12 Electromagnetic Interactions

We have discussed the charged (complex) scalar field and the electromagnetic (photon) field but not the interactions between them.

Interactions are introduced by "minimal coupling", in which the partial derivative ∂_μ, is replaced by $\partial_\mu + ieA_\mu$ when acting on a complex field ϕ representing a particle of electric charge e

The Lagrangian density for a charged scalar field interacting with a photon field is

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \left(\partial_\mu - ieA_\mu\right)\phi^\dagger\left(\partial_\mu + ieA_\mu\right)\phi - m^2\phi^\dagger\phi.$$

Note that the canonical momentum π is no longer $\dot{\phi}^*$ but $\dot{\phi}^\dagger - iA_0\phi^\dagger$.

The Euler Lagrange equation of motion for the photon field becomes

$$\partial_\mu F^{\mu\nu} = j^\nu,$$

where the electromagnetic current $(\rho, \mathbf{j})$ is defined as

$$j_\mu \;=\; -ie\left(\phi\left(\partial_\mu - ieA_\mu\right))\phi^\dagger - \phi^\dagger\left(\partial_\mu + ieA_\mu\right)\right)\phi.$$

In components, this represents Maxwell's equations in the presence of a current and charge density

$$\nabla\cdot\mathbf{E} \;=\; \rho$$

$$\nabla\times\mathbf{B} - \frac{d\mathbf{E}}{dt} = \mathbf{j}$$

12.1 Feynman Rules for Scalar Electrodynamics

The interaction part of the Lagrangian density is

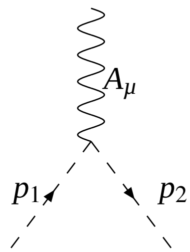
$$\mathcal{L}_I \;=\; -ieA_\mu\left(\phi^*\partial^\mu\phi - \phi\partial^\mu\phi^*\right) + e^2A_\mu A^\mu\phi^*\phi$$

The first term is cubic but unlike the ϕ^3 theory it contains a derivative, which, in momentum space, pulls down a factor of$-ip_\mu$ where p_μ is the momentum of the charge line.

In order to establish the sign (or direction of flow) consider the propagator

$$\langle 0|T\phi(x)\phi^*(y)|0\rangle \;=\; i\int\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2 - m^2}.$$

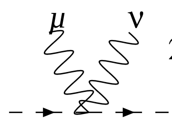
If $x_0 > y_0$ this represents a particle flowing from y to x. Thus $\partial_\mu \phi(x)$ gives $-ip_\mu$, where p_μ is the momentum of the charged particle flowing *into* the vertex. Similarly it could be the momentum of the antiparticle flowing out of the vertex. This then gives us the Feynman rule



$$-ie(p_1 + p_2)_\mu$$

The arrow on the charge line flows along the direction of charge. If the outgoing line is to be interpreted as an ingoing antiparticle with momentum p_2 then the sign of p_2 is reversed in the Feynman rule.

The other term in the interaction Lagrangian is $e^2 A_\mu A^\mu \phi^* \phi$ and gives rise to the so-called "seagull" vertex.



$$2ie^2 g_{\mu\nu}$$

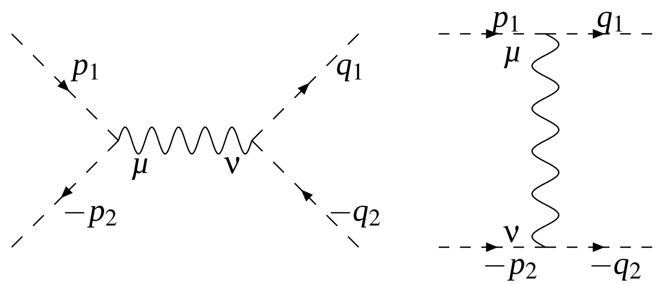
The $g_{\mu\nu}$ is present because the A_μ's are contracted with each other and the factor of 2 arises because there are two ways of contracting the A_μ in the interaction term with external photon fields.

12.2 Examples of Scalar Electrodynamics

π^+, π^- are examples of charged scalar particles (they are also strongly interacting and the strong interactions will dominate their scattering but in these examples we will consider only their electromagnetic interactions.)

$$(1)\;\; \pi^+ + \pi^- \;\to\; \pi^+ + \pi^-$$

There are two Feynman graphs for this process



The lines marked $-p_2$ and $-q_2$ represent antiparticles (π^-) with momenta p_2 and q_2 in the *opposite* direction from the arrows on the charge lines.

For the first diagram we have:
Left-hand vertex: $-ie(p_1 - p_2)^\mu$
Right-hand vertex: $-ie(q_1 - q_2)^\nu$
Internal propagator: $-ig_{\mu\nu}/(p_1 + p_2)^2$

The contribution to this diagram (omitting the energy-momentum conserving delta-function) is therefore

$$\mathcal{M}_1 = ie^2 \frac{(p_1 - p_2) \cdot (q_1 - q_2)}{(p_1 + p_2)^2}$$

In terms of Mandelstam variables we have

$$p_1 \cdot p_2 = q_1 \cdot q_2 = \frac{s}{2} - m^2$$

$$p_1 \cdot q_1 = p_2 \cdot q_2 = m^2 - \frac{t}{2}$$

$$p_1 \cdot q_2 = p_2 \cdot q_1 = m^2 - \frac{u}{2}$$

and

$$p_1^2 = p_2^2 = q_1^2 = q_2^2 = m^2$$

so that

$$\mathcal{M}_1 = ie^2 \frac{(u - t)}{s}$$

For the second diagram we have:
Upper vertex: $-ie(p_1 + q_1)^\mu$
Lower vertex: $+ie(p_2 + q_2)^\nu$
Internal propagator: $-ig_{\mu\nu}/(p_1 - q_1)^2$

So this diagram contributes

$$\mathcal{M}_2 = -ie^2 \frac{(p_1 + q_1) \cdot (p_2 + q_2)}{(p_1 - q_1)^2}$$

Again, writing the scalar products of the momenta in terms of Mandelstam variables this becomes

$$\mathcal{M}_2 \;=\; -ie^2\frac{(s-u)}{t}$$

So that the combined square-matrix-element is

$$|\mathcal{M}|^2 \;=\; e^4\left(\frac{u-t}{s}-\frac{s-u}{t}\right)^2$$

The cross-section is obtained by integrating over the phase-space for the outgoing pions and dividing by the flux-factor F. In the case of equal mass incoming particles the flux facto is given by

$$F \;\equiv\; 4\sqrt{(p_1\cdot p_2)^2-p_1^2p_2^2} \;=\; 2\sqrt{s(s-4m^2)},$$

so that

$$\sigma = \frac{e^4}{2\sqrt{s(s-4m^2)}}\int\frac{d^3\mathbf{q_1}}{(2\pi)^3 2E_{q_1}}\frac{d^4q_2}{(2\pi)^3}\delta(q_2^2-m^2)\theta(q_2^0)(2\pi)^4\delta^4(p_1+p_2-q_1-q_2)\left(\frac{u-t}{s}-\frac{s-u}{t}\right)^2$$

Integrating over q_2 to absorb the energy-momentum conserving delta-function and writing

$$\frac{d^3\mathbf{q_1}}{2E_{q_1}} \;=\frac{1}{2}\,d\phi d\cos\theta|\mathbf{q_1}|dE_{q_1},$$

we arrive at

$$\sigma = \frac{e^4}{(4\pi)^2\sqrt{s(s-4m^2)}}\int d\phi d\cos\theta|\mathbf{q_1}|dE_{q_1}\delta((p_1+p_2-q_1)^2-m^2)\left(\frac{u-t}{s}-\frac{s-u}{t}\right)^2$$

In the centre-of-mass frame

$$\delta((p_1+p_2-q_1)^2-m^2) \;=\; \delta(s-2\sqrt{s}E_{q_1})$$

and

$$t \;=\; 2m^2-2E_{p_1}E_{q_1}+2|\mathbf{p_1}||\mathbf{q_1}|\cos\theta$$

so that

$$d\cos\theta \;=\; \frac{dt}{2|\mathbf{p_1}||\mathbf{q_1}|}$$

and

$$|\mathbf{p_1}| \;=\; |\mathbf{q_1}| \;=\; \frac{1}{2}\sqrt{s-4m^2}$$

We end up with the differential cross-section

$$\frac{d\sigma}{dt} = \frac{e^4}{16\pi^2\sqrt{s}(s-4m^2)}\int d\phi dE_{q_1}\delta\left(s-2\sqrt{s}E_{q_1}\right)\left(\frac{u-t}{s}-\frac{s-u}{t}\right)^2$$
$$= \frac{e^4}{16\pi}\frac{1}{s(s-4m^2)}\left(\frac{u-t}{s}-\frac{s-u}{t}\right)^2$$

We usually express electromagnetic cross-sections in terms of the fine-structure constant [†]

$$\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}$$

$$\frac{d\sigma}{dt} = \pi\alpha^2\frac{1}{s(s-4m^2)}\left(\frac{u-t}{s}-\frac{s-u}{t}\right)^2$$

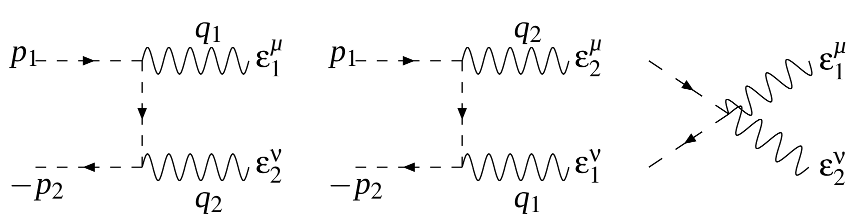
Recall that u can be replaced by $4m^2-s-t$.

(2) $\pi^+\pi^-$ annihilation into two photons

$$\pi^+\pi^- \to \gamma\gamma$$

π^+ and π^- can annihilate, but the minimum number of photons in the final state is two in order to be able to conserve energy and momentum.

There are three Feynman diagrams for this process



Again the lines marked $-p_2$ represent π^- with momentum p_2. The third graph comes form the "seagull" interaction term.

For the first diagram we have:
Upper vertex: $-ie(2p_1-q_1)\cdot\varepsilon_1^*(q_1,\lambda_1)$

[†]Remember that we are using units for which $\hbar = c = 1$ and also the permittivity of the vacuum $\varepsilon_0 = 1$.

Lower vertex: $-ie(p_1 - q_1 - p_2) \cdot \varepsilon_2^*(q_2, \lambda_2)$
Internal propagator: $i/((p_1 - q_1)^2 - m^2)$

For the second diagram we have:
Upper vertex: $-ie(2p_1 - q_2) \cdot \varepsilon_2^*(q_2, \lambda_2)$
Lower vertex: $-ie(p_1 - q_2 - p_2) \cdot \varepsilon_1^*(q_1, \lambda_1)$
Internal propagator: $i/((p_1 - q_2)^2 - m^2)$

For the third diagram we have:
Vertex: $2ie^2 \varepsilon_1^*(q_1, \lambda_1) \cdot \varepsilon_2^*(q_2, \lambda_2)$

We need to add these contributions together and square them. We also assume that the detector does *not* distinguish between left- and right- circular-polarised photons and so we sum over the helicities λ_1 and λ_2. In Feynman gauge this means making the replacement

$$\varepsilon_1^\mu(q_1, \lambda)\varepsilon_1^{\nu *}(q_1, \lambda) \rightarrow -g^{\mu\nu}$$

and similarly for λ_2.

The algebra is cumbersome but straightforward, and nowadays is best conducted using one of several computer packages which can handle the necessary substitutions (FORM is the most suitable for this type of manipulation but it can also be done in MATHEMATICA, MAPLE, or REDUCE).

Expressing the scalar products of the momenta in terms of Mandelstam variables the result is (after summing over the helicities of the final-state photons)

$$|\mathcal{M}|^2 = 4e^2 \left\{ 1 + \left(1 - \frac{2m^2 s}{(t - m^2)(u - m^2)} \right)^2 \right\}$$

Once again the flux factor is $2\sqrt{s(s - 4m^2)}$, so the cross-section is

$$\begin{aligned} \sigma &= \frac{4e^4}{2\sqrt{s(s-4m^2)}} \int \frac{d^3\mathbf{q_1}}{(2\pi)^3 2E_{q_1}} \frac{d^4 q_2}{(2\pi)^3} \delta(q_2^2)(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2) \left\{ 1 + \left(1 - \frac{2m^2 s}{(t-m^2)(u-m^2)} \right)^2 \right\} \\ &= \frac{e^4}{8\pi^2 \sqrt{s(s-4m^2)}} \int d\phi \, d\cos\theta \frac{|\mathbf{q_1}|}{2} \delta((p_1 + p_2 - q_1)^2) \left\{ 1 + \left(1 - \frac{2m^2 s}{(t-m^2)(u-m^2)} \right)^2 \right\} \end{aligned}$$

In the centre-of-mass frame

$$t = m^2 - 2|\mathbf{q_1}| \, ((E_{p_1} - |\mathbf{p_1}| \cos\theta)$$

and

$$\delta((p_1 + p_2 - q_1)^2) = \delta(s - 2\sqrt{s} E_{q_1})$$

$$|\mathbf{p_1}| = \frac{\sqrt{s-4m^2}}{2}$$

so that the differential cross-section is (using $\alpha = e^2/(4\pi)$)

$$\frac{d\sigma}{dt} = \frac{4\pi\alpha^2}{s(s-4m^2)}\left\{1+\left(1-\frac{2m^2 s}{(t-m^2)(u-m^2)}\right)^2\right\},$$

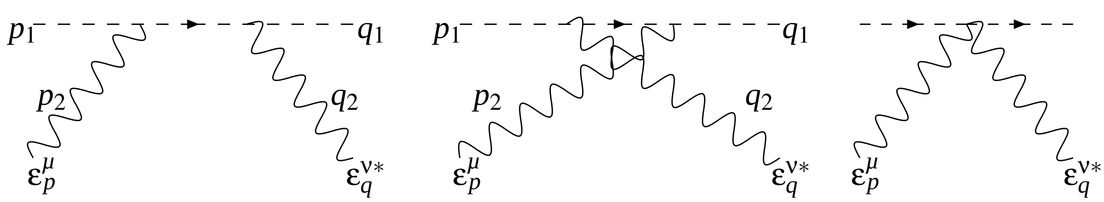
and in this case $u = 2m^2 - s - t$.

(3) Compton Scattering

This is the process

$$\pi^+\gamma \rightarrow \pi^+\gamma$$

There are three Feynman diagrams



Note that these graphs are the same as the ones for the process $\pi^= + \pi^- \rightarrow \gamma\gamma$, but turned on their side.

The scattering matrix element is an analytic function of the Mandelstam variables s,t, which means that they may be continued into an "unphysical region" where s is negative and t is positive. But such an unphysical region is the physical region for the process in which an incoming particle is exchanged for an outgoing antiparticle (or vice versa) and an outgoing particle is exchanged for an incoming antiparticle (or vice versa). This is precisely the interchange that transforms from the process $\pi^+\pi^- \rightarrow \gamma\gamma$ to the Compton-scattering process $\pi^+\gamma \rightarrow \pi^+\gamma$ - note that the incoming π^- in the former process is replaced by an outgoing π^+ in the latter process. This is known as "crossing symmetry".

This means that we do not have to calculate the square-matrix element again but merely interchange the Mandelstam variables s and t. There is also a factor of $\frac{1}{2}$, which arises because for $\pi^+\pi^- \rightarrow \gamma\gamma$ we sum over the helicities of the final-state photons, whereas for the Compton scattering process, we sum over the helicities of the outgoing photon but *average* over the helicities for the incoming photon (assuming that the incoming photon beam is unpolarised). The square matrix element, summed over final-state photon helicities and averaged over initial state helicities is therefore

$$|\mathcal{M}|^2 = 2e^4\left\{1+\left(1-\frac{2m^2 t}{(s-m^2)(u-m^2)}\right)^2\right\}.$$

In the case of Compton scattering, it is usual to express the cross-section in terms of the scattering angle of the photon in the rest-frame of the target particle (known as the "lab-frame") , rather than the centre-of-mass frame. In such a frame we have

$$\begin{aligned} p_1 &= (m,0,0,0) \\ p_2 &= (p,0,0,p) \\ q_2 &= (q, q\sin\theta\cos\phi, q\sin\theta\sin\phi, q\cos\theta) \end{aligned}$$

where θ is the scattering angle of the photon (initially along the z-direction), and we have introduced p and q as the energies of the initial and final photon respectively.

$$\begin{aligned} s &= m^2+2mp \\ u &= m^2-2mq \\ t &= 2m(q-p) = -2pq(1-\cos\theta) \end{aligned}$$

The relation

$$(1-\cos\theta) = m\left(\frac{1}{q}-\frac{1}{p}\right),$$

being the well-known formula for the wavelength shift in Compton scattering.

In terms of these quantities, the square-matrix-element becomes

$$|\mathcal{M}|^2 = 2e^4\left(1+\cos^2\theta\right).$$

Since one of the incoming particles is massless the flux factor is

$$F = 4p_1\cdot p_2$$

and in the lab frame this becomes

$$F = 4mp$$

The cross section is

$$\sigma = \frac{1}{4mp}\int \frac{d^3\mathbf{q_2}}{(2\pi)^3 2q}\frac{d^4q_1}{(2\pi)^3}\delta(q_1^2-m^2)\theta(q_1^0)(2\pi)^4\delta^4(p_1+p_2-q_1-q_2)\,2e^4(1+\cos^2\theta)$$

Note that this time we have chosen to use the integration over q_1 to absorb the energy-momentum conserving delta-function as we wish to calculate the differential cross-section w.r.t $\cos\theta$, the scattering angle of the photon.

Performing the integral over q_1 and writing

$$d^3\mathbf{q_2} = d\phi d\cos\theta q^2 dq,$$

we get

$$\sigma = \frac{e^4}{4\pi^2 mp}\int d\phi d\cos\theta q dq \delta((p_1+p_2-q)^2-m^2)2e^4(1+\cos^2\theta)$$

In the lab-frame

$$\delta((p_1+p_2-q)^2-M^2) \;=\; \delta(2mp-2q(m+p(1-\cos\theta))\,,$$

so that the integration over q absorbs the remaining delta-function and gives a factor $1/(2(m+p(1-\cos\theta)))$. Using (from the Compton scattering formula)

$$q \;=\; \frac{mp}{(m+p(1-\cos\theta))}$$

we finally end up with a differential cross-section w.r.t the lab-frame scattering angle of

$$\frac{d\sigma}{d\cos\theta} \;=\; \frac{\pi\alpha^2(1+\cos^2\theta)}{(m+p(1-\cos\theta))^2} \;=\; \frac{\pi\alpha^2}{m^2}(1+\cos^2\theta)\omega^2$$

where ω is the ratio of the outgoing to incoming photon momenta, q/p