilon \to 0} G(x) = 0 \quad \text{for} \quad x \neq 0, \quad \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} G(x) \, dx = 1$$ (564)

for the following functions:

(a) $G(x) = \frac{\epsilon}{\pi \epsilon^2 + x^2}$,

(b) $G(x) = \begin{cases} 1/2\epsilon, & |x| < \epsilon \\ 0, & |x| > \epsilon \end{cases}$,

(c) $G(x) = \begin{cases} (\epsilon - |x|)/\epsilon^2, & |x| < \epsilon \\ 0, & |x| > \epsilon \end{cases}$,

(d) $G(x) = \frac{1}{\sqrt{\pi \epsilon^2}} e^{-x^2/\epsilon^2}$. (565)
46. Show that
\[ \int_a^b \delta(x) \, dx = H(b) - H(a), \quad \int_{-\infty}^{\infty} \delta''(x) \phi(x) \, dx = \phi''(0) \] (367)
\[ \int_{-\infty}^{\infty} \delta^{(n)}(x) \phi(x) \, dx = (-1)^n \phi^{(n)}(0) \] (368)
where \( \phi^{(n)}(x) \) denotes the \( n \)-th derivative of the test function \( \phi(x) \).

47. Sketch the following functions and find both of their first and second derivatives:
\[ H(x)e^{2x}, \quad H(x)(3 - x), \quad H(x) - H(x - 1), \quad e^{xH(x)}, \quad [H(x) - H(x - \pi/2)]\sin(x), \quad H(1 + x^2). \] (369) (370)

48. Find the numerical values of
\[ \int_{-1}^{1} \delta(x) \, dx, \quad \int_{-5}^{3} (2\delta(x) + 3\delta'(x)) \, dx, \quad \int_{-1}^{1} e^{5x} \delta(x) \, dx, \] (371)
\[ \int_{-1}^{1} (3 + x^2) \delta(x) \, dx, \quad \int_{-1}^{1} e^{2x} \delta'(x) \, dx, \quad \int_{0}^{3} x^2 \delta'(x - 2) \, dx, \] (372)
\[ \int_{-\infty}^{\infty} \cos^2(x) \delta''(x) \, dx, \quad \int_{-107}^{32} (x^2 - 3x + 5) \delta''(x - 1/2) \, dx. \] (373)

49. Simplify
\[ \delta(2x - 8), \quad \delta([x - a](x - b)). \] (374)

50. Using the generalized definition of derivative, show that
\[ \frac{d |x|}{dx} = \text{sgn}(x), \quad \frac{d^2 |x|}{dx^2} = 2\delta(x). \] (375)

51. Define the generalised sine and cosine functions, \( S(x) \) and \( C(x) \), by
\[ \int_{-\infty}^{\infty} S(x) \phi(x) \, dx = \int_{-\infty}^{\infty} \sin(x) \phi(x) \, dx, \] (366)
\[ \int_{-\infty}^{\infty} C(x) \phi(x) \, dx = \int_{-\infty}^{\infty} \cos(x) \phi(x) \, dx. \] (377)
(Here \( \sin(x) \) and \( \cos(x) \) are the standard sine and cosine functions.) Show that, in the generalized sense,
\[ S'(x) = C(x), \quad C'(x) = -S(x). \] (378)

52. Verify that the 2-dimensional \( \delta \)-function
\[ \delta(x, y) := \delta(x)\delta(y) \] (379)
can also be defined as either of the limits
\[ \lim_{\epsilon \to 0} \begin{cases} 1/4\epsilon^2 & |x| < \epsilon \text{ and } |y| < \epsilon \\ 0 & |x| > \epsilon \text{ or } |y| > \epsilon \end{cases} \] (380)
or
\[ \lim_{\epsilon \to 0} \frac{\epsilon}{2\pi(x^2 + y^2 + \epsilon^2)^{3/2}} \] (381)

Hint: use polar coordinates and recall that
\[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dx \, dy = \int_{0}^{\infty} \int_{0}^{2\pi} f(r, \theta) \, d\theta \, dr. \] (382)

53. By a change of integration variable, show that
\[ \delta[f(x)] = \sum_i \delta(x - x_i) \left| f'(x_i) \right|, \] (383)
where \( x_i \) are the zeros of \( f(x) \), so that \( f(x_i) = 0 \).

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6 Green’s functions for ODEs

In this Section we introduce the concept of Green’s function, which can be used to solve inhomogeneous differential equations. Since the technique readily generalises from ODEs to PDEs we will first consider the simpler case of ODEs. We begin with the general definition, and then consider some examples.

**Definition 6.1.** A Green’s function for the inhomogeneous linear ODE

\[ Ly(t) = f(t), \]  

(384)

where \( L = L(t, d/dt) \) is any homogeneous linear ordinary differential operator, is a generalised function \( G(t, s) \) of two variables that satisfies

\[ L_t G(t, s) = \delta(t - s), \]  

(385)

where \( L_t \) signifies that the differential operator \( L \) acts on the variable \( t \), not \( s \).

**Remark 6.2.** The reason for this definition is this: if we put

\[ y(t) = \int_{-\infty}^{\infty} G(t, s)f(s) \, ds, \]  

(386)

we find that

\[ Ly(t) = \int_{-\infty}^{\infty} L_t G(t, s)f(s) \, ds = \int_{-\infty}^{\infty} \delta(t - s)f(s) \, ds = f(t), \]  

(387)

so that (386) is a solution of (384).

**Remark 6.3.** The above definition of a Green’s function becomes unique only when we complement it with appropriate boundary conditions, either on the ODE, or equivalently on the Green’s function. Clearly, two different Green’s functions for the same problem must differ by a solution of the homogeneous problem \( L_t G = 0 \).

### 6.1 A simple example: first-order linear ODE with constant coefficients

**Example 6.4.** A Green’s function for the first-order linear ODE with constant coefficients,

\[ \dot{y} + ay = f(t) \]  

(388)

is a generalised function \( G(t, s) \) of two variables \( t \) and \( s \) that satisfies

\[ G_t + aG = \delta(t - s). \]  

(389)

To find the Green’s function \( G(t, s) \) for (388), we note that for \( t \neq s \) we have

\[ G_t + aG = 0, \]  

(390)

and hence we have

\[ G(t, s) = \begin{cases} A(s)e^{-at} & \text{for } t < s \\ B(s)e^{-at} & \text{for } t > s, \end{cases} \]  

(391)

where \( A(s) \) and \( B(s) \) are integration constants. (They are functions of \( s \), rather than true constants because, although \( G(t, s) \) obeys an ODE in the single variable \( t \), it also depends on the parameter \( s \), and so these integration “constants” can depend on \( s \). You may remember seeing something similar when you learned how to solve exact PDEs by integration.)

We now recall that a \( \delta \)-function is the derivative of the Heaviside function, i.e. a jump discontinuity of unit size. Thus it will be enough to make \( G(t, s) \) jump by one as we go across \( t = s \). Thus we want

\[ \lim_{t \to s^-} G(t, s) = \lim_{t \to s^-} G(t, s) + 1, \]  

(392)
or

\[ B(s)e^{-as} = A(s)e^{-as} + 1, \]  
(393)

from which it follows that

\[ B(s) = A(s) + e^{as}. \]  
(394)

Thus we have

\[ G(t, s) = \begin{cases}  
  A(s)e^{-at} & t < s \\
  A(s)e^{-at} + e^{a(s-t)} & t > s,
\end{cases} \]  
(395)

which can be written as

\[ G(t, s) = A(s)e^{-at} + H(t-s)e^{-a(t-s)}. \]  
(396)

Then we see that (386) implies

\[ y(t) = \int_{-\infty}^{\infty} G(t, s)f(s) \, ds 
= e^{-at} \int_{-\infty}^{\infty} A(s)f(s) \, ds + \int_{-\infty}^{\infty} H(t-s)e^{-a(t-s)}f(s) \, ds 
= Ce^{-at} + \int_{-\infty}^{t} e^{-a(t-s)}f(s) \, ds. \]  
(397)

The term \( Ce^{-at} \) is of course the general solution of (388) with \( f(t) = 0 \), or in other words, the complementary function. Like an ordinary solution to this ODE, the Green’s function needs boundary conditions to be uniquely defined.

If we choose \( A(s) = 0 \), and hence \( C = 0 \), then

\[ G(t, s) = H(t-s)e^{-a(t-s)} = \begin{cases}  
  0, & t < s \\
  e^{-a(t-s)}, & t > s
\end{cases} \]  
(398)

This Green’s function represents the reaction of the system to a unit impulse at time \( t = s \) where the system is at rest prior to the impulse; see Figs. 5 and 6. It is called the causal Green’s function. The solution (386) with (398) is

\[ y(t) = \int_{-\infty}^{t} e^{-a(t-s)}f(s) \, ds. \]  
(399)
Figure 6: The Green’s function $H(t-s)e^{-(t-s)}$ for $-2 < s < 2$, $-2 < t < 2$.

It is called the **causal solution** because it depends only on $f(s)$ for $s < t$, that is, it depends only on the values of $f(s)$ before the present time. Thus this solution can be said to be *caused* by the driving force $f(t)$. This is in contrast to the general case (386), where $y(t)$ can depend on the values of $f(s)$ for both $s < t$ (i.e., values that have already occurred) and $t < s$ (i.e., values that have yet to happen). One is typically mainly interested in the causal solution to a physical problem. In general, we obtain such solutions by imposing $G(t, s) = 0$ for $t < s$. The factor $H(t - s)$ in (398) obviously makes sure of that.

### 6.2 Another example: the harmonic oscillator

**Definition 6.5.** The causal Green’s function for the harmonic oscillator problem

\[
\ddot{y} + \omega^2 y = f(t) \tag{400}
\]

is a function $G(t, s)$ that satisfies

\[
G_{tt} + \omega^2 G = \delta(t - s). \tag{401}
\]

and $G = 0$ for $t < s$.

We take $G(t, s) = 0$ for $t < s$. For $t > s$ we have

\[
G_{tt} + \omega^2 G = 0, \tag{402}
\]

so that

\[
G(t, s) = A(s)e^{it\omega} + B(s)e^{-it\omega}. \tag{403}
\]

The $\delta$-function in (401) must come from the term $G_{tt}$, which implies that $G_t$ must jump by one as we go across $t = s$. This implies that $G(t, s)$ must be continuous across $t = s$. If $G(t, s)$ was discontinuous then $G_t$ would have a $\delta$-function and $G_{tt}$ would contain $\delta'(t)$.

This can be derived more formally in the following way. Assume that the Green’s function is continuous, but that its derivative may have discontinuities. Then integrate (401) from $s - \epsilon$ to $s + \epsilon$. This gives

\[
\int_{s-\epsilon}^{s+\epsilon} \left[ G_{tt} + \omega^2 G \right] dt = G_t|_{s+\epsilon} - G_t|_{s-\epsilon} = \int_{s-\epsilon}^{s+\epsilon} \delta(t - s) dt = 1. \tag{404}
\]
(In the limit $\epsilon \to 0$, the integral over $\omega^2 G$ vanishes). This method can be used also for more complicated equations.

As we let $t \to s+$, we have

$$
\lim_{t \to s^+} G(t, s) = A(s)e^{i\omega s} + B(s)e^{-i\omega s},
$$

which must be zero since $G(t, s) = 0$ for $t < s$ and we want $G(t, s)$ to be continuous. Thus

$$
A(s)e^{i\omega s} + B(s)e^{-i\omega s} = 0.
$$

We have $G_t = 0$ for $t < s$ and we want $G_t$ to jump by one as we go across $t = s$, so we must have

$$
\lim_{t \to s^+} G_t = i\omega A(s)e^{i\omega s} - i\omega B(s)e^{-i\omega s} = 1.
$$

Solving for $A(s)$ and $B(s)$ we find that

$$
A(s) = \frac{1}{2\omega i}e^{-i\omega s}, \quad B(s) = -\frac{1}{2\omega i}e^{i\omega s},
$$

and hence (see Fig. 7)

$$
G(t, s) = \begin{cases} 
0, & t < s \\
\frac{1}{2\omega} \sin \omega(t - s), & t > s 
\end{cases}
$$

$$
= \frac{1}{\omega} H(t - s) \sin \omega(t - s).
$$

Thus, the causal solution of (400) is

$$
y(t) = \frac{1}{\omega} \int_{-\infty}^{t} f(s) \sin \omega(t - s) \, ds.
$$

![Figure 7: The Green’s function $H(t - s)e^{-(t-st)}$ as a function of $s$ and $t$.](image)

### 6.3 The general second order linear ODE with constant coefficients

The procedure from the previous Section works for more general equations. Take the general linear second order ODE with constant coefficients,

$$
\ddot{y} + \alpha \dot{y} + \beta y = f(t).
$$

The causal Green’s function is the solution of

$$
G_{tt} + \alpha G_t + \beta G = \delta(t - s),
$$

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which satisfies \( G(t, s) = 0 \) for \( t < s \). Finding \( G(t, s) \) involves solving
\[
G_{tt} + \alpha G_t + \beta G = 0
\]
for \( t > s \), and the solution is
\[
G(t, s) = A(s)e^{\lambda_1 t} + B(s)e^{\lambda_2 t},
\]
where \( \lambda_1 \) and \( \lambda_2 \) are the roots of the quadratic equation
\[
\lambda^2 + \alpha \lambda + \beta = 0.
\]
Thus \( G(t, s) \) may be written as
\[
G(t, s) = \begin{cases} 
0 & t < s \\
A(s)e^{\lambda_1 t} + B(s)e^{\lambda_2 t} & t > s.
\end{cases}
\]
(Here we have assumed that \( \lambda_1 \neq \lambda_2 \).) In order to obtain the term \( \delta(t-s) \) we need \( G(t, s) \) to be continuous at \( t = s \) and \( G_t(t, s) \) to have a jump of magnitude one. Thus we find that
\[
A(s)e^{\lambda_1 s} + B(s)e^{\lambda_2 s} = 0
\]
from the continuity of \( G(t, s) \) at \( t = s \), and that
\[
\lambda_1 A(s)e^{\lambda_1 s} + \lambda_2 B(s)e^{\lambda_2 s} = 1
\]
from the jump in \( G_t \). These are two equations for the functions \( A(s) \) and \( B(s) \) which we can solve to find them. Once we have done this, the Green’s function is
\[
G(t, s) = H(t-s) \left[ A(s)e^{\lambda_1 t} + B(s)e^{\lambda_2 t} \right].
\]

6.4 Initial-value problems

Rather than starting at \( t = -\infty \) and looking for the causal solution, we often want to solve the initial value problem. We will illustrate this with the example of the general second-order linear ODE (not restricting to constant coefficients). It should be clear afterwards how to apply the same method to a first-order or higher-order linear ODE.

So consider the initial value problem
\[
\ddot{y} + \alpha(t)\dot{y} + \beta(t)y = f(t), \quad y(0) = A, \quad \dot{y}(0) = B
\]
for \( t > 0 \). We can extend the Green’s function methods described above to deal with this situation. We define a new function
\[
z(t) = H(t)y(t),
\]
so that \( z(t) \) equals \( y(t) \) for \( t \geq 0 \) and is zero for \( t < 0 \). Then
\[
\ddot{z}(t) = y(0)\delta(t) + \dot{y}(t)H(t) = A\delta(t) + \dot{y}(t)H(t)
\]
and
\[
\dddot{z}(t) = A\delta'(t) + \ddot{y}(t)\delta(t) + \dot{y}(t)H(t) = A\delta'(t) + B\delta(t) + \dot{y}(t)H(t),
\]
and thus
\[
\dddot{z} + \alpha(t)\ddot{z} + \beta(t)z = A\delta'(t) + [\alpha(0)A + B]\delta(t) + H(t)f(t): = \tilde{f}(t).
\]
Now assume that \( G(t, s) \) is the causal Green’s function, so that the causal solution of (424) is
\[
z(t) = \int_{-\infty}^{t} G(t, s)\tilde{f}(s) \, ds
\]
\[
= \int_{-\infty}^{t} G(t, s) \{ A\delta'(s) + [\alpha(0)A + B]\delta(s) + H(s)f(s) \} \, ds
\]
\[
= H(t) \left\{ -AG_{\lambda}(t, 0) + [\alpha(0)A + B]G(t, 0) + \int_{0}^{t} G(t, s)f(s) \, ds \right\}.
\]
The factor $H(t)$ multiplies the first two terms because $\delta(s)$ and $\delta'(s)$ both vanish for $s \neq 0$, and hence for $t < 0$ they vanish everywhere inside the integration range. Similarly, $H(t)$ also multiplies the third term because $H(s) = 0$ for $s < 0$, and so if $t < 0$, it vanishes everywhere inside the integration range. Finally, if $t > 0$, $H(s) = 1$ still only for $s > 0$, and so we can remove it if we limit the integration range to $s > 0$ and $s < t$.

For $t > 0$ we have $g(t) = z(t)$ and hence

$$y(t) = -AG_s(t,0) + [\alpha(0)A + B]G(t,0) + \int_0^t G(t,s)f(s)\,ds. \quad (426)$$

$G(t,0)$ obeys the homogeneous equation $y'' + \alpha y' + \beta y = 0$ for $t > 0$ because $G(t,s)$ does for $t > s$ and $s = 0$ here. Similarly, $G_s(t,0)$ also obeys the homogeneous equation. The third term obeys the inhomogeneous equation $y'' + \alpha y' + \beta y = f(t)$ for $t > 0$ by construction. Therefore the first two terms can be considered as the “complementary function” and the third term as the “particular integral”.

(426) obeys the initial conditions by construction, but it is instructive to check this explicitly. Note that at $t = s$, $G(t,s)$ is continuous, and $G_+(t,s)$ jumps by 1. This implies that

$$G(t,s) = H(t-s) \left[(t-s) + (t-s)^2g(t,s)\right] \quad (427)$$

where $g(t,s)$ is some regular function of $t$ and $s$. But from this we have that

$$G(0+,0) = 0, \quad G_s(0+,0) = -1, \quad G_+(0+,0) = 1, \quad G_{ts}(0+,0) = -2g(0,0) = -G_++.$$ \quad (428)

Furthermore, $G(t,s)$ obeys the homogenous ODE $G_{tt} + \alpha G_t + \beta G = 0$ for $t > s$. Therefore

$$G_+(0+,0) = -\alpha(0)G_+(0+,0) - \beta(0)G(0+,0) = -\alpha(0). \quad (429)$$

Using these results, we have

$$y(0) = -AG_s(0+,0) + (\alpha(0)A + B)G(0+,0) = A, \quad (430)$$

$$y'(0) = -AG_{ts}(0+,0) + (\alpha(0)A + B)G_t(0+,0) = 0.$$ \quad (431)

6.5 Exercises

54. **Homework 20:** a) Find the causal Green’s function for $dy/dt + y/t = f(t)$. b) Use it to solve $dy/dt + y/t = f(t)$ for $t > 1$, with $y(1) = A$.

55. **Homework 21:** Check that

$$G(t,s) = H(t-s)\frac{1}{\omega} \sin \omega (t-s) \quad (432)$$

obeys

$$G_{tt} + \omega^2 G = \delta(t-s). \quad (433)$$

56. **Check that**

$$G(t,s) = A(s)e^{-\alpha t} + H(t-s)e^{-\alpha(t-s)} \quad (434)$$

obeys

$$G_t + aG = \delta(t-s). \quad (435)$$

57. Show that the Green’s function $G(t,s) = H(t-s)e^{-\alpha(t-s)}$ satisfies

$$G_+ + aG = \delta(t-s) \quad (436)$$

in the sense that

$$\int_{-\infty}^\infty G(t,s) \left(a\phi(t) - \dot{\phi}(t)\right)\,dt = \phi(s) \quad (437)$$

for every test function $\phi(t)$. 62
58. Show that the Green’s function 
\[ G(t,s) = \frac{1}{\omega} H(t-s) \sin \omega(t-s) \]

in the sense that
\[ \int_{-\infty}^{\infty} G(t,s) \left( \ddot{\phi}(t) + \omega^2 \phi(t) \right) dt = \phi(s) \]

for every test function \( \phi(t) \).

59. Find the Green’s function and causal solutions of
\[ \ddot{y} + 5\dot{y} + 6y = f(t) \]

and find the particular causal solutions when \( f(t) = e^{-t} \) and \( f(t) = H(t)e^{-t} \).

60. Use the method of Green’s functions to find the causal solutions of:

(a) \( \ddot{y} + 4\dot{y} + 4y = H(t) + H(-t)e^{2t} \);
(b) \( \ddot{y} + 4\dot{y} + 4y = H(-t) \sin t \);
(c) \( \ddot{y} + 4\dot{y} + 3y = 2 \sin 5t \);
(d) \( \ddot{y} + 6\dot{y} + 9y = t^2 \);
(e) \( \ddot{y} + 3\dot{y} + 2y = \delta(t) \);
(f) \( y''' + \dot{y} = f(x) (*) \)

(*) Hint: you can either solve \( \partial^3 G/\partial x^3 + \partial G/\partial x = \delta(x-y) \) with \( G = 0 \) for \( x < y \), or put \( f(x) = F'(x) \) and integrate the equation once.

61. Use the method of Green’s functions to solve
\[ \ddot{y} + \omega^2 y = f(t), \quad t > 0 \]

subject to the initial conditions
\[ y(0) = A, \quad \dot{y}(0) = B. \]

62. Complete Section 6.3 from (416) for the special case where \( \lambda_1 = \lambda_2 \).
7 Green’s functions for the Poisson and Helmholtz equations

After warming up with inhomogeneous linear ODEs, we now turn to inhomogeneous PDEs. In this Section we consider

\[ Lu = f(x) \tag{443} \]

where \( L \) is the Laplacian \( \Delta \) or the Helmholtz operator \( \Delta + k^2 \) and \( f(x) \) is a given function. The nature of the solution depends on the boundary conditions and in this Section we assume that solutions are required in unbounded space.

7.1 Three-dimensional \( \delta \)-function

Definition 7.1. The three-dimensional \( \delta \)-function can be defined by

\[ \delta(x) = \delta(x_1)\delta(x_2)\delta(x_3), \tag{444} \]

or by the properties

\[ \delta(x) = 0 \text{ for } x \neq 0, \text{ and } \int \delta(x) \, dx^3 = 1, \tag{445} \]

or by the property

\[ \int \delta(x)\phi(x) \, dx^3 = \phi(0) \tag{446} \]

for all test functions \( \phi(x) \).

Proposition 7.2. In the spherical polar coordinates (25), with \( r^2 = x_1^2 + x_2^2 + x_3^2 \), we have

\[ \delta(r) = \frac{\delta(r)}{4\pi r^2}. \tag{447} \]

Proof. We will show that (447) obeys (446). It makes sense to carry out the integral \( \int d^3x \) in spherical polar coordinates. The integration range is \( 0 \leq r < \infty \), and we face the problem that \( \int_0^\infty \delta(r) \, dr \) is not defined. But \( \int_0^\infty \delta(r-\epsilon) \, dr = 1 \) for all \( \epsilon > 0 \) as now \( r-\epsilon = 0 \) occurs inside the integration range.

So instead of (447) we consider

\[ F_\epsilon(x) = \frac{\delta(r-\epsilon)}{4\pi r^2} \tag{448} \]

for \( \epsilon > 0 \), and then take \( \epsilon \to 0_+ \) at the end. Now

\[ \int F_\epsilon(x)\phi(x) \, dx^3 = \int_0^{2\pi} \int_0^\pi \int_0^\infty \frac{\delta(r-\epsilon)}{4\pi r^2} \phi(r,\theta,\varphi) r^2 \sin \theta \, d\theta \, d\varphi \]

\[ = \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \phi(\epsilon,\theta,\varphi) \sin \theta \, d\theta \, d\varphi, \tag{449} \]

where the two factors of \( r^2 \) have cancelled, and we have carried out the integration over \( r \). The remaining integral over \( \theta \) and \( \varphi \) is the average of \( \phi \) over a sphere of radius \( \epsilon \). As \( \epsilon \to 0 \), and the sphere shrinks to a point, this average becomes simply \( \phi(0) \). We have shown that \( \lim_{\epsilon \to 0} F_\epsilon(x) = \delta(x) \). We shall use this trick of “protecting” \( \delta(r) \) again. \( \square \)

Remark 7.3. Although we have said that

\[ g(x)\delta(x) = g(0)\delta(x), \tag{451} \]

this only applies to functions \( g(x) \) which are continuous at \( x = 0 \). An expression of the form \( g(x)\delta(x) \) when \( g(x) \) is not continuous at \( x = 0 \) must be interpreted as a generalised function in its own right, that is, under an integral. Thus \( \delta(r)/r^2 \) is a generalised function and it is certainly not equal to \( \delta(r)/0^2 \).
7.2 Free space Green’s function for the Poisson equation

Definition 7.4. The free space problem for the Poisson equation is

\[ \Delta u = f(x) \]  

with the boundary condition at infinity

\[ u(x) \to 0 \text{ as } |x| \to \infty. \]  

We assume that \( f(x) \to 0 \) as \( |x| \to \infty \) so that there are no sources at infinity. That is, we consider the effect of a source distribution that falls off far away from these sources. This fall-off condition takes the place of the Dirichlet, Neumann or Robin boundary conditions. We can solve this problem in much the same way as we found causal solutions for ODEs.

Suppose \( G(x, y) \) satisfies

\[ \Delta_x G(x, y) = \delta(x - y), \quad G(x, y) \to 0 \text{ as } |x - y| \to \infty, \]  

where \( \Delta_x \) indicates differentiation with respect to \( x = (x_1, x_2, x_3) \) and not with respect to \( y = (y_1, y_2, y_3) \). If we put

\[ u(x) = \int G(x, y)f(y)\,d^3y, \]  

then, since the integral is with respect to \( y \) and the derivatives are with respect to \( x \),

\[ \Delta_x u = \int \Delta_x G(x, y)f(y)\,d^3y = \int \delta(x - y)f(y)\,d^3y = f(x). \]

Because the boundary conditions are at infinity, the Green function \( G(x, y) \) depends only on the relative position of \( x \) and \( y \) and so \( G(x, y) = G(x - y) = G(x') \). Then the problem for \( G \) becomes

\[ \Delta_{x'} G = \delta(x'). \]  

Because the problem is invariant under rotations, we expect \( G \) to be spherically symmetric, since \( \Delta_{x'} \) and \( \delta(x') \) are, so we look for a generalised function

\[ G(x') = G(|x'|) =: G(r). \]  

Then, using (26) and (447), (457) becomes

\[ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial G}{\partial r} \right) = \frac{1}{4\pi} \frac{\delta(r)}{r^2}. \]  

For \( r > 0 \), \( \delta(r) = 0 \), so we find that

\[ G = \frac{A}{r} + B \quad (\text{for } r > 0), \]  

where \( A \) and \( B \) are constants. Since \( G \to 0 \) as \( r \to \infty \), \( B = 0 \).

To find \( A \) we argue as follows. Since \( \Delta_{x'} G = \delta(x') \), we must have

\[ \int_V \Delta_{x'} G \,d^3x' = 1 \]  

for any volume \( V \) that includes the origin. By the divergence theorem we also have

\[ \int_V \Delta_{x'} G \,d^3x = \int_S G_{,n} \,d^2x', \]  

where \( S \) is the surface of \( V \).
Figure 8: Sphere of radius $R$

Now, choose $V$ to be a sphere of radius $R$ (see Fig. 8). The outward normal to the surface is just the unit vector pointing from the origin to the point on the surface and so

$$G_{,n} = G_{,r} = -\frac{A}{r^2} = -\frac{A}{R^2}$$  \hspace{1cm} (463)

on the surface $r = R$ of the sphere. Also, on the surface $r = R$, the surface element $d^2x'$ is given by $R^2 \sin \theta \, d\theta \, d\varphi$, so

$$\int_0^\pi \int_0^{2\pi} \frac{\partial G}{\partial r} \bigg|_{r=R} R^2 \sin \theta \, d\theta \, d\varphi = 1.$$  \hspace{1cm} (464)

Thus

$$-A \int_0^\pi \int_0^{2\pi} \sin \theta \, d\theta \, d\varphi = 1,$$  \hspace{1cm} (465)

and so

$$A = -\frac{1}{4\pi}. \hspace{1cm} (466)$$

Thus

$$G = -\frac{1}{4\pi} r = -\frac{1}{4\pi} \frac{1}{|x'|} = -\frac{1}{4\pi} \frac{1}{|x-y|} = -\frac{1}{4\pi} \frac{1}{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}}, \hspace{1cm} (467)$$

and the solution of the problem

$$\Delta u = f(x), \quad u \to 0 \text{ as } |x| \to \infty$$  \hspace{1cm} (468)

is

$$u(x) = -\frac{1}{4\pi} \int \frac{f(y) \, d^3y}{|x-y|},$$  \hspace{1cm} (469)

or

$$u(x_1, x_2, x_3) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{f(y_1, y_2, y_3) \, dy_1 \, dy_2 \, dy_3}{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}}.$$  \hspace{1cm} (470)

Note that $G(x, y)$ is not defined at $x = y$. Rather, it is a generalised function. One useful interpretation (see Section 7.4) is

$$G(x, y) = \lim_{\epsilon \to 0} -\frac{1}{4\pi} \frac{H(|x-y| - \epsilon)}{|x-y|}$$  \hspace{1cm} (471)
as shown in Fig. 9.

Figure 9: Generalised function interpretation of Green’s function. We consider $r = |x - y|$ and illustrate $H(r - \epsilon)/r$ for a small non-zero $\epsilon$.

### 7.3 Free space Green’s function for the Helmholtz equation

Recall the derivation of the Helmholtz equation for time-periodic solutions with angular frequency $\omega$ of the wave equation in Sec. 1.5.1.

The Green’s function for the Helmholtz equation satisfies

$$ (\Delta x + k^2)G(x, y) = \delta(x - y). $$

Hence

$$ u(x) = \int G(x, y)f(y)\,d^3y $$
is a solution of (51). However, this solution depends on the boundary conditions, which we have not yet specified. For the Helmholtz equation, it is natural to impose this boundary condition directly on the Green’s function. Physically, we expect waves to propagate away from the disturbance generating them and not towards it. This gives us a radiation boundary condition, which replaces the fall-off condition that $u \to 0$ as $|x| \to \infty$ used for Poisson’s equation. We shall see shortly what form this radiation condition takes for the Green’s function.

As before, it is convenient to introduce $x' = x - y$, in which case the problem becomes

$$ (\Delta_{x'} + k^2)G = \delta(x'), $$

which clearly has spherical symmetry. So, we look for a solution with $G(x') = G(r)$, and the problem is then

$$ \frac{1}{r} \left[ \frac{\partial^2}{\partial r^2} (rG) + k^2 (rG) \right] = \frac{\delta(r)}{4\pi r^2}, $$

since

$$ \Delta_3 f(r) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial}{\partial r} f\right) = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rf) $$

when acting on a spherically symmetric function $f(x) = f(|x|) = f(r)$. (The second equality is a useful identity to remember.) So for $r > 0$ we have

$$ \frac{\partial^2}{\partial r^2} (rG) + k^2 (rG) = 0, $$

which implies that $rG = \alpha e^{ikr} + \beta e^{-ikr}$ or

$$ G = \frac{A}{4\pi r} e^{ikr} + \frac{B}{4\pi r} e^{-ikr}. $$

If we consider $G e^{-ikct}$, which is a solution of the wave equation, we have

$$ G e^{-ikct} = \frac{A}{4\pi r} e^{ik(r-ct)} + \frac{B}{4\pi r} e^{-ik(r+ct)}. $$
Now any function \( f(r - ct) \) represents a wave moving away from \( r = 0 \) towards \( r \to \infty \) with speed \( c \) as \( t \) increases, because \( f \) is constant on lines \( r - ct = C \). Fig. 11). On the other hand a function \( g(r + ct) \) represents a wave moving inwards, see Fig. 11. The \( \delta \)-function in the problem for \( G \) represents a disturbance at the origin. Physically we expect waves to propagate outward away from this disturbance and not inward from infinity towards the disturbance. So the radiation condition tells us that \( B = 0 \). Hence

\[
G = \frac{A}{4\pi r} e^{ikr}, \quad r > 0.
\]

We extend this to all values of \( r \) by defining \( G \) to be the generalised function

\[
G = \lim_{\epsilon \to 0} \left( \frac{AH(r - \epsilon)}{4\pi r} e^{ikr} \right).
\]

Using the result (490), we find that

\[
\Delta G = -\frac{Ak^2 e^{ikr}}{4\pi r} - A\delta(x'),
\]

so that

\[
(\Delta + k^2)G = -A\delta(x'),
\]

and hence we must take \( A = -1 \).

We obtain

\[
G(x') = -\frac{1}{4\pi} e^{ik|x'|},
\]

or

\[
G(x, y) = -\frac{1}{4\pi |x - y|} e^{ik|x - y|}
\]

\[
= -\frac{1}{4\pi} \exp \left( ik \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2} \right)
\]

Note that as \( k \to 0 \) we recover the Green’s function for the Poisson equation.

To summarize: The solution of the inhomogeneous Helmholtz problem

\[
(\Delta + k^2)u = f(x), \quad u(\mathbf{x}) \to 0, \quad |\mathbf{x}| \to \infty
\]

that satisfies the outgoing radiation boundary condition is given by

\[
u(x) = -\frac{1}{4\pi} \int \frac{f(y)}{|x - y|} e^{ik|x - y|} d^3 y.
\]

This represents the (spatial part of) an outgoing train of waves caused by a disturbance in the region where \( f(x) \neq 0 \).

7.4 An alternative derivation

It is instructive to check by direct differentiation that the free space Green’s functions we have derived for the Poisson and Helmholtz equations actually obey the inhomogeneous PDEs they are supposed to.

Let \( Z(r) \) be the (generalised) function defined by

\[
Z(r) := \lim_{\epsilon \to 0} Z_\epsilon(r), \quad Z_\epsilon(r) := \frac{H(r - \epsilon)}{4\pi r} f(r),
\]

where \( f(r) \) smooth and bounded (so that \( f(r) \) times any test function is again a test function). This is similar to the trick we have used in 7.1, where we replaced \( \delta(r) \) by \( \delta(r - \epsilon) \) and took \( \epsilon \to 0 \) at the end.
We shall now prove that
\[ \Delta Z = \frac{1}{4\pi r} f''(r) - f(0)\delta(x). \] (490)
Then for \( f(r) = -1 \) we have
\[ \Delta \left( -\frac{1}{4\pi r} \right) = \delta(x) \] (491)
and for \( f(r) = -e^{ikr} \) we have
\[ (\Delta + k^2) \left( -\frac{e^{ikr}}{4\pi r} \right) = \delta(x). \] (492)

To see this note that
\[ \Delta Z_\epsilon = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \left( \frac{H(r - \epsilon)}{4\pi r} f(r) \right) \right) \]
\[ = \frac{1}{4\pi r^2} \frac{\partial}{\partial r} \left[ rf(r)\delta(r - \epsilon) + (rf'(r) - f(r))H(r - \epsilon) \right]. \] (493)

To evaluate the derivative of the term with the \( \delta \)-function correctly, we integrate it over a test function \( \phi(r) \):
\[ \int \frac{1}{4\pi r^2} [rf(r)\delta(r - \epsilon)]' \phi(r) 4\pi r^2 dr = - \int rf(r)\delta(r - \epsilon)\phi'(r) dr \]
\[ = -\epsilon f(\epsilon)\phi'(\epsilon) = \int \frac{1}{4\pi r^2} rf(\epsilon)\delta'(r - \epsilon)\phi(r) 4\pi r^2 dr. \] (494)
Therefore
\[ \Delta Z_\epsilon = \frac{1}{4\pi r^2} [\epsilon f(\epsilon)\delta'(r - \epsilon) + (\epsilon f'(\epsilon) - f(\epsilon))\delta(r - \epsilon) + (rf''(r) + f'(r) - f'(r))H(r - \epsilon)]. \] (496)

In the limit \( \epsilon \to 0 \)
\[ \Delta Z = \frac{f''(r)}{4\pi r} - \frac{f(0)}{4\pi r} \delta(r) = \frac{1}{4\pi r} f''(r) - f(0)\delta(x). \] (497)

\section{7.5 The large distance and long wavelength approximations}

If the source distribution is nonzero only in a bounded region, or if it falls off sufficiently rapidly with distance, one can deduce the approximate behaviour of the solution at large distance from the source without solving the full problem.

It is important that two related but separate approximations are necessary here. For the Poisson equation, the point where we evaluate the solution must be much further away from the source than the size of the source (large distance approximation). For the Helmholtz equation, we need the large distance approximation and a separate approximation, namely that the size of the source is much smaller than the wavelength (long wavelength approximation). (Recall that the Helmholtz equation is about waves of a specific frequency and hence wavelength).

We study these two approximations separately.

\textbf{Example 7.5.} Consider the free-space Poisson problem
\[ \Delta u = e^{-|x|^2/\ell^2} =: f(x), \quad u(x) \to 0 \quad \text{as} \quad |x| \to \infty. \] (498)
What approximation can we make for \( u(x) \) as \( |x| \gg \ell? \)

The exact solution is
\[ u(x) = -\frac{1}{4\pi} \int_{y \in \mathbb{R}^3} \frac{f(y)}{|x - y|} d^3y. \] (499)
But let us assume we cannot or do not want to evaluate this approximately. The key observation is that in this example, although the source \( f(x) \) is nowhere zero, it falls off very rapidly with \( |x| \), and we can approximate it as zero for \( |x| > R \), where \( R \) is the “size” of the source region, chosen...
to be a few \(\ell\). [Of course if \(f(x)\) is nonzero only on a bounded region, say \(f(x) = H(R - |x|)\), the size of the source is defined unambiguously]. So we approximate

\[
    u(x) \simeq -\frac{1}{4\pi} \int_{|y|< R} \frac{f(y)}{|x - y|} \, d^3 y.
\]

(500)

For any two vectors \(x, y \in \mathbb{R}^n\) and the Euclidean norm \(| \cdot |\), we have the triangle equalities

\[
    |x| - |y| \leq |x - y| \leq |x| + |y|.
\]

(501)

The outer bracket in the first expression is just the absolute value of a real number. (As an exercise, try deriving these from the form \(|x + y| \leq |x| + |y|\) of the triangle inequality that is part of the definition of any norm. To just convince yourself that it is true, draw some vectors \(x\) and \(y\) in \(\mathbb{R}^2\).)

Now we consider the limit \(|x| \gg R\) (and hence in particular \(|x| > R\)). In the integral (500), \(|y| < R\). Hence we have

\[
    |x| - R \leq |x - y| \leq |x| + R.
\]

(502)

We can write this as

\[
    |x - y| = |x| + O(R) \quad \text{as} \quad |x| \to \infty.
\]

(503)

The symbol \(O(\cdot)\) is pronounced “order of” or “big-O of”. Its formal definition is that

\[
    f(x) = O[g(x)] \quad \text{as} \quad x \to \infty \quad \text{if} \quad cg(x) < |f(x)| < Cg(x)
\]

for two constants \(0 < c < C\) as \(x \to \infty\). Its intuitive meaning is “grows or decays like”. Note that a limit is always part of the definition of \(O\).

We now rewrite the estimate (7.5) of an absolute error as an estimate of a relative error,

\[
    |x - y| = |x| \left[ 1 + O \left( \frac{R}{|x|} \right) \right]
\]

(504)

We can approximate this by \(|x|\) if and only if \(R/|x| \ll 1\), that is when the relative error in \(|x - y|\) is small. Clearly, it is the relative error (in percent) that matters here, not the absolute error (in meters).

We can now continue from (500) as

\[
    u(x) \simeq -\frac{1}{4\pi} \int_{|y|< R} \frac{f(y)}{|x|} \, d^3 y
    = -\frac{1}{4\pi |x|} \int_{|y|< R} f(y) \, d^3 y
    \simeq -\frac{1}{4\pi |x|} \int_{y \in \mathbb{R}^3} f(y) \, d^3 y
    = -\frac{A}{4\pi |x|}
\]

(505)

where

\[
    A := \int f(y) \, d^3 y
\]

(506)

is the magnitude of the source. This is the large distance approximation: the source term is approximated as a point source of magnitude \(A\) located at the origin.

**Example 7.6.** Consider the free-space Helmholtz problem

\[
    (\Delta + k^2) u = e^{-|x|^2/\ell^2} =: f(x), \quad \text{outgoing wave BC as} \quad |x| \to \infty.
\]

(507)

What approximation can we now make for \(u(x)\) as \(|x| \gg \ell\)?

The exact solution is now

\[
    u(x) = -\frac{1}{4\pi} \int_{y \in \mathbb{R}^3} \frac{e^{ik|x-y|}}{|x-y|} f(y) \, d^3 y.
\]

(508)
We now want to approximate $|x - y|$ by $x$ in two different places, namely the amplitude $1/|x - y|$ and the complex phase $\exp ik|x - y|$. For the amplitude, we have once again (504), where it matters that the relative error in amplitude (in percent) is small. So, as in the Poisson example, we need the large distance condition
\[
\frac{R}{|x|} \ll 1.
\] (509)
But the complex phase, measured in radians, is
\[
k|x - y| = k|x| \left[ 1 + O \left( \frac{R}{|x|} \right) \right] = k|x| + O(kR).
\] (510)
We are clearly completely out of phase when the error in the phase approaches $2\pi$. The relative error in the phase (first equality above) is completely irrelevant, what matters is the absolute phase error (second equality). Hence we can approximate $\exp ik|x - y|$ by $\exp ik|x|$ if and only if
\[
kR \ll 2\pi.
\] (511)
(You will also find $kR \ll 1$ in the literature, which is equally good within the approximation implied by $\ll$.) As $k = 2\pi/\lambda$, this means that the wavelength of the waves is much larger than the size of the source,
\[
\frac{R}{\lambda} \ll 1,
\] (512)
the long wavelength condition.
As a real world example, consider the loudspeaker of a radio, with a diameter of 10cm. Hence the large distance approximation holds if we look at the sound field at distances much larger than 10cm from the loud speaker, and the long wavelength approximation holds for wavelengths much larger than 10cm, or frequencies much lower than $(340\text{m/s})/(10\text{cm})=3400\text{Hz}$.

7.6 Uniqueness of the solution to the free-space problem

Given the Green’s function we deduced in Sec. 5.3, we know that the solution of the free space Poisson problem is
\[
u(x) = -\frac{1}{4\pi} \int \frac{f(y) d^3 y}{|x - y|}
\] (513)

**Theorem 7.7.** The solution (513) of the free space Poisson problem (452,453) is unique.

**Proof.** Assume there are two solutions $u_1$ and $u_2$, and let $u := u_1 - u_2$. The proof is then identical to the proof for a bounded volume $V$ given in Sec. 2.4.3 up to Eq. (149).

Now consider $S$ in that equation to be a sphere of radius $R$, and let $R \to \infty$. On the sphere
\[
u \sim \frac{1}{R}, \quad u, u, n = \frac{\partial u}{\partial r} \bigg|_{r=R} \sim \frac{1}{R^2}, \quad d^2 x = R^2 \sin \theta d\theta d\varphi,
\] (514)
and so
\[
\int_S uu, n d^2 x \sim \int_0^\pi \int_0^{2\pi} \frac{1}{R} \sin \theta d\theta d\varphi = \frac{4\pi}{R} \to 0.
\] (515)
Thus, in the limit $R \to \infty$, we get
\[
\int |\nabla u|^2 d^3 x = 0,
\] (516)
so that $|\nabla u|^2 = 0$ at all points. Hence $u$ must be constant. Now note that since $u \to 0$ as $|x| \to \infty$, this constant must be zero. Hence $u_1 = u_2$, and the solution is unique. 

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7.7 Exercises

63. Homework 22: Check that
\[ \Delta f(r) = \frac{\partial^2 f}{\partial r^2} + \frac{2}{r} \frac{\partial f}{\partial r} = \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial f}{\partial r} \right) = \frac{1}{r} \frac{\partial^2}{\partial r^2} (rf) \] (517)
when acting on a spherically symmetric function \( f(x) = f(r) \).

64. Homework 23: Show by differentiation, and using \( \Delta f = r^{-1}(rf)'' \) for \( f(x) = f(r) \), that
\[ \frac{\Delta f(r)}{4\pi r} = -\frac{\delta(r) f(0)}{4\pi r^2} + \frac{f''(r)}{4\pi r} \] (518)

65. Homework 24: We originally met the triangle inequality as
\[ ||X + Z|| \leq ||X|| + ||Z||. \] (519)
Make two appropriate choices of \( Z \) to obtain the equivalent triangle inequalities
\[ ||X|| - ||Y|| \leq ||X - Y|| \leq ||X|| + ||Y||. \] (520)

66. Show that in two spatial dimensions
\[ \delta(x) = \frac{1}{2\pi} \frac{\delta(r)}{r}, \] (521)
where \( r = \sqrt{x_1^2 + x_2^2} \).

67. Show that in spherical polar coordinates the problem
\[ \Delta u = \frac{1}{\ell^2} e^{-|x|/\ell}, \quad u(x) \to 0 \quad \text{as} \quad |x| \to \infty \] (522)
becomes
\[ \frac{1}{r} \frac{\partial^2}{\partial r^2} (ru) = \frac{1}{\ell^2} e^{-r/\ell}, \quad u(r) \to 0 \quad \text{as} \quad r \to \infty. \] (523)
Hence deduce that
\[ u = e^{-r/\ell} + \frac{2\ell}{r} \left( e^{-r/\ell} - 1 \right). \] (524)

68. Suppose that \( f(x) = 0 \) if \( |x| > R \) where \( R \) is a constant. Moreover suppose that
\[ \int f(y) \, d^3y = \int_{|y| < R} f(y) \, d^3y = A, \] (525)
where \( A \) is a constant. Show that if \( u(x) \) is the solution of
\[ \Delta u = f(x), \quad u(x) \to 0 \quad \text{as} \quad |x| \to \infty \] (526)
then for \( |x| \gg R \)
\[ u(x) \sim A G(x, 0). \] (527)
That is, show that if you are far enough away from a distributed source (i.e., \( f(x) \)) then it looks like a point source at the origin (i.e., \( G(x, 0) \)) of strength \( A \).

69. Given that
\[ f(x) = \begin{cases} \frac{R}{\pi |x|} \sin \left( \frac{\pi |x|}{R} \right) & \text{if } |x| < R, \\ 0 & \text{if } |x| > R \end{cases} \] (528)
show that if
\[ \Delta u = f(x) \] (529)
then
\[ u \sim -\frac{R^3}{\pi^2 |x|} \quad \text{if} \quad |x| \gg R. \] (530)
8 Green’s functions for bounded regions

So far we have only considered free-space problems. However, in many situations of interest we are looking for a solution that satisfies both the PDE and given boundary conditions. This Section discusses how we can find a Green’s function solution that satisfies the relevant boundary conditions. We focus on the Helmholtz equation

\[(\Delta + k^2)u = f(x),\]  

but the method is readily generalised to other problems. For example, results for the Poisson equation follow by taking the limit \(k \to 0\).

Our main mathematical tool will be the Kirchhoff-Helmholtz formula. To derive it, we need two major ingredients, Green’s theorem and the reciprocal theorem.

8.1 Green’s theorem

**Theorem 8.1.** Suppose that \(G(x)\) and \(u(x)\) are functions with continuous second derivatives on a region \(V\) with surface \(S\). Then

\[
\int_V (G \Delta u - u \Delta G) \, d^3x = \int_S (G u_n - uG_n) \, d^2x.
\]  

**Proof.** Recalling that

\[
\Delta u = \nabla \cdot (\nabla u)
\]  

and likewise for \(G\), and that

\[
\nabla \cdot (G \nabla u) = G \Delta u + (\nabla G) \cdot (\nabla u),
\]  

\[
\nabla \cdot (u \nabla G) = u \Delta G + (\nabla u) \cdot (\nabla G),
\]  

we find that

\[
G \Delta u - u \Delta G = \nabla \cdot (G \nabla u - u \nabla G).
\]  

Thus, integrating over \(V\) we have

\[
\int_V (G \Delta u - u \Delta G) \, d^3x = \int_V \nabla \cdot (G \nabla u - u \nabla G) \, d^3x.
\]  

By the divergence theorem

\[
\int_V \nabla \cdot (G \nabla u - u \nabla G) \, d^3x = \int_S (G \nabla u - u \nabla G) \cdot \mathbf{n} \, d^2x = \int_S (G u_n - uG_n) \, d^2x,
\]  

which establishes Green’s theorem.

8.2 The reciprocal theorem

In all of the problems in this Section, the Green’s function is symmetric, that is

\[
G(x, y) = G(y, x).
\]  

The physical meaning of this symmetry is that hitting the system at \(x\) produces the same effect at \(y\) as the other way around. Mathematically speaking, Green’s functions are symmetric if they correspond to self-adjoint differential operators with homogeneous (that is, zero) boundary conditions. Not all physical systems have this property. Note also that the causal Green’s function for a time evolution problem (or for an ODE) does not have this symmetry under the interchange of \(t\) and \(s\).

We illustrate this symmetry for the Dirichlet problem for the Helmholtz equation:
**Theorem 8.2.** If \( G(x, y) \) is the solution of
\[
(\Delta + k^2)G(x, y) = \delta(x - y) \quad \text{for } x, y \in V, \tag{540}
\]
\[
G(x, y) = 0 \quad \text{for } x \text{ on } S, \tag{541}
\]
then
\[
G(x, y) = G(y, x). \tag{542}
\]

**Proof.** Consider the two problems
\[
(\Delta_x + k^2)G(x, y_1) = \delta(x - y_1), \quad (\Delta_x + k^2)G(x, y_2) = \delta(x - y_2), \tag{543}
\]
and multiply the first equation by \( G(x, y_2) \), multiply the second equation by \( G(x, y_1) \), subtract and integrate over \( V \):
\[
\int_V \left[ G(x, y_2)\Delta_x G(x, y_1) - G(x, y_1)\Delta_x G(x, y_2) \right] d^3x = \int_V \left[ G(x, y_2)\delta(x - y_1) - G(x, y_1)\delta(x - y_2) \right] d^3x. \tag{544}
\]
(The terms containing \( k^2 \) have cancelled). Now apply Green’s theorem to the left-hand side and integrate out the \( \delta \)-function on the right-hand side. This gives
\[
\int_S [G(x, y_2)G_{n_x}(x, y_1) - G(x, y_1)G_{n_x}(x, y_2)] d^2x = G(y_1, y_2) - G(y_2, y_1). \tag{545}
\]
But from
\[
G(x, y_1) = G(x, y_2) = 0 \quad \text{for } x \text{ on } S, \tag{546}
\]
this surface integral vanishes and we conclude that
\[
G(y_2, y_1) - G(y_1, y_2) = 0. \tag{547}
\]
This proves the theorem since \( y_1 \) and \( y_2 \) can be any points inside \( V \). \( \square \)

**Corollary 8.3.**
\[
(\Delta_y + k^2)G(x, y) = \delta(x - y). \tag{548}
\]

Proof from the reciprocity theorem \( G(x, y) = G(y, x), \delta(x - y) = \delta(y - x) \), and interchanging \( x \) and \( y \).

**8.3 The Kirchhoff-Helmholtz formula**

We want to derive a formula for the solution \( u(x) \) of a Helmholtz problem
\[
(\Delta + k^2)u(x) = f(x) \tag{549}
\]
on a bounded domain \( V \), with some boundary conditions on \( S = \partial V \). We leave these boundary conditions unspecified for now.

To obtain the desired formula, consider
\[
\int_V G(x, y)f(y) d^3y - u(x) \tag{550}
\]
\[
= \int_V [G(x, y)f(y) - u(y)\delta(x - y)] d^3y \tag{551}
\]
\[
= \int_V \left[ G(x, y)(\Delta + k^2)u(y) - u(y)(\Delta_y + k^2)G(x, y) \right] d^3y \tag{552}
\]
\[
= \int_V \left[ G(x, y)\Delta u(y) - u(y)\Delta_y G(x, y) \right] d^3y \tag{553}
\]
\[
= \int_S [G(x, y)u_n(y) - u(y)G_{n_y}(x, y)] d^2y \tag{554}
\]
In (551) we have used the definition of the $\delta$-function. In (552), we have used (549), but with $x$ renamed to $y$, to replace $f(y)$, and we have used (548) to replace the $\delta$-function. In (553), we have cancelled the term proportional to $k^2$. In (554), we have used Green’s theorem in the variable $y$. Here $u_{n,n}$ denotes the normal derivative with respect to the $y$ variables, i.e.,

$$G_{n,n} := n \cdot (\nabla_y G).$$

Combining (550) and (554) and rearranging, we obtain the Kirchhoff-Helmholtz representation

$$u(x) = \int_V G(x,y)f(y)\,d^3y + \int_S [G_{n,n}(x,y)u(y) - G(x,y)u_{n,n}(y)]\,d^2y.\tag{556}$$

This formula gives the value of $u(x)$ inside the region $V$ in terms of the source distribution $f(x)$ in $V$ and the values of $u$ and $u_{n,n}$ on the surface $S$. It is true for any $G(x,y)$ that is symmetric.

**Remark 8.4.** When attempting to solve (549) analytically, we must choose $G$ so that we minimise the amount of information we need to know about $u$ and $u_{n,n}$ on the boundary. For example, if we are given a Dirichlet problem, where $u$ is prescribed on the boundary, we try to find $G$ so that $G(x,y) = 0$ when $y$ is on the boundary. This eliminates the unknown $u_{n,n}$ and allows us to calculate $u$ in terms of known quantities. Similarly, for the Neumann problem where we know $u_{n,n}$ on the boundary we try to find $G$ so that $G_{n,n}(x,y) = 0$ when $y$ is on the boundary. This eliminates the unknown $u$ from the integral over the surface.

**Remark 8.5.** (556) can also be used numerically. We choose a simple $G$, say the free space Green’s function, and this gives us an integral equation to solve numerically. For example if we are given a Neumann problem with $u_{n,n}$ specified on the boundary (but where we do not know $u$ on the boundary) then by choosing $x$ to be a point on the boundary, (549) becomes an integral equation for the unknown $u(x)$ on the boundary. This is solved numerically, and once we know $u(x)$ on the boundary, then (549) tells us the value of $u(x)$ at all points inside the boundary. This is the essence of boundary integral methods. Note that the integral equation is 2-dimensional whereas the original problem is 3-dimensional. This reduction in dimensionality is why boundary integral methods are useful.

### 8.4 Problems on bounded regions

#### 8.4.1 The Dirichlet problem

This is the problem of finding $u$ in $V$ given that

$$(\Delta + k^2)u(x) = f(x) \quad \text{in } V,$$

$$u(x) = g(x) \quad \text{on } S.\tag{557}$$

We solve this in terms of the Kirchhoff-Helmholtz representation by eliminating the unknown $u_{n,n}$ from the integral, that is, we attempt to find a Green’s function such that

$$(\Delta_y + k^2)G(x,y) = \delta(x - y) \quad \text{in } V,$$

$$G(x,y) = 0 \quad \text{when } y \text{ on } S.\tag{558}$$

In practice it may be difficult to find such a $G$, but assuming $G$ is known the solution is then, from the Kirchhoff-Helmholtz representation,

$$u(x) = \int_V G(x,y)f(y)d^3y + \int_S G_{n,n}(x,y)g(y)d^2y.\tag{559}$$

#### 8.4.2 The Neumann problem for the Helmholtz equation

This is the problem of finding $u$ in $V$ given that

$$(\Delta + k^2)u(x) = f(x) \quad \text{in } V, \quad u_{n,n}(x) = g(x) \quad \text{on } S.\tag{560}$$
We solve this in terms of the Kirchhoff–Helmholtz representation by eliminating the unknown $u$ from the integral, that is, we attempt to find a Green’s function such that

$$\sum y + k^2 G(x, y) = \delta(x - y) \quad \text{in} \ V, \quad G_n(x, y) = 0 \quad \text{when} \ y \ \text{on} \ S. \quad (561)$$

Assuming $G$ can be found, the solution is then

$$u(x) = \int_V G(x, y)f(y) \, d^3 y - \int_S G(x, y)g(y) \, d^2 y. \quad (562)$$

### 8.4.3 The Neumann problem for the Poisson equation

As we have already discussed in Remark 1.5, the Neumann problem for the Poisson equation ($k = 0$),

$$\Delta u(x) = f(x) \quad \text{in} \ V, \quad u_n(x) = g(x) \quad \text{on} \ S, \quad (563)$$

is posed consistently only if

$$\int_V f(x) \, d^3 x = \int_V \Delta u \, d^3 x = \int_S u_n \, d^2 x = \int_S g(x) \, d^2 x. \quad (564)$$

That is, $f(x)$ and $g(x)$ must satisfy the compatibility condition for the data of the Poisson equation with Neumann boundary conditions,

$$\int_V f(x) \, d^3 x = \int_S g(x) \, d^2 x. \quad (565)$$

By the same reasoning, for $x \in V$ we find another compatibility condition, this time for the Green’s function $G(x, y)$ (and integrating over $y$),

$$\int_S G_n(x, y) \, d^2 y = \int_V \Delta y G(x, y) \, d^3 y = \int_V \delta(x - y) \, d^3 y = 1. \quad (566)$$

Hence we cannot impose $G_n(x, y) = 0$ for $y$ on $S$!

However, it turns out that this is not necessary. Instead, consider the Green’s function defined by

$$\Delta y G(x, y) = \delta(x - y) \quad \text{in} \ V, \quad (567)$$

$$G_n(y, x) = \frac{1}{A} \quad \text{for} \ y \ \text{on} \ S, \quad (568)$$

where

$$A := \int_S d^2 y \quad (569)$$

is the area of $S$. Then the compatibility condition (566) for the Green’s function is satisfied. The solution of the Neumann problem is, from the Kirchhoff–Helmholtz representation (556), given by

$$u(x) = \int_V G(x, y)f(y) \, d^3 y + \int_S \left( \frac{u(y)}{A} - G(x, y)g(y) \right) \, d^2 y, \quad (570)$$

but

$$\frac{1}{A} \int_S u(y) \, d^2 y := C \quad (571)$$

is independent of $x$, and so it does not matter that we do not know $u(y)$ in this integral. Instead, we just write

$$u(x) = \int_V G(x, y)f(y) \, d^3 y - \int_S G(x, y)g(y) \, d^2 y + C. \quad (572)$$

$C$ is now an unknown constant, and in hindsight we expect it to be there, as the solution of the Neumann problem is defined only up to an additive constant.

The compatibility conditions (565) for the data or (566) for the Green’s function only apply if the volume $V$ and hence its boundary $S$ are finite. If all or part of the boundary are at infinity, with the usual condition that $u \to 0$ at infinity, then we must have $C = 0$. 

77
8.4.4 Robin boundary conditions

This is the problem of finding \( u \) in \( V \) given that

\[
(\Delta + k^2)u(x) = f(x) \quad \text{in } V, \\
u_n(x) + \lambda(x)u(x) = g(x) \quad \text{on } S,
\]

(573)

where \( f(x) \), \( g(x) \) and \( \lambda(x) \) are all given functions.

We solve this in terms of the Kirchhoff–Helmholtz representation by eliminating the unknown \( u \) from the problem using the fact that

\[
u_n(x) = g(x) - \lambda(x)u(x) \quad \text{on } S.
\]

(574)

so the Kirchhoff-Helmholtz representation (556) becomes

\[
u(x) = \int_V G(x, y) f(y) \, d^3 y + \int_S \left[ G, n(y) + \lambda(y)G(x, y) \right] u(y) \, d^2 y - \int_S G(x, y) g(y) \, d^2 y.
\]

(575)

Then, as we do not know \( u \) on the surface \( S \), we choose \( G \) so that this term is eliminated. That is, we choose \( G \) to be a solution of

\[
(\Delta_y + k^2)G(x, y) = \delta(x - y) \quad \text{in } V, \\
G, n(y) + \lambda(y)G(x, y) = 0 \quad \text{when } y \text{ on } S.
\]

(576)

Assuming \( G \) can be found, the solution is then

\[
u(x) = \int_V f(y)G(x, y) \, d^3 y - \int_S yG(x, y) \, d^2 y.
\]

(577)

8.5 The method of images

8.5.1 Example: Laplace equation with Neumann BCs on a plane

It remains to find a Green’s function that obeys the homogeneous version of the boundary condition we want to impose. As an example, consider the half-space Neumann problem (see Fig. 12) for Laplace’s equation:

\[
\Delta u = 0, \quad x_3 > 0, \\
u_{x_3} = v(x_1, x_2) \quad \text{on } x_3 = 0, \\
u \to 0 \quad \text{as } x_3 \to \infty.
\]

(578)

As we are given \( u_{x_3} = -u_n \) on the boundary, we choose a Green’s function that satisfies

\[
G, n_y(x, y) = 0 \quad \text{on } y_3 = 0.
\]

(579)

Technically, the surface we need to integrate over also includes \( y_3 \to \infty \). However, we can assume that \( G \to 0 \) and \( G, n \to 0 \) as \( y_3 \to \infty \), hence eliminating the integral over those parts of the surface at \( \infty \).

The Kirchhoff-Helmholtz representation (556) becomes

\[
u(x) = -\int_S G(x, y)u_n(y) \, d^2 y
\]

(580)

or, with \( u_n = -u_{x_3} = -v \),

\[
u(x) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, (y_1, y_2, 0)) v(y_1, y_2) \, dy_1 \, dy_2.
\]

(581)
We still need to find a Green’s function that obeys the homogeneous Neumann condition (579). This can be done by the method of images. We start with the free space Green’s function

\[ G_F(x, y) = -\frac{1}{4\pi} \frac{1}{|x - y|}, \tag{582} \]

This represents the effect at \( x \) of a unit source at \( y \). Now consider what would happen if there were another point source at the image point

\[ y' := (y_1, y_2, -y_3). \tag{583} \]

Because the sources are now symmetric under a reflection in the \( x_3 \)-plane (see Fig. 12), so is the solution \( u(x) \). Hence its \( x_3 \)-derivative vanishes on the \( x_3 = 0 \) plane.

To add in this fictitious image charge, we use the Green’s function

\[ G(x, y) = G_F(x, y) + G_F(x, y') = -\frac{1}{4\pi} \left( \frac{1}{|x - y|} + \frac{1}{|x - y'|} \right) \tag{584} \]

\[ = -\frac{1}{4\pi} \left( \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}} + \frac{1}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 + y_3)^2}} \right). \tag{585} \]

Clearly this obeys

\[ G(x, (y_1, y_2, y_3)) = G(x, (y_1, y_2, -y_3)), \tag{586} \]

and so \( G_{y_3} = 0 \) on \( y_3 = 0 \).

When we evaluate \( G(x, y) \) on \( y_3 = 0 \), the two terms in (585) become identical, and substituting into (581) we obtain

\[ u(x_1, x_2, x_3) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{v(y_1, y_2) dy_1 dy_2}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^2}}. \tag{587} \]

One might object that \( G \) obeys \( \Delta G = \delta(x - y) + \delta(x - y') \). But when \( y \) is inside the domain of the PDE then \( y' \) is outside and so we do not integrate over the region containing \( y' \), and so \( \delta(x - y') \) does not make a contribution.

![Figure 12: Method of images for the Neumann problem for Laplace’s equation.](image-url)
8.5.2 Example: Helmholtz equation with Dirichlet BCs on a plane

As a second example, consider now the (homogeneous) Helmholtz equation with (inhomogeneous) Dirichlet BCs on a plane,

\[(\Delta + k^2)u = 0, \quad x_3 > 0, \quad (588)\]
\[u = v(x_1, x_2), \quad x_3 = 0, \quad (589)\]
\[u \to 0, \quad |x| \to \infty. \quad (590)\]

We need \(G\) to obey \(G(x, y) = 0\) for \(y \in S\), that is for \(y_3 = 0\). The required Green’s function is given by

\[G(x, y) = G_F(x, y) - G_F(x, y'), \quad (591)\]

where \(G_F\) is the Green’s function for the free space Helmholtz problem,

\[G_F(x, y) = -\frac{e^{ik|x-y|}}{4\pi|x-y|}, \quad (592)\]

and where the image point \(y' = (y_1, y_2, -y_3)\).

We then have

\[u(x) = \int_S G_{n_y}(x, y)u(y) \, d^2y = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} -G_{y_3}(x, (y_1, y_2, 0)) \, v(y_1, y_2) \, dy_1 \, dy_2, \quad (593)\]

where the sign comes from \(n = (0, 0, -1)\). (The rest of the example is left as an exercise)

8.6 Exercises

70. **Homework 25**: Complete the proof of the reciprocity theorem begun in the lecture by showing that

\[\int_S [G(x, y_2)G_{n_y}(x, y_1) - G(x, y_1)G_{n_y}(x, y_2)] \, d^2x = 0, \quad (594)\]

if \(G\) obeys Dirichlet, Neumann or Robin boundary conditions.

71. **Homework 26**: Solve the PDE problem

\[(\Delta + k^2)u = 0, \quad x_3 > 0, \quad (595)\]
\[u = v(x_1, x_2), \quad x_3 = 0, \quad (596)\]
\[u \to 0, \quad |x| \to \infty. \quad (597)\]

72. * Find the solution \(u(r)\) of

\[\Delta u = f(r), \quad 0 \leq r < R \quad (598)\]
\[\alpha u(0) + \beta u'(0) = 0 \quad (599)\]

(Poisson problem in three space dimensions inside a sphere with spherically symmetric source and spherically symmetric Robin BC) in the form

\[u(r) = \int_0^R G(r, s)f(s) \, ds. \quad (600)\]

Hint: Write the PDE in spherical coordinates. We effectively now have an ODE problem. Proceed from first principles, as we have done for constructing the Green’s function for ODEs. It is implicit in the problem that \(u(r)\) must obey the boundary condition \(u'(0) = 0\). If \(u'(0) \neq 0\), \(u(x, y, z)\) would have a conical shape at the origin \(r = 0\) (or \(x = y = z = 0\), and this would correspond to a \(\delta\)-function source.
73. Show, by differentiation, that

\[ u(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{v(y_1, y_2)}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^3}} dy_1 dy_2 \]  

(601)
satisfies

\[ u_{x_1 x_1} + u_{x_2 x_2} + u_{x_3 x_3} = 0. \]  

(602)

Further, show that

\[ \lim_{x_3 \to 0} u_{x_3} = v(x_1, x_2). \]  

(603)

Hint: recall from an earlier exercise that

\[ \lim_{\epsilon \to 0} \frac{\epsilon}{2\pi (x_1^2 + x_2^2 + \epsilon^2)^{3/2}} = \delta(x_1)\delta(x_2). \]  

(604)

74. Use the Kirchhoff-Helmholtz representation and the method of images to show that the solution of

\[ \Delta u = 0, \quad x_3 > 0, \]

\[ u(x_1, x_2, 0) = v(x_1, x_2) \quad \text{on} \quad x_3 = 0, \]

\[ u \to 0 \quad \text{as} \quad x_3 \to \infty. \]  

(605)

is

\[ u(x) = \frac{x_3}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{v(y_1, y_2)}{\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + x_3^3}} dy_1 dy_2. \]  

(606)

Verify that this solution does indeed satisfy Laplace’s equation and that as \( x_3 \to 0, u(x_1, x_2, x_3) \to v(x_1, x_2) \). Hint: See the hint in the previous question.

75. Consider the problem

\[ \Delta u = 0, \quad x_3 > 0, \]

\[ u_{x_3} = \begin{cases} 1 & x_1^2 + x_2^2 \leq 1 \\ 0 & x_1^2 + x_2^2 > 1 \end{cases} \quad \text{on} \quad x_3 = 0 \]

\[ u \to 0 \quad \text{as} \quad x_3 \to \infty. \]  

(607)

Show that for \( |x| \gg 1 \)

\[ u(x) \sim -\frac{1}{2|x|}. \]  

(608)

76. Consider the problem

\[ (\Delta + k^2)u = 0, \quad x_3 > 0, \]

\[ u_{x_3}(x_1, x_2, 0) = \begin{cases} 1 & x_1^2 + x_2^2 \leq 1 \\ 0 & x_1^2 + x_2^2 > 1 \end{cases} \]  

(609)

which corresponds to a circular piston oscillating in a baffle attached to a wall. Show that in the large wave length limit, \( k \ll 1 \), we have

\[ u(x) \sim -\frac{1}{2|x|} e^{ik|x|} \]  

(610)

for \( x \gg 1 \). What happens if \( k \sim 1 \)?
77. Use the Kirchhoff-Helmholtz representation and the method of images to show that the solution of

\[(\Delta + k^2)u = 0, \quad x_3 > 0,\]
\[u(x_1, x_2, 0) = u(x_1, x_2)\] (611)

with outgoing wave behaviour is

\[u(x) = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u(y_1, y_2) G_{y_3}(x, y_1, y_2, 0) dy_1 dy_2,\] (612)

where

\[G(x, y) = -\frac{1}{4\pi} \left( \frac{e^{i|x-y|}}{|x-y|} - \frac{e^{i|x'-y|}}{|x'-y|} \right),\] (613)

and where the image point \(x'\) and \(x\) are related by

\[x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad x' = \begin{pmatrix} x_1 \\ x_2 \\ -x_3 \end{pmatrix}.\] (614)

78. Show that the solution of

\[(\Delta + k^2)u = 0 \quad \text{for} \quad x_3 > 0,\]
\[u_{,x_3}(x_1, x_2, 0) = v(x_1, x_2), \quad u \to 0 \quad \text{as} \quad x_3 \to \infty\] (615)

is

\[u(x) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{ik\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + x_3^2}} v(y_1, y_2)}{\sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + x_3^2}} dy_1 dy_2.\] (616)
9 The diffusion equation

In the previous Sections we have set out the general principles involved in solving a PDE by means of the Green’s function technique. The method readily generates to PDEs other than the ones we have considered so far. The only additional complication is that we need to also consider time-dependent problems, like diffusion or wave equations. These are the topics of this and the following Section.

The diffusion equation (or heat equation) is

\[ u_t = \kappa \Delta u + \hat{f}(x, \hat{t}) \quad (617) \]

where \( \kappa \) is the diffusion constant (with dimension length²/time). This can be reduced to the form

\[ u_t = \Delta u + f(x, t) \quad (618) \]

by making the change of variable

\[ t := \kappa \hat{t}, \quad f(x, t) := \frac{1}{\kappa} \hat{f}(x, \hat{t}). \quad (619) \]

We shall assume that this change of variable has been made and only consider (618) in these notes.

We shall begin with the heat equation in one space dimension. This is interesting in its own right, but as we shall see, it is easy to find the Green’s function in \( n \) space dimensions from the one in one space dimension.

9.1 The one-dimensional diffusion equation

The one-dimensional diffusion equation is

\[ u_t = u_{xx} + f(x, t), \quad (620) \]

that is, we assume that \( u \) depends only on \( t \) and \( x \). The associated causal Green’s function, \( G(x, t; y, \tau) \) satisfies

\[ G_t - G_{xx} = \delta(t - \tau) \delta(x - y), \quad G = 0 \text{ if } t < \tau. \quad (621) \]

The (causal) solution of (620) is given by

\[ u(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} G(x, t; y, \tau) f(y, \tau) \, dy \, d\tau, \quad (622) \]

see Exercise 83.

In order to find \( G(x, t; y, \tau) \) we introduce the variables

\[ z = x - y, \quad T = t - \tau. \quad (623) \]

In terms of these variables, the problem for \( G(x, t; y, \tau) = G(z, T) \) becomes

\[ G_T - G_{zz} = \delta(T) \delta(z), \quad G(z, T) = 0 \text{ if } T < 0. \quad (624) \]

In order to accommodate the condition \( G(z, T) = 0 \) if \( T < 0 \) we put

\[ G(z, T) = H(T) g(z, T). \quad (625) \]

Recalling that \( dH(T)/dT = \delta(T) \) and \( \delta(T) g(z, T) = \delta(T) g(z, 0) \), we find that

\[ G_T - G_{zz} = \delta(T) g(z, 0) + H(T) \left(g_T - g_{zz}\right), \quad (626) \]

so that problem (624) for \( G(z, T) \) is satisfied if \( g(z, T) \) satisfies

\[ g_T = g_{zz}, \quad g(z, 0) = \delta(z). \quad (627) \]
There are many ways of solving (627) to find \( g(z,T) \). One particularly elegant way involves the use of **similarity variables**, leading to a **similarity ansatz**.

The basic idea behind the similarity method is to notice that if \( \lambda > 0 \) is a constant and we put

\[
\bar{z} = \lambda z, \quad \bar{T} = T^{2/3}, \quad \bar{g}(\bar{z}, \bar{T}) = \lambda^{-1/2} g(z, T),
\]

then

\[
\bar{g},T = \lambda^{-1} \frac{T}{2} \bar{g} = \lambda^{-1} \frac{T}{2} g_{zz} = \bar{g}_{zz},
\]

and

\[
\bar{g}(\bar{z}, 0) = \lambda^{-1} g(z, 0) = \lambda^{-1} \delta(z) = \delta(\bar{z}),
\]

That is, the problem (627) is invariant under the change of variables (628), in the sense that it is the same in barred variables and unbarred variables. That is, if (in unbarred variables),

\[
g, T = G_{zz}, \quad g(z, 0) = \delta(z)
\]

then (in barred variables)

\[
\bar{g}, \bar{T} = \bar{g}_{zz}, \quad \bar{g}(\bar{z}, 0) = \delta(\bar{z})
\]

Given that the problem is invariant under the transformation (628), it is sensible to look for solutions of the problem in terms of variables which are invariant under the same transformation. It is fairly obvious that both of

\[
\psi = \sqrt{T} g = \sqrt{T^2} (\lambda^{-1} g) = \sqrt{T} \bar{g}
\]

and

\[
\xi = z/\sqrt{T} = \lambda z/\sqrt{T^2} = \bar{z}/\sqrt{\bar{T}}
\]

are invariant under the transformation (628). As \( x \) and \( T \) are independent variables and \( g \) is a dependent variable, it is reasonable to look for a solution, in terms of the invariants \( \psi \) and \( \xi \), in the form

\[
\psi = \psi(\xi).
\]

Since \( \psi = \sqrt{T} g \) and \( \xi = z/\sqrt{T} \) this amounts to looking for a solution of the form

\[
g(z, T) = \frac{1}{\sqrt{T}} \psi(\xi), \quad \text{where } \xi = \frac{z}{\sqrt{T}}.
\]

If we look for a solution of this form we find that

\[
g, T = -\frac{1}{2T^{1/2}} \psi(\xi) + \frac{1}{T^{1/2}} \xi \frac{d\psi}{d\xi}(\xi)
\]  
\[
= -\frac{1}{2T^{3/2}} \psi(\xi) + \frac{1}{T^{1/2}} \left( -\frac{z}{2T^{3/2}} \right) \psi'(\xi)
\]  
\[
= -\frac{1}{2T^{3/2}} [\psi(\xi) + \xi \psi'(\xi)],
\]

and that

\[
g_{zz} = \frac{1}{T^{1/2}} \frac{\partial^2}{\partial z^2} \psi(\xi) = \frac{1}{T^{3/2}} \psi''(\xi),
\]

since \( \partial / \partial z = (\partial \xi / \partial z) d / d \xi = T^{-1/2} d / d \xi \).

This shows that

\[
-\frac{1}{2T^{3/2}} [\psi(\xi) + \xi \psi'(\xi)] = \frac{1}{T^{3/2}} \psi''(\xi),
\]

so that \( \psi(\xi) \) satisfies the ordinary differential equation

\[
\psi''(\xi) + \frac{1}{2} [\xi \psi'(\xi) + \psi(\xi)] = 0,
\]

which can be written as

\[
\psi''(\xi) + \frac{1}{2} \xi' [\psi(\xi)]' = 0.
\]
This can be integrated once to give
\[ \psi'(\xi) + \frac{1}{2} \psi(\xi) = C. \] (642)

This is a first order linear ordinary differential equation for \( \psi(\xi) \). With the integrating factor \( \exp(\xi^2/4) \) it is equivalent to
\[ \frac{d}{d\xi} \left( e^{\frac{\xi^2}{4}} \psi \right) = Ce^{\frac{\xi^2}{4}}. \] (643)

This integrates to give
\[ \psi(\xi) = Ce^{-\frac{\xi^2}{4}} \int_{0}^{\xi} e^{\eta^2/4} d\eta + Be^{-\frac{\xi^2}{4}}. \] (644)

In order to get \( \psi(\xi) \) to vanish fast enough as \( \xi \to \pm \infty \) (that is, in order to get \( g(x,0) = \delta(x) \)) we have to take \( C = 0 \) so that
\[ \psi(\xi) = Be^{-\frac{\xi^2}{4}}. \] (645)

Subsituting this into (636) gives
\[ g(z,T) = \frac{B}{\sqrt{T}} \exp \left( -\frac{z^2}{4T} \right). \] (646)

To determine the constant \( B \) we note that
\[ \int_{-\infty}^{\infty} g(z,0) dz = \int_{-\infty}^{\infty} \delta(z) dz = 1. \] (647)

We also have
\[ \int_{-\infty}^{\infty} g(z,T) dz = \frac{B}{\sqrt{T}} \int_{-\infty}^{\infty} \exp \left( -\frac{z^2}{4T} \right) dz \] (648)

and putting \( q = z/2\sqrt{T} \) this becomes
\[ \int_{-\infty}^{\infty} g(z,T) dz = 2B \int_{-\infty}^{\infty} e^{-q^2} dq = 2B\sqrt{\pi}. \] (649)

This is valid for any \( T > 0 \), so taking the limit \( T \to 0 \)
\[ \int_{-\infty}^{\infty} g(z,0) dz = 2B\sqrt{\pi} = 1 \] (650)

and hence
\[ B = \frac{1}{2\sqrt{\pi}}. \] (651)

Recalling that \( G(z,T) = H(T)g(z,T), \) \( z = x-y \) and \( T = t-\tau \) we conclude that the one dimensional causal Green’s function for the heat equation is
\[ G(x,t;y,\tau) = \frac{H(t-\tau)}{2\sqrt{\pi(t-\tau)}} \exp \left[ -\frac{(x-y)^2}{4(t-\tau)} \right]. \] (652)

9.2 The initial-value problem in one dimension

In practice, we are often interested in finding the solutions of initial value problems, rather than causal solutions. Consider the initial value problem
\[ \begin{align*}
  u_t &= u_{xx}, \quad t > 0, \\
  u(x,0) &= g(x).
\end{align*} \] (653)

Since we are only interested in \( u(x,t) \) for \( t > 0 \), we use again a trick we have seen before, and write
\[ \psi(x,t) = H(t)u(x,t), \] (654)
so $\psi = 0$ for $t < 0$ and $\psi = u$ for $t > 0$. Then

$$\psi_t = \delta(t)u(x, t) + H(t)u_t,$$  \hfill (655)

and since $\delta(t)u(x,t) = \delta(t)u(x,0)$ (this is just $\delta(t)f(t) = \delta(t)f(0)$ – think of $x$ as some fixed parameter), this becomes

$$\psi_t = \delta(t)g(x) + H(t)u_t.$$  \hfill (656)

As $H(t)$ does not depend on $x$,

$$\psi_{xx} = H(t)u_{xx}$$  \hfill (657)

and hence

$$\psi_t - \psi_{xx} = \delta(t)g(x) + H(t)(u_t - u_{xx})$$  \hfill (658)

so that

$$\psi_t - \psi_{xx} = \delta(t)g(x).$$  \hfill (659)

The causal solution of this problem is

$$\psi(x,t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-\frac{(x-y)^2}{4(t-\tau)}} \delta(\tau)u_0(y) \, dy \, d\tau.$$  \hfill (660)

For $t > 0$, the integral $\int_{\tau=0}^{t} \ldots \delta(\tau) \, d\tau$ simply picks out the value of the integrand at $\tau = 0$, while for $t < 0$ the point $\tau = 0$ is not in the integration range, so that

$$\psi(x,t) = H(t) \int_{-\infty}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} u_0(y) \, dy,$$  \hfill (661)

and hence for $t > 0$ we have

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} u_0(y) \, dy.$$  \hfill (662)

**Example 9.1.** Consider the particular case $g(x) = H(x)$. Since $H(y) = 0$ for $y < 0$ and $H(y) = 1$ for $y > 0$,

$$u(x,t) = \int_{0}^{\infty} \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-y)^2}{4t}} \, dy.$$  \hfill (663)

Now put $q = (y-x)/2\sqrt{t}$ to obtain

$$u(x,t) = \frac{1}{2\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2} \, dq.$$  \hfill (664)

This integral cannot be evaluated in terms of elementary functions, but we can express it in terms of a special function, the complementary error function.

$$u(x,t) = \frac{1}{2} \text{erfc} \left( -\frac{x}{2\sqrt{t}} \right).$$  \hfill (665)

The **error function** is defined as

$$\text{erf}(x) := \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-q^2} \, dq,$$  \hfill (666)

and it has the properties (see Fig. 13)

- $\text{erf}(-x) = -\text{erf}(x)$,
- $\text{erf}(0) = 0$,
- $\text{erf}(\infty) = 1$,
- $\text{erf}(-\infty) = -1$. 

86
• \( \text{erf}(x) \) is a monotonically increasing function of \( x \).

The complementary error function is defined as

\[
\text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-q^2} dq.
\] (667)

The error and complementary error functions are related by

\[
\text{erf}(x) + \text{erfc}(x) = 1,
\] (668)

because

\[
\frac{2}{\sqrt{\pi}} \left( \int_0^x e^{-q^2} dq + \int_x^\infty e^{-q^2} dq \right) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-q^2} dq = 1.
\] (669)
Example 9.2. Show that if \( u(x, t) \) satisfies
\[
\begin{align*}
  u_t & = u_{xx}, \quad t > 0, \\
  u(x, 0) & = e^{-\frac{x^2}{\ell}}.
\end{align*}
\] (670)
then
\[
  u(0, t) = \frac{\ell}{\sqrt{t^2 + 4t}}.
\] (671)
We have
\[
  u(x, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4t}} e^{-\frac{y^2}{\ell}} dy.
\] (672)
Hence
\[
  u(0, t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{y^2}{\ell}} dy = \frac{\ell}{\sqrt{t^2 + 4t}}.
\] (673)
and putting \( y\sqrt{t^2 + 1}/4t = q \) this becomes
\[
  u(0, t) = \frac{\ell}{2\sqrt{\pi t}} \frac{1}{\sqrt{t^2 + 4t}} \int_{-\infty}^{\infty} e^{-q^2} dq = \frac{\ell}{\sqrt{t^2 + 4t}}.
\] (674)

9.3 The three-dimensional problem

The (causal) Green’s function, \( G(x, t; y, \tau) \) for the diffusion equation is defined by
\[
  G_t - \Delta_x G = \delta(t - \tau)\delta(x - y), \quad G = 0 \text{ if } t < \tau.
\] (675)
To find it, let \( G(x, t, y, \tau) = G(z, T) \), where \( z := |x - y| \) and \( T := t - \tau \), and recall that
\[
  g(z, T) = \frac{1}{2\sqrt{\pi T}} \exp \left(-\frac{z^2}{4T}\right)
\] (676)
obey
\[
  g(z, 0) = \delta(z).
\] (677)
Therefore \( G(z, T) = H(T)g(z, T) \) obeys
\[
  G_T - G_{zz} = \delta(T)g(z, 0) + H(T) (g_T - g_{zz}) = \delta(T)\delta(z).
\] (678)
In three dimensions, try
\[
  G(z, T) = H(T)g(z_1, T)g(z_2, T)g(z_3, T).
\] (679)
Clearly this vanishes for \( T < 0 \) as required. Furthermore, it also obeys
\[
  \begin{align*}
  G_T - \Delta G & = G_T - (G_{z_1z_1} + G_{z_2z_2} + G_{z_3z_3}) \\
  & = \delta(T) g(z_1, 0) g(z_2, 0) g(z_3, 0) \\
  & \quad + H(T) \{[g_{z_1}(z_1, t) - g_{z_1z_1}(z_1, t)] g(z_2, T)g(z_3, T) + 2 \text{ more terms}\} \\
  & = \delta(T)\delta(z_1)\delta(z_2)\delta(z_3) = \delta(T)\delta(z).
  \end{align*}
\] (680)
We have shown that the causal Green’s function for the heat equation in three dimensions in free space is
\[
  G(x, t; y, \tau) = \frac{H(t - \tau)}{8\pi(t - \tau)^{3/2}} \exp \left[-\frac{|x - y|^2}{4(t - \tau)}\right].
\] (681)
By a similar argument, the two dimensional Green’s function is given by
\[
  G(x, t; y, \tau) = \frac{H(t - \tau)}{4\pi(t - \tau)} \exp \left[-\frac{|x - y|^2}{4(t - \tau)}\right]
\] (682)
where now \( x = (x_1, x_2), y = (y_1, y_2) \). Clearly this method works in any number of space dimensions.
Remark 9.3. Consider the initial value problem for the heat equation in three dimensions,

\[
\begin{align*}
    u_t &= \Delta u, \\
    u(x, 0) &= f(x).
\end{align*}
\]  

(683)

We turn this initial value problem into a causal problem in the usual way, that is, we assume \( t > 0 \) and write

\[
\psi(x, t) = H(t)u(x, t).
\]  

(684)

Then \( \psi \) satisfies

\[
\psi_t - \Delta \psi = \delta(t) f(x)
\]  

(685)

Thus

\[
\psi(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, t; y, \tau) \delta(\tau) f(y) d^3y. d\tau
\]  

(686)

For \( t > 0 \) the \( \delta(\tau) \) in the integral \( \int_{-\infty}^{t} \) simply picks out the value of the integrand at \( \tau = 0 \), and for \( t < 0 \) we get nothing, so

\[
\psi(x, t) = H(t) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, t; y, 0) f(y) d^3y.
\]  

(687)

Thus for \( t > 0 \) (when \( \psi = u \)) we have

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, t; y, \tau) \delta(\tau) f(y) d^3y. d\tau
\]  

(688)

Example 9.4. Suppose that \( u(x, t) \) satisfies the initial value problem

\[
\begin{align*}
    u_t &= \Delta u, \quad t > 0, \quad u(x, 0) = (1 - |x|)H(1 - |x|). \\
    u(0, t) &= \text{erf}\left(\frac{1}{2\sqrt{t}}\right) + 4\sqrt{\frac{t}{\pi}}\left(e^{-1/4t} - 1\right).
\end{align*}
\]  

(689)

Show that for \( t > 0 \)

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{|x-y|^2}{4t}\right) (1 - |y|)H(1 - |y|) d^3y.
\]  

(690)

The solution of the initial value problem is

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{|x-y|^2}{4t}\right) (1 - |y|)H(1 - |y|) d^3y.
\]  

(691)

and hence

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left(-\frac{|y|^2}{4t}\right) (1 - |y|)H(1 - |y|) d^3y.
\]  

(692)

As this integral depends only on \( r = |y| \) we change to spherical polar coordinates;

\[
\int_{0}^{2\pi} \int_{0}^{\pi} \int_{0}^{\infty} \exp\left(-\frac{r^2}{4t}\right) (1 - r)H(1 - r)r^2 \sin \theta d\theta d\varphi dr.
\]  

(693)

The integrals separate and we have

\[
\int_{0}^{2\pi} \int_{0}^{\pi} \sin \theta d\theta d\varphi = 4\pi,
\]  

(694)

so that

\[
\int_{0}^{\infty} \exp\left(-\frac{r^2}{4t}\right) (1 - r)H(1 - r)r^2 dr.
\]  

(695)
Since $H(1 - r) = 0$ for $r > 1$ and $H(1 - r) = 1$ for $r < 1$ this becomes

$$u(0, t) = \frac{1}{2\sqrt{\pi t^{3/2}}} \int_{0}^{1} \exp\left(-\frac{r^2}{4t}\right) (1 - r)r^2 \, dr.$$  \hfill (696)

If we put $q = r/2\sqrt{t}$ this becomes

$$u(0, t) = \frac{4}{\sqrt{\pi}} \int_{0}^{1/2\sqrt{t}} e^{-q^2} \left(q^2 - 2\sqrt{t}q^3\right) dq,$$  \hfill (697)

and hence, integrating by parts, we obtain the desired solution (690).

Note that we can always integrate an integral of the form $\int e^{-q^2} q^n \, dq$ by parts to get either an error function if $n$ is even, or an exponential function if $n$ is odd.

9.4 Exercises

79. **Homework 27**: Show that an alternative similarity ansatz to the one in the lectures is \( \xi := z^2/T \) and \( \psi := zg \), and use this to find \( g(z, T) \).

80. **Homework 28**: Find the solution of

$$u_{,t} = u_{,xx}, \quad t > 0,$$

$$u \to 0, \quad x \to \pm \infty,$$

$$u(x, 0) = H(x + 1)H(1 - x)$$ \hfill (700)

in terms of error functions.

81. **Homework 29**: Evaluate

$$u(0, t) = \frac{1}{2\sqrt{\pi t^{3/2}}} \int_{0}^{1} \exp\left(-\frac{r^2}{4t}\right) (1 - r)r^2 \, dr.$$  \hfill (701)

82. Suppose that \( u(x, t) \) satisfies

$$u_{,t} = \Delta u,$$

$$u(x, 0) = f(x), \quad u(x, t), \nabla u(x, t) \to 0 \text{ as } |x| \to \infty.$$  \hfill (702)

We can show that this solution is unique as follows:

Suppose there are two solutions, \( u_1 \) and \( u_2 \). Put \( u = u_1 - u_2 \) so that

$$u_{,t} = \Delta u,$$

$$u(x, 0) = 0, \quad u(x, t), \nabla u(x, t) \to 0 \text{ as } |x| \to \infty.$$  \hfill (703)

Define \( Q(t) \) by

$$Q(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (u(x, t))^2 \, d^3x,$$  \hfill (704)

and deduce that \( Q(t) \geq 0 \) for all \( t \). Further, by using the identity \( u\Delta u = \nabla \cdot (u \nabla u) - |\nabla u|^2 \), deduce that

$$\frac{dQ}{dt} = -\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla u|^2 \, d^3x \leq 0, \quad Q(0) = 0.$$  \hfill (705)

Hence conclude that \( Q(t) = 0 \) for all \( t \geq 0 \) and that therefore \( u = 0 \).

83. Show that if \( G(x, t; y, \tau) \) satisfies

$$G_{,t} - G_{,xx} = \delta(t - \tau)\delta(x - y), \quad G = 0 \text{ if } t < \tau,$$  \hfill (706)

then the causal solution of

$$u_{,t} = u_{,xx} + f(x, t)$$  \hfill (707)

is given by

$$u(x, t) = \int_{-\infty}^{t} \int_{-\infty}^{\infty} G(x, t; y, \tau)f(y, \tau) \, dy \, d\tau.$$  \hfill (708)
84. The concentration of a diffusing dye satisfies
\[ u_t = \Delta u, \ t > 0, \ u(x, 0) = e^{-|x|^2}. \tag{709} \]

Use the Green’s function for the diffusion equation to write down the solution of this problem in terms of a triple integral. Deduce that the concentration at the origin is given by
\[ u(0, t) = (1 + 4t)^{-3/2}. \tag{710} \]

85. Find the solution of the diffusion problem
\[ u_t = \Delta u, \ t > 0, \ u(x, 0) = \begin{cases} 1 & x_1 > 0, \ x_2 > 0, \ x_3 > 0, \\ 0 & \text{otherwise} \end{cases} \tag{711} \]

Express your solution in terms of error functions \( \text{erf}(x) \) or complementary error functions \( \text{erfc}(x) \).

86. Find the solution of the initial value problem
\[ u_t = \Delta u, \ t > 0, \ u(x, 0) = H(1 - x_1^2)H(1 - x_2^2)H(1 - x_3^2). \tag{712} \]
10 The wave equation

10.1 One space dimension

10.1.1 The Green’s function in one space dimension

Consider the Green’s function for the wave equation in one space dimension,

\[ \frac{1}{c^2} G_{,tt} - G_{,xx} = \delta(x - y)\delta(t - \tau). \]  

(713)

If we put \( z = x - y \) and \( T = t - \tau \) this becomes

\[ \frac{1}{c^2} G_{,TT} - G_{,zz} = \delta(z)\delta(T). \]  

(714)

As a boundary condition, we want the solution \( G(z, T) \) which has outgoing wave behaviour, that is, the solution for which waves move outwards from \( z = 0 \) towards infinity.

Outside the point \( (z = 0, T = 0) \), the d’Alembert solution applies, which can be written as

\[ G(z, T) = F(T - z/c) + E(T + z/c). \]  

(715)

Here \( F(T - z/c) \) represents a wave travelling towards increasing \( z \), while \( E(T + z/c) \) represents a wave travelling towards decreasing \( z \).

Now, if we want a wave that travels away from its source at \( z = 0 \), then for \( z > 0 \) we need the wave moving towards increasing \( z \), but for \( z < 0 \) we need it to travel towards decreasing \( z \). Consider therefore the ansatz

\[ G(z, T) = f\left(T - \frac{|z|}{c}\right). \]  

(716)

Note that \( |z| = z \) for \( z > 0 \) and \( |z| = -z \) for \( z < 0 \). It is easy to see that (716) has the property of travelling away from \( z = 0 \) for either \( z < 0 \) or \( z > 0 \). However, at \( z \) it does not obey the wave equation. But that may be to the good, because \( d\frac{|z|}{dz} = \text{sgn}(z) \) and \( d\text{sgn}(z)/dz = 2\delta(z) \).

Taking two time derivatives of (716), we easily find

\[ \frac{1}{c^2} G_{,TT} = \frac{1}{c^2} f''(T - |z|/c). \]  

(717)

Taking a first space derivative, we find

\[ G_{,z} = -\frac{\text{sgn}(z)}{c} f'(T - |z|/c), \]  

(718)

and hence taking another space derivative we find

\[ G_{,zz} = -\frac{2\delta(z)}{c} f'(T - |z|/c) + \left(\frac{-\text{sgn}(z)}{c}\right)^2 f''(T - |z|/c). \]  

(719)

Since \( \text{sgn}(z)^2 = 1 \) whatever \( z \) is and \( \delta(z)f(z) = \delta(z)f(0) \), this becomes

\[ G_{,zz} = -\frac{2\delta(z)}{c} f'(T) + \frac{1}{c^2} f''(T - |z|/c). \]  

(720)

Hence

\[ \frac{1}{c^2} G_{,TT} - G_{,zz} = \frac{2\delta(z)}{c} f'(T). \]  

(721)

Comparing with (714), we have

\[ F'(T) = \frac{c}{2} \delta(T) \]  

(722)

and hence

\[ F(T) = \frac{c}{2} H(T). \]  

(723)

Thus

\[ G(z, T) = F(T - |z|/c) = \frac{c}{2} H(T - |z|/c) \]  

(724)

so that

\[ G(x, t; y, \tau) = \frac{c}{2} H \left[(t - \tau) - |x - y|/c\right]. \]  

(725)
10.1.2 The initial value problem in one space dimension

Often, instead of wanting the causal solution, we want to solve the initial value problem.

\[ \frac{1}{c^2}u_{tt} - u_{xx} = 0, \quad t > 0, \quad u(x, 0) = f(x), \quad u_t(x, 0) = g(x). \] \quad (726)

We can turn this into a problem solvable with the causal Green’s function by the usual method. That is, put

\[ \psi(x, t) = H(t)u(x, t), \] \quad (727)

so that \( \psi = u \) for \( t > 0 \). Then, as usual, we have

\[ \psi_t = \delta(t)u(x, 0) + H(t)u_t, \] \quad (728)

where we have used that \( \delta(t)u(x, t) = \delta(t)u(x, 0) \),

\[ \psi_{tt} = \delta'(t)u(x, 0) + \delta(t)u_t(x, 0) + H(t)u_{tt}, \] \quad (729)

and note that the naive Leibniz rule does not apply. Also

\[ \psi_{xx} = H(t)u_{xx}. \] \quad (730)

Thus

\[ \frac{1}{c^2}\psi_{tt} - \psi_{xx} = \frac{1}{c^2}\left[\delta'(t)f(x) + \delta(t)g(x) + H(t)(u_{tt} - c^2u_{xx})\right] \] \quad (731)

and hence

\[ \frac{1}{c^2}\psi_{tt} - \psi_{xx} = \frac{1}{c^2}(\delta'(t)f(x) + \delta(t)g(x)). \] \quad (732)

The source term vanishes except at \( t = 0 \). If we solve this with the causal Green’s function \( 725 \) we therefore get a solution \( \psi(x, t) \) that vanishes for \( t < 0 \), which is precisely what we want, so the causal Green’s function is the correct one to use.

For \( t > 0 \), \( \psi = u \) and we have

\[ u = \frac{1}{c^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, t; y, \tau) (\delta'(\tau)f(y) + \delta(\tau)g(y)) \, dy \, d\tau. \] \quad (733)

where \( G \) is the one dimensional Greens function. Using the definitions of \( \delta(\tau) \) and its derivative \( \delta'(\tau) \), we obtain

\[ u(x, t) = \frac{1}{c^2} \int_{-\infty}^{\infty} [G(x, t; y, 0)g(y) - G_{,\tau}(x, t; y, 0)f(y)] \, dy. \] \quad (734)

Now from (725) we find

\[ G_{,\tau}(x, t; y, 0) = -\frac{c}{2}\delta(t - |x - y|/c), \] \quad (735)

and so

\[ u(x, t) = \frac{1}{2c} \left[ \int_{-\infty}^{\infty} H(t - |x - y|/c)g(y) \, dy + \int_{-\infty}^{\infty} \delta(t - |x - y|/c)f(y) \, dy \right]. \] \quad (736)

Finally note that

\[ t - |x - y|/c < 0 \iff y < x - ct \text{ or } y > x + ct, \] \quad (737)

\[ t - |x - y|/c = 0 \iff y = x - ct \text{ or } y = x + ct, \] \quad (738)

\[ t - |x - y|/c > 0 \iff x - ct < y < x + ct, \] \quad (739)

so that

\[ u(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x - ct}^{x + ct} g(y) \, dy. \] \quad (740)

It is interesting to write this explicitly in the d’Alembert form. In terms of the primitive function \( \hat{g} \) of \( g \), defined by

\[ \hat{g}(z) := \frac{1}{c} \int_{z}^{\infty} g(y) \, dy, \] \quad (741)

we can write

\[ u(x, t) = \frac{1}{2} [f(x + ct) + \hat{g}(x + ct)] + \frac{1}{2} [f(x - ct) - \hat{g}(x - ct)], \] \quad (742)

which is of the form

\[ u(x, t) = F(x - ct) + E(x + ct). \] \quad (743)
10.2 The three-dimensional problem

10.2.1 The Green’s function in three space dimensions

The three-dimensional wave equation problem is

$$\frac{1}{c^2}u_{tt} = \Delta u + f(x, t),$$  \hspace{1cm} (744)

together with a radiation condition. Specifically, waves should travel outwards from points where $f(x, t) \neq 0$ towards infinity rather than travel in from infinity.

The associated Green’s function, $G(x, t; y, \tau)$ satisfies

$$\frac{1}{c^2}G_{tt} - \Delta x G = \delta(t - \tau)\delta(x - y),$$  \hspace{1cm} (745)

together with the radiation condition that disturbances should radiate away from the point of disturbance, $x = y$, rather than towards it. If we introduce $z = x - y$ and $T = t - \tau$ then (745) becomes

$$\frac{1}{c^2}G_{TT} - \Delta z G = \delta(T)\delta(z).$$  \hspace{1cm} (746)

Since $\delta(z) = \delta(r)/4\pi r^2$, where $r = |z|$ it follows that $G = G(r, T)$. Using this, and the identity (476), we find

$$\frac{1}{c^2}(rG)_{TT} - \frac{1}{r}(rG)_{rr} = \delta(T)\frac{\delta(r)}{4\pi r^2},$$  \hspace{1cm} (747)

For $r \neq 0$ we have $\delta(r) = 0$, and after multiplying through by $r$ we obtain

$$\frac{1}{c^2}(rG)_{TT} - (rG)_{rr} = 0,$$  \hspace{1cm} (748)

which is the one-dimensional wave equation for $rG$. Its general solution can be written as

$$G(r, T) = \frac{F(T - r/c)}{4\pi r} + \frac{E(T + r/c)}{4\pi r},$$  \hspace{1cm} (749)

for some functions $F(T - r/c)$ and $E(T + r/c)$. (The factor of $1/4\pi$ is introduced purely for convenience in what follows.)

The radiation condition that disturbances should radiate away from $x = y$ rather than towards it becomes the condition that disturbances should move away from $r = 0$ towards increasing $r$. Hence we must take

$$G(r, T) = \frac{F(T - r/c)}{4\pi r}.$$  \hspace{1cm} (750)

This solution is singular as $r \to 0$, so we define a “generalised version” of it by

$$G(r, T) = \lim_{\epsilon \to 0} \left( H(r - \epsilon) \frac{F(T - r/c)}{4\pi r} \right),$$  \hspace{1cm} (751)

where the point $r = 0$ is excluded. We then use the result from Sec. 7.4 that

$$\Delta z \left[ \lim_{\epsilon \to 0} \left( H(r - \epsilon) \frac{f(r)}{4\pi r} \right) \right] = \frac{f''(r)}{4\pi r} - f(0)\delta(z),$$  \hspace{1cm} (752)

where $r := |z|$, to find

$$\Delta z G = \frac{F''(T - r/c)}{4c^2\pi r} - F(T)\delta(z).$$  \hspace{1cm} (753)

We also have

$$G_{TT} = \frac{F''(T - r/c)}{4\pi r},$$  \hspace{1cm} (754)

and putting the two together, we have

$$\frac{1}{c^2}G_{TT} - \Delta z G = F(T)\delta(z).$$  \hspace{1cm} (755)
Thus we need
\[ F(T) = \delta(T) \] (756)
and so
\[ G(r, T) = \frac{\delta(T - r/c)}{4\pi r} . \] (757)
Recalling our shorthands \( T = t - \tau, \ z = x - y \) and \( r = |z| \), written out in full this is
\[ G(x, t; y, \tau) = \frac{1}{4\pi|x - y|} \delta \left( t - \tau - \frac{1}{c} |x - y| \right) . \] (758)
Note that \( G = 0 \) except where \( t = \tau + \frac{1}{c} |x - y| \). The \( \delta \)-function here is the scalar one, not the three-dimensional one.

10.2.2 Retarded potentials

The causal solution of the problem
\[ \frac{1}{c^2}u_{tt} - \Delta_3 u = f(x, t) \] (759)
is
\[ u(x, t) = \int_\infty^\infty \int_\infty^\infty \int_\infty^\infty f(y, \tau) \frac{1}{4\pi |x - y|} \delta \left( t - \tau - \frac{1}{c} |x - y| \right) d^3 y \ d\tau. \] (760)
The integral over \( \tau \) simply picks out the value of the integrand at \( \tau = t - |x - y| / c \), and so
\[ u(x, t) = \frac{1}{4\pi} \int_\infty^\infty \int_\infty^\infty \int_\infty^\infty f(y, t - |x - y| / c) \frac{1}{|x - y|} d^3 y. \] (761)
This form of the solution is called a retarded potential. Essentially, the solution at point \( x \) and time \( t \) represents the superposition (in the form of an integral over space) of disturbances at points \( y \), and this is attenuated by a factor of \( 1/\text{distance} \). This looks very similar to the Greens’ function solution of the Poisson equation; hence the name “potential”. But this source is evaluated not at time \( t \), but at an earlier time, namely earlier by the time takes for the disturbance to travel in a straight line at speed \( c \); hence the name “retarded”.

Example 10.1. Find the field generated by a time harmonic point source at the origin which is switched on at time \( t = 0 \):
\[ \frac{1}{c^2}u_{tt} - \Delta_3 u = H(t) \delta(x) e^{i\omega t} . \] (762)
Using the retarded potential integral (761) with \( f(x, t) = \delta(x)H(t) e^{i\omega t} \), we have
\[ u(x, t) = \frac{1}{4\pi} \int_\infty^\infty \int_\infty^\infty \int_\infty^\infty \frac{\delta(y)H(t - |x - y| / c) e^{i\omega(t - |x - y| / c)}}{|x - y|} d^3 y = \frac{H(t - |x| / c)}{4\pi |x|} e^{i\omega(t - |x| / c)}. \] (763)
The Heaviside function confines the solution to the expanding sphere
\[ |x| \leq ct \] (764)
whose radius expands at the wave speed \( c \). The surface \( |x| = ct \) of this sphere represents the wave front of the expanding sphere of disturbance. Once the wave front has passed a given point \( x \), so that \( x \) is inside the sphere, the solution is simply
\[ u = \frac{1}{4\pi |x|} e^{i\omega(t - |x| / c)} \] (765)
and the phase of the oscillation at \( x \) differs from the phase of the oscillation at the source of the disturbance, \( 0 \), by \( \omega |x| / c = k|x| \). This phase difference is simply the time it takes the
disturbance to propagate from \( 0 \) to \( x \), travelling at speed \( c \). The factor of \( 1/(4\pi |x|) \) says that the amplitude decays inversely with the distance from the source.

Discarding the time factor in (765), this has the same form as the Green’s function for the Helmholtz equation (but with the opposite sign). This is not surprising since, for \(|x| < ct\), we have
\[
\frac{1}{c^2} u_{,tt} = -\frac{\omega^2}{c^2} u = -k^2 u
\]
and hence
\[
\frac{1}{c^2} u_{,tt} - \Delta u = H(t)\delta(x)e^{i\omega t}
\]
reduces to
\[
(\Delta + k^2)u = -H(t)\delta(x)e^{i\omega t}
\]
and writing \( u = H(t)e^{i\omega t}\psi \) this becomes the problem for (minus) the Helmholtz Green’s function;
\[
(\Delta + k^2)\psi = -\delta(x)
\]

10.3 The method of descent
10.3.1 From three to one dimensions
The one dimensional wave equation, with a source term, is
\[
\frac{1}{c^2} u_{,tt} = u_{,xx} + f(x_1, t)
\]
We can think of this as a three dimensional equation in planar symmetry
\[
\frac{1}{c^2} u_{,tt} = \Delta u + f(x_1, t)
\]
where, since the source term depends only on \( x_1 \) and \( t \), \( u(x, t) \) also depends only on \( x_1 \) and \( t \). Thus the solution can be written as
\[
u(x_1, t) = \int_{-\infty}^{\infty} \int_{y_1 = -\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{y_2 = -\infty}^{\infty} G_3(x, t; y, \tau) dy_2 dy_3 \right) f(y_1, \tau) dy_1 d\tau,
\]
where \( G_3(x, t; y, \tau) \) is the three dimensional Green’s function (758). Because \( f \) depends only on \( y_1 \) and \( \tau \), we can evaluate the term in round brackets first.

But because the solution of the one-dimensional problem is also, by definition, given by
\[
u(x_1, t) = \int_{-\infty}^{\infty} \int_{y_1 = -\infty}^{\infty} G_1(x_1, t; y_1, \tau)f(y_1, \tau) dy_1 d\tau,
\]
we have an expression for the one-dimensional Greens’ function \( G_1 \) as an integral over the three-dimensional one \( G_3 \), namely
\[
G_1(x_1, t; y_1, \tau) = \int_{y_2 = -\infty}^{\infty} \int_{y_3 = -\infty}^{\infty} G_3(x, t; y, \tau) dy_2 dy_3
\]
This is called the method of descent (in the number of dimensions). It is equally valid for the Poisson, Helmholtz and diffusion equations (although it is rather pointless in the case of the diffusion equation, where we used the one-dimensional Green’s function to find the three-dimensional one).

We now evaluate this integral. First we write \( z = x - y \) and \( T = t - \tau \) so that the problem becomes
\[
G_1(z_1, T) = \int_{z_2 = -\infty}^{\infty} \int_{z_3 = -\infty}^{\infty} \frac{1}{4\pi |z|} \delta(T - |z|/c) dz_2 dz_3, 
\]
Now introduce cylindrical polar coordinates in which \( z_1 \) is the axial direction and

\[
\begin{aligned}
z_2 &= r \cos \theta, \\
z_3 &= r \sin \theta
\end{aligned}
\]

so that

\[
|z| = \sqrt{z_1^2 + z_2^2 + z_3^2} = \sqrt{z_1^2 + r^2}
\]

and

\[
dz_2 dz_3 = r \, d\theta \, dr.
\]

with \(-\infty < z_1 < \infty\), \(0 \leq r < \infty\) and \(0 \leq \theta < 2\pi\). Then

\[
G_1(z_1, T) = \int_0^\infty \int_0^{2\pi} \frac{1}{4\pi \sqrt{z_1^2 + r^2}} \delta \left( T - \frac{1}{c} \sqrt{z_1^2 + r^2} \right) r \, d\theta \, dr.
\]

As the integrand does not depend on \( \theta \), the integration over \( \theta \) simply gives a factor of \( 2\pi \), or

\[
G_1(z_1, T) = \frac{c}{2} \int_0^\infty \frac{1}{\sqrt{z_1^2 + r^2}} \delta \left( T - \frac{1}{c} \sqrt{z_1^2 + r^2} \right) r \, dr.
\]

Now note that

\[
\frac{d}{dr} \sqrt{z_1^2 + r^2} = \frac{r}{\sqrt{z_1^2 + r^2}}
\]

so that we can make the substitution

\[
q = \frac{1}{c} \sqrt{z_1^2 + r^2} \quad \Rightarrow \quad dq = \frac{1}{c} \frac{d}{dr} \left( \sqrt{z_1^2 + r^2} \right) dr
\]

to obtain

\[
G_1(z_1, T) = \frac{c}{2} \int_{z_1/c}^\infty \delta(T - q) \, dq.
\]

If \(|z_1|/c > T\) the integral is zero since \( T - q \) is not in the integration range, and if \(|z_1|/c < T\) the integral is one, since now the zero of the \( \delta \)-function is inside the integration range. Thus

\[
G_1(z_1, T) = \frac{c}{2} H(T - |z_1|/c),
\]

and we have recovered (724), as expected.

10.3.2 From three to two dimensions

We can use the method of descent to deduce the two dimensional Greens function

\[
G_2(x_1, x_2, t; y_1, y_2, \tau)
\]

for the two dimensional wave equation

\[
\frac{1}{c^2} u_{tt} - (u_{x_1 x_1} + u_{x_2 x_2}) = f(x_1, x_2, t).
\]

By starting from the three-dimensional wave equation in cylindrical symmetry we find that

\[
G_2(x_1, x_2, t; y_1, y_2, \tau) = \int_{y_3 = -\infty}^\infty G_3(x, t; y, \tau) \, dy_3.
\]

Evaluating this integral is in principle similar to the calculation given above. Writing \( x = (x_1, x_2) \) and \( y = (y_1, y_2) \), it can be shown that

\[
G_2(x, t; y, \tau) = \frac{H(t - \tau - |x - y|/c)}{2\pi \sqrt{(t - \tau)^2 - |x - y|^2/c^2}}.
\]

(In fact, the method of descent was invented not to find the one dimensional Green’s function but rather, as the easiest way of finding the two dimensional Green’s function!)
10.4 Exercises

87. **Homework 30**: Use the causal Green’s function for the wave equation in one space dimension to solve the Cauchy problem

\[ c^2u_{tt} - u_{xx} = s(x, t), \]  
\[ u(x, 0) = f(x), \]  
\[ u_t(x, 0) = g(x). \]  

88. **Homework 31**: 
a) Find the causal solution of

\[ c^2u_{tt} - \Delta u = H(R - |x|)H(t)\sin \omega t \]  
in three-dimensional free space, where \( R > 0 \) and \( \omega \) are constants, and simplify as much as possible (but not more). 
b) State the long wavelength and large distance approximations for this problem and find \( u(x, t) \) using these approximations. 
c) Find \( u(0, t) \) in closed form. [Hint: This is a bit fiddly. Use polar coordinates for \( y \), and in your final expression distinguish the three cases \( t < 0, 0 < t < c^{-1}R \) and \( t > c^{-1}R \).]

89. **Homework 32**: Write out the causal solution of

\[ c^2u_{tt} - \Delta u = f(x, t) \]  
in two-dimensional free space. Then eliminate the Heaviside function by restricting the integration domain instead. [Hint: The restricted integration domain can be described as the interior of the past lightcone of \((x, t)\). What we are looking for is the two-dimensional equivalent of the retarded potential for the three-dimensional wave equation.]

90. Consider the wave equation problem

\[ u_{tt} = \Delta u, \quad u(x, 0) = 0, \quad u_t(x, 0) = H(1 - |x|), \]  
where \( H(\cdot) \) is the unit Heaviside function. Use the transformation \( \psi(x, t) = H(t)u(x, t) \) to convert the problem into the form

\[ \psi_{tt} - \Delta \psi = f(x, t). \]  
Use the three dimensional Green’s function to find the solution of this problem in integral form and hence show

\[ u(0, t) = tH(1 - t), \quad t > 0. \]  
[Note: the \( \delta \)-function in the Green’s function for the three-spatial dimensional wave equation is a one-dimensional Green’s function, not a vector Green’s function. You will have to use spherical polar coordinates to get the correct answer.]

91. Find the solution of the problem

\[ u_{tt} = u_{xx}, \quad u(x, 0) = \cos(x), \quad u_t(x, 0) = 2x e^{-x^2} \]  
for \( t > 0 \). Show that this can be written as the sum of two waves, one moving to the left and one moving to the right, both at unit speed.

92. By writing \( \psi = H(t)u \), show that the solution of

\[ u_{tt} - u_{xx} = \delta(x)\sin \omega t, \quad t > 0, \]  
with

\[ u(x, 0) = \cos(x), \quad u_t(x, 0) = 0 \]  
is

\[ u(x, t) = \frac{1}{2} \left[ \cos(x + t) + \cos(x - t) + \frac{1}{\omega}H(t - |x|)(1 - \cos(\omega(t - |x|))) \right]. \]
93. Show that the integral we need to solve in Sec. 10.3.2 can be written as

\[
\frac{1}{4\pi} \int_{-\infty}^{\infty} \frac{\delta \left( t - \sqrt{\rho^2 + z^2} / c \right)}{\sqrt{\rho^2 + z^2}} \, dz = \frac{1}{2\pi} \frac{H(t - \rho)}{\sqrt{t^2 - \rho^2 / c^2}},
\]  

(801)

where \( \rho = \sqrt{x^2 + y^2} \), and carry out the integration. Hint: Distinguishing the two cases \( t > \rho \) and \( t < \rho \) will give you the Heaviside function. Carry out the integration over \( z \) by using the identity

\[
\delta[f(x)] = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|},
\]  

(802)

where the \( x_i \) are the zeros of \( f(x) \).

94. Using a Green’s function, show that the causal solution of

\[
\frac{1}{c^2} u_{tt} - u_{,xx} = H(t)\delta(x)e^{i\omega t}
\]  

(803)

is

\[
u(x, t) = \frac{ic}{2\omega} H(t - |x| / c) \left(1 - e^{i\omega(t - |x| / c)}\right).
\]  

(804)
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