8 Feynman Diagrams

We now return to the scattering problem

$$p_1 + p_2 \;\rightarrow\; q_1 + q_2$$

Setting Z to 1, the LSZ reduction and perturbation formula gives

$$\begin{aligned}\langle q_1, q_2, out|p_1, p_2, in\rangle \;&=\; \int d^4x_1 d^4x_2 d^4y_1 d^4y_2 e^{iq_1\cdot y_1 + q_2\cdot y_2 - p_1\cdot x_1 - p_2\cdot x_2}\left(\Box_{x_1} + m^2\right)\left(\Box_{x_2} + m^2\right)\\ &\left(\Box_{y_1} + m^2\right)\left(\Box_{y_2} + m^2\right)\langle 0|Te^{-i\int d^4x g\phi(x)^3/3!}\phi(x_1)\phi(x_)\phi(y_1)\phi(y_2)|0>_0,\end{aligned}$$

where the suffix 0 means that the fields are free fields, and we have taken

$$\mathcal{L}_I \;=\; -g\frac{\phi^3}{3!}$$

If we take the term 1 of the exponential we get disconnected products of Green functions such as

$$\langle 0|T\phi(x_1)\phi(y_1)|0\rangle\langle 0|T\phi(x_2)\phi(y_2|0\rangle.$$

These do not involve interactions and do *not* contribute to the scattering amplitude. In fact, using the Green function equations

$$\left(\Box_x + m^2\right)\langle 0|T\phi(x)\phi(y)|0\rangle \;=\; i\delta^4(x-y),$$

this boils down to an uninteresting

$$\sim (p_1^2 - m^2)(p_2^2 - m^2)\delta^4(p_1 - q_1)\delta^4(p_2 - q_2),$$

representing particles going straight through without scattering.

If we expand the potential to order g we get a term with the vacuum expectation value of an odd number of fields, which vanishes.

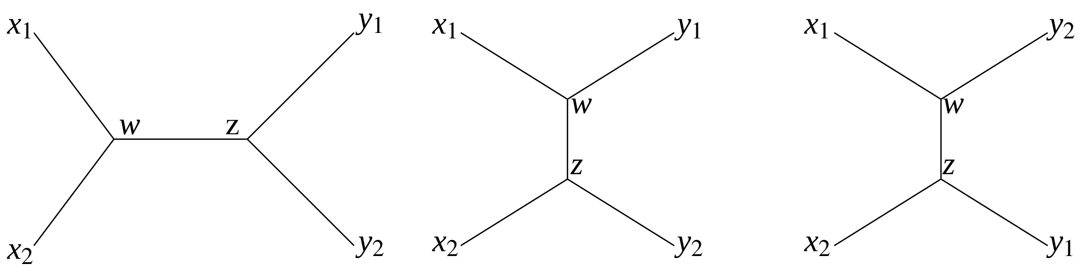
Thus the lowest non-trivial term comes in order g^2 and this is

$$\begin{aligned}=\;& \frac{-g^2}{2!3!3!}\int d^4x_1 d^4x_2 d^4y_1 d^4y_2 d^4z d^4w e^{i(q_1\cdot y_1 + q_2\cdot y_2 - p_1\cdot x_1 - p_2\cdot x_2)}\left(\Box_{x_1} + m^2\right)\left(\Box_{x_2} + m^2\right)\\ &\left(\Box_{y_1} + m^2\right)\left(\Box_{y_2} + m^2\right)\langle 0|T\phi^3(z)\phi^3(w)\phi(x_1)\phi(x_)\phi(y_1)\phi(y_2)|0>_0,\end{aligned}$$

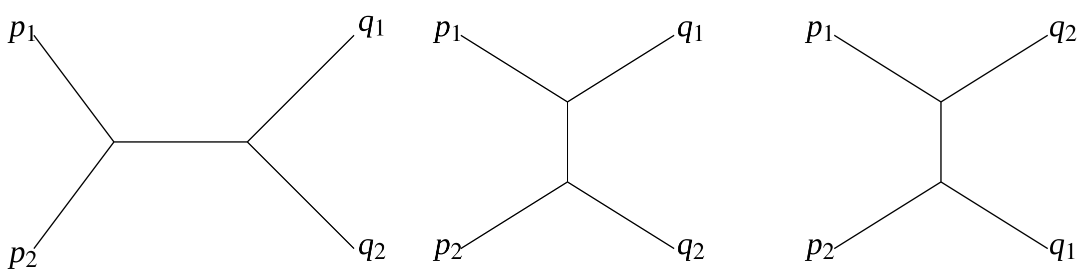
We now perform the Wick contraction, but since we are only interested in connected terms we only keep terms in which one of the external fields $\phi(x_1)$, $\phi(x_2)$, $\phi(y_1)$, $\phi(y_2)$ are contracted with one of the internal fields $\phi(z)$ or $\phi(w)$ coming from the interaction Lagrangian. Using the symmetry under $z \leftrightarrow w$, we end up with three distinct contractions

$$\begin{aligned}&\Delta_F(x_1,w)\Delta_F(x_2,w)\Delta_F(w,z)\Delta_F(z,y_1)\Delta_F(z,x_2)\\ &\Delta_F(x_1,w)\Delta_F(x_2,z)\Delta_F(w,z)\Delta_F(w,y_1)\Delta_F(z,y_2)\\ &\Delta_F(x_1,z)\Delta_F(x_2,w)\Delta_F(w,z)\Delta_F(w,y_1)\Delta_F(z,y_2)\end{aligned}$$

By drawing a line to represent each Feynman propagator and a vertex for each interaction these three contractions may be represented diagrammatically as



It is more useful *not* to label the space-time coordinates, but their associated momenta (read off from the term $e^{i(q_1 \cdot y_1 + q_2 \cdot y_2 - p_1 \cdot x_1 - p_2 \cdot x_2)}$ with $e^{-ip\cdot x}$ meaning an incoming momentum, p, and $e^{+iq\cdot y}$ meaning an outgoing momentum, q). Thus we redraw these diagrams as



These diagrams are called "Feynman diagrams" or "Feynman graphs". The momenta going along the internal lines are determined by the fact that mom netum is conserved at each vertex, so in the 3 above diagrams the internal momenta are respectively $(p_1 + p_2)$, $(p_1 - q_1)$ and $(p_1 - q_2)$.

In higher order of perturbation theory the diagrams would contain loops of particles. These leading order graphs are called "tree-level"graphs (we confine ourselves to such tree-level graphs in these lectures).

Now if we insert this back into the LSZ reduction formula, and use the Green function equations

$$\left(\Box_{x_1} + m^2\right) \Delta_F(x_1, w) \; = \; -\delta^4(x_1 - w),$$

etc. and then perform the integration over x_1, x_2, y_1, y_2 to absorb the delta functions, we end up with

$$\begin{aligned} -ig^2 \int d^4z e^{-i(p_1+p_2-q_1-q_2)\cdot z} \{ \quad & \int d^4w \quad e^{i(p_1+p_2)\cdot(w-z)} \Delta_F(z,w) + \\ & \int d^4w \quad e^{i(p_1-q_1)\cdot(w-z)} \Delta_F(z,w) + \\ & \int d^4w \quad e^{i(p_1-q_2)\cdot(w-z)} \Delta_F(z,w) \; \} \end{aligned}$$

The terms inside $\{\cdots\}$ are the Fourier transforms of the scalar field propagators with momenta being equal to the momenta along the internal lines in the three Feynman graphs.

The the three Feynman graphs give contributions to the scattering amplitude of

$$\frac{-ig^2}{(p_1+p_2)^2-m^2}$$
$$\frac{-ig^2}{(p_1-q_1)^2-m^2}$$
$$\frac{-ig^2}{(p_1-q_2)^2-m^2},$$

respectively. †

The remaining integral over the variable z generates an energy-momentum conserving delta-function $(2\pi)^4\delta^4(p_1+p_2-q_1-q_2)$

8.1 Feynman Rules for Scalar ϕ^3 Theory

$$\mathcal{L}_I = -\frac{g}{3!}\phi^3(x)$$

The quantum amplitude can be obtained by the application of the following Feynman rules:

1. Draw *all* connected Feynman graphs up to a given order in the coupling, for the process required.
2. Label the momenta of all lines, conserving momentum at each vertex.
3. Write a factor of $-ig$ for each vertex.
4. A propagator factor of
$$\frac{i}{p^2-m^2+i\varepsilon}$$
for each *internal* line with momentum p along it.
5. A factor of $(2\pi)^4\delta(\sum(incoming\,momenta)-\sum(outgoing\;momenta))$.

Note that the momentum, p, going along the internal lines are *not* such that $p^2=m^2$ for example for the first diagram we have

$$p^2=(p_1+p_2)^2=2m^2+2(E_{p_1}E_{p_2}-\mathbf{p_1}\cdot\mathbf{p_2}) \;>\; m^2,$$

and for the second diagram we have

$$p^2=(p_1-q_1)^2=2m^2-2(E_{p_1}E_{q1}-\mathbf{p_1}\cdot\mathbf{q_1}) \;<\; 0,$$

†We have dropped the $i\varepsilon$ in the propagators, but for future reference we should remember that they are there and we mean the limit as $\varepsilon\to 0$.

and similarly for the third diagram.

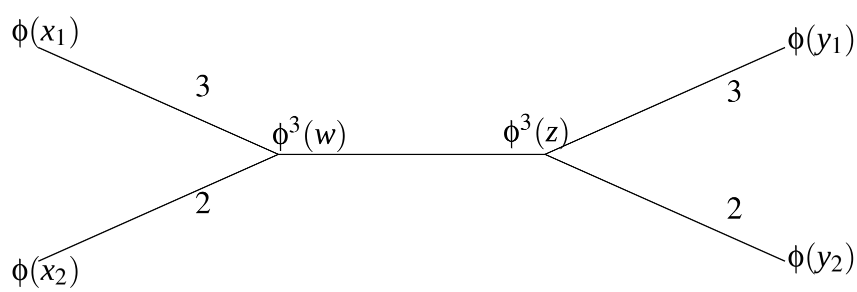
Such particles are said to be "off mass-shell" or "virtual particles". Qualitatively what is happening is that since these particles only exist for a very short period of time - e.g. in the first Feynman graph they are formed by the annihilation of two incoming particles and a moment later they decay into the two outgoing particles - conservation of energy can be violated by an amount determined by Heisenberg's Uncertainty Principle and the uncertainty in the energy is such that the energy of the internal particles need *not* be equal to $\sqrt{p^2+m^2}$. On the other hand the external particles, which travel for a long time before they interact are "on mass-shell" so that $p_1^2 = p_2^2 = q_1^2 = q_2^2 = m^2$.

8.2 Combinatorics

The factor 3! in the denominator of $\mathcal{L}_I = -g/3!\phi^3$ has cancelled against a factor of 3! from the 3! ways in which the 3 ϕ's in ϕ^3 can be contracted with external fields. Hence the Feynman rule for the interaction vertex is $-ig$ and *not* $-ig/3!$.

Nevertheless, it is good practice to check that the allowed number of contractions introduces a factor which cancels the factors in the denominator.

To order g^2 we have a factor of $1/2!$ from the expansion of the exponential and two factors of $1/3!$ from the two powers of the interaction Lagrangian. How many ways can we make the first Feynman diagram ?



- 3 ways to couple x_1 to w.
- 2 ways to couple x_2 to the remaining w
- 3 ways to couple y_1 to z.
- 2 ways to couple y_2 to the remaining z
- One remaining way to couple the remaining w to the remaining z
- A factor of 2 for the interchange of z and w

This gives a combinatorial factor of $3\times2\times3\times2\times2$, which exactly cancels the denominator of $1/(2!3!3!)$.

The other Feynman diagrams also have a combinatorial factor of 1. However, it can sometimes happen that this combinatorics introduces factors of 2 or $\frac{1}{2}$

8.3 Mandelstam Variables

We consider a two-particle to two-particle scattering with incoming momenta $p_1,\ p_2$ and outgoing momenta $q_1\ q_2$.

The Mandelstam variables, s,t and u are defined as

$$\begin{aligned} s &= (p_1+p_2)^2 = (q_1+q_2)^2 \\ t &= (p_1-q_1)^2 = (p_2-q_2)^2 \\ u &= (p_1-q_2)^2 = (p_2-q_2)^2 \end{aligned}$$

Not all of these are independent. In fact we can show using conservation of energy momentum that

$$s+t+u = m_1^2+m_2^2+m_3^3+m_4^2,$$

(where we are now allowing for the possibility that the four particles involved have different masses).

If the incoming particles have masses, m_1 and m_2 then we can move into the centre-of-mass frame of these two particles, moving in opposite directions along the z-axis, with momentum p for which

$$p_1 = \left(\sqrt{p^2+m_1^2}, 0, 0, p\right)$$

$$p_2 = \left(\sqrt{p^2+m_2^2}, 0, 0, -p\right)$$

so that

$$s = \left(\sqrt{p^2+m_1^2}+\sqrt{p^2+m_2^2}\right)^2,$$

i.e. it is equal to the total energy in the centre-of-mass frame. In terms of s, m_1, m_2 we may express the magnitude of the three-momentum, p, in this frame as

$$p = \lambda^{1/2}(s, m_1, m_2),$$

where

$$\lambda(x,y,z) = x^2+y^2+z^2-2xy-2xz-2yz$$

This simplifies considerably if the two masses are equal for which

$$p_1 = \frac{1}{2}\left(\sqrt{s}, 0, 0\sqrt{s-4m^2}\right)$$

$$p_2 = \frac{1}{2}\left(\sqrt{s}, 0, 0-\sqrt{s-4m^2}\right)$$

Likewise, in the case where all four masses are equal, the quantity t is the 4-momentum transfer and is related to the scattering angle θ in the centre-of-mass frame

$$t = 2\left(m^2 - p_1 \cdot q_1\right) = 2\left(m^2 - E_{p_1}E_{q_1} + \mathbf{p_1} \cdot \mathbf{q_1}\right) = \left(2m^2 - \frac{s}{2} + \frac{(s-4m^2)}{2}\cos\theta\right)$$

Similarly

$$u = \left(2m^2 - \frac{s}{2} - \frac{(s-4m^2)}{2}\cos\theta\right)$$

and we can see that $s + t + u = 4m^2$.

Note that s, t, u are the square-momenta of the internal lines in the three Feynman graphs, so that in terms of these Mandelstam variables the three contributions to the scattering amplitude are

$$\frac{-ig^2(2\pi)^4\delta(p_1 + p_2 - q_1 - q_2)}{(s-m^2)},$$

$$\frac{-ig^2(2\pi)^4\delta(p_1 + p_2 - q_1 - q_2)}{(t-m^2)},$$

$$\frac{-ig^2(2\pi)^4\delta(p_1 + p_2 - q_1 - q_2)}{(u-m^2)}.$$

The diagrams are referred to as the $s-$channel, t-channel and u-channel exchange graphs respectively.

Note also that s, t, u are Lorentz invariant quantities whereas other directly measured variables such as the scattering angle are frame dependent and would be different, for example, in a fixed target scattering experiment where one of the incident particles is at rest.