9 Lorentz Invariant phase-Space

9.1 Cross-sections

The scattering amplitude

$$M \equiv \langle q_1, q_2, out | p_1, p_2, in \rangle$$

is the amplitude for a state $|p_1, p_2\rangle$ to make a transition into the state $|q_1, q_2\rangle$. The transition probability is the square modulus of this quantity. But here we have a problem. Let us write

$$M = \mathcal{M}(2\pi)^4\delta^4(p_1 + p_2 - q_1 - q_2).$$

The square of the energy-momentum conserving delta-function is *not* defined.

The problem arises because we do not have incoming states which are perfect eigenstates of momentum, but rather a wave-packet, which is a weighted superposition of such states, so that in "in"-state is really

$$|in> = \int \frac{d^3\mathbf{p_1}}{(2\pi)^3 2E_1}\frac{d^3\mathbf{p_2}}{(2\pi)^3 2E_2} f_1(\mathbf{p_1}) f_2(\mathbf{p_2}) |p_1, p_2\rangle,$$

where f_1 and f_2 are the Fourier transforms of the wavefunctions of the incident particles. The transition probability, W, is now given by

$$\begin{aligned} W &= \int \frac{d^3\mathbf{p_1}}{(2\pi)^3 2E_1}\frac{d^3\mathbf{p_2}}{(2\pi)^3 2E_2} \int \frac{d^3\mathbf{p_1'}}{(2\pi)^3 2E_1'}\frac{d^3\mathbf{p_2'}}{(2\pi)^3 2E_2'} f_1(\mathbf{p_1}) f_2(\mathbf{p_2}) f_1^*(\mathbf{p_1'}) f_2^*(\mathbf{p_2'}) \\ &\quad (2\pi)^8\delta^4(q_1+q_2-p_1-p_2)\delta^4(p_1+p_2-p_1'-p_2')|\mathcal{M}|^2 \end{aligned}$$

We can write the second delta-function as

$$(2\pi)^4\delta^4(q_1+q_2-p_1'-p_2') = \int d^4x e^{i(p_1+p_2-p_1'-p_2')\cdot x},$$

and perform the integration over $\mathbf{p_1}', \mathbf{p_2}'$ (the inverse Fourier transform) to get an expression in terms of the wave-functions, $\psi_1(x)$, $\psi_2(x)$ of the incoming particles. For incoming wavepackets which are sharply peaked at $\mathbf{p_1}$ and $\mathbf{p_2}$, this integration approximates to

$$W = \int d^4x \frac{|\psi_1(x)|^2}{2E_1}\frac{|\psi_2(x)|^2}{2E_2}(2\pi)^4\delta^4(q_1+q_2-p_1-p_2)|\mathcal{M}|^2$$

The transition *rate* per unit volume is

$$\begin{aligned} \frac{dW}{d^3\mathbf{x}dt} &= \frac{(2\pi)^4\delta^4(q_1+q_2-p_1-p_2)}{4E_1E_2}|\mathcal{M}|^2|\psi_1(x)|^2|\psi_2(x)|^2 \\ &= d\sigma \times \text{flux} \end{aligned}$$

$d\sigma$ is the differential cross-section for the initial state to go into the state $|q_1, q_2\rangle$, and the flux factor, F, is the probability to find particle 1 per unit volume multiplied by the probability to find particle 2 per unit volume multiplied by their relative velocity, v.

$$F = |\psi_1(x)|^2 |\psi_2(x)|^2 v.$$

In the rest-frame of one of the particles (2) the relative velocity is given by $v = \frac{|\mathbf{p_1}|}{E_1}$. Remembering that the states are relativistically normalised, the square-wavefunction for an (almost) momentum eigenstate can be replaced by $2E$ and so we have, in the rest frame of particle 2

$$F = 4E_1 E_2 v = 4|\mathbf{p_1}|E_2 = 4|\mathbf{p_1}|m_2.$$

This can be written in manifest Lorentz invariant form as

$$F = 4m_2\sqrt{E_1^2 - m_1^2} = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$$

Since this latter expression is in terms of masses and Lorentz-invariant scalar products of 4-momenta, it is a Lorentz invariant expression. We can write

$$F = 2\lambda^{1/2}(s, m_1^2, m_2^2),$$

with λ (as before) given by

$$\lambda(x,y,z) = x^2 + y^2 + z^2 - 2xy - 2xz - 2yz.$$

Thus finally we end up with an expression for thew differential cross-section

$$d\sigma = \frac{(2\pi)^4 \delta^4(q_1 + q_2 - p_1 - p_2)|\mathcal{M}|^2}{F}.$$

9.2 Lorentz-invariant phase-space (LIPS) integration

$d\sigma$ is the cross-section for a transition into the state $|q_1, q_2\rangle$. The total cross-section is obtained by integrating over all possible final state momenta using the Lorentz invariant measure.

$$DLIPS = \frac{d^4q_1}{(2\pi)^3}\frac{d^4q_2}{(2\pi)^3}\delta(q_1^2 - m_3^2)\theta(q_1^0)\delta(q_2^2 - m_4^2)\theta(q_2^0),$$

where we have taken the masses of the outgoing particles to be m_3 and m_4. In general, if we have n final-state particles the Lorentz-invariant phase-space is given by

$$DLIPS = \prod_{i=1}^{n} \frac{d^4q_1}{(2\pi)^3}\delta(q_i^2 - m_i^2)\theta(q_i^0).$$

It will be convenient to write some of these factors in the non-manifestly Lorenz invariant form

$$\frac{d^3\mathbf{q_1}}{(2\pi)^3 2E_{q_1}},$$

and choose a suitable frame in which to perform the integration.

Thus the rules for calculating the total cross-section are

1. Calculate the matrix element $\mathcal{M}$ from the Feynman rules, omitting the energy-momentum delta-function $(2\pi)^4\delta^4(\sum_i(q_i) - p_1 - p_2)$.

2. The cross-section for n-particles in the final state is

$$\sigma = \prod_{i=1}^{n}\int \frac{d^4q_i}{(2\pi)^3}\delta(q_i^2 - m_i^2)\theta(q_i^0)(2\pi)^4\delta^4(\sum_i(q_i) - p_1 - p_2)\frac{|\mathcal{M}|^2}{F}.$$

Returning to the case of two final-state particles, we may not want the total cross section but a quantity such as $\frac{d\sigma}{d\theta}$, where θ is the scattering angle. Since this is frame-dependent it would be better to calculate a quantity such as $\frac{d\sigma}{dt}$, and then transform the result into the differential cross-section w.r.t scattering angle in a chosen frame.

Since we are then calculating a Lorentz invariant quantity, we are at liberty to consider the system in a convenient frame of reference. For the two final-state case the easiest frame is the centre-of-mass frame for which the incoming momenta p_1, p_2 are given by

$$\begin{aligned} p_1^\mu &= \left(\sqrt{p^2 + m_1^2}, 0, 0, p\right) \\ p_2^\mu &= \left(\sqrt{p^2 + m_2^2}, 0, 0, -p\right) \end{aligned}$$

Using the definitions of the Mandelstam variable s and λ this can be written as

$$\begin{aligned} p_1^\mu &= \left(\frac{s + m_1^2 - m_2^2}{2\sqrt{s}}, 0, 0, \frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}\right) \\ p_2^\mu &= \left(\frac{s + m_2^2 - m_1^2}{2\sqrt{s}}, 0, 0, -\frac{\lambda^{1/2}(s, m_1^2, m_2^2)}{2\sqrt{s}}\right) \end{aligned}$$

Likewise the outgoing momenta may be written as

$$\begin{aligned} q_1^\mu &= \left(\sqrt{q^2 + m_3^2}, q\sin\theta\cos\phi, q\sin\theta\sin\phi, q\cos\theta\right) \\ q_2^\mu &= \left(\sqrt{q^2 + m_4^2}, -q\sin\theta\cos\phi, -q\sin\theta\sin\phi, -q\cos\theta\right), \end{aligned}$$

where θ, ϕ are the polar angles of the outgoing particle with momentum q_1. Since s may also be written $s = (q_1 + q_2)^2$ we can perform the same manipulations to obtain

$$q_1^\mu = \left(\frac{s + m_3^2 - m_4^2}{2\sqrt{s}}, \frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{2\sqrt{s}} \mathbf{n} \right)$$

$$q_1^\mu = \left(\frac{s + m_4^2 - m_3^2}{2\sqrt{s}}, -\frac{\lambda^{1/2}(s, m_3^2, m_4^2)}{2\sqrt{s}} \mathbf{n} \right)$$

with the unit 3-vector $\mathbf{n}$ given by

$$\mathbf{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

and the Mandelstam variable t is

$$t = (p_1 - q_1)^2 = m_1^2 + m_3^2 - 2E_{p_1}E_{q_1} + 2|\mathbf{p_1}||\mathbf{q_1}|\cos\theta$$

Now write the expression for the cross-section as

$$\sigma = \int \frac{d^3\mathbf{q_1}}{(2\pi)^3 2E_{q_1}} \frac{d^4 q_2}{(2\pi)^3} (\delta(q_2^2 - m_4^2)(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2) \frac{|\mathcal{M}|^2}{F},$$

where we have written the phase-space measure for q_1 in non-relativistic form. We can now use the integral over $d^4 q_2$ to absorb the energy-momentum conserving delta-function, but remember that q_2 must be replaced by $p_1 + p_2 - q_1$ inside the delta-function $\delta(q_2^2 - m_4^2)$, so that we are now left with

$$\sigma = \frac{1}{(2\pi)^2} \int \frac{d^3\mathbf{q_1}}{2E_{q_1}} \delta((p_1 + p_2 - q_1)^2 - m_4^2) \frac{|\mathcal{M}|^2}{F}$$

$$d^3\mathbf{q_1} = d\cos\theta d\phi |\mathbf{q_1}|^2 d|\mathbf{q_1}|$$

The integration over ϕ introduces a factor of 2π. We want to replace the integral over $\cos\theta$ by an integral over t. From the expression for t we have

$$d\cos\theta = \frac{dt}{2|\mathbf{p_1}||\mathbf{q_1}|}$$

In the centre-of-mass frame,

$$(p_1 + p_2)^\mu = (s, 0, 0, 0),$$

so that the argument of the remaining delta-function is

$$(s - 2\sqrt{s}E_{q_1} + m_3^2 - m_4^2)$$

Furthermore since $E_{q_1}^2 = m_3^3 + |\mathbf{q_1}|^2$, we have

$$|\mathbf{q_1}| d|\mathbf{q_1}| = E_{q_1} dE_{q_1},$$

so that

$$\frac{d^3\mathbf{q_1}}{2E_{q_1}} = \frac{d\phi dt}{2|\mathbf{p_1}||\mathbf{q_1}|}|\mathbf{q_1}|\frac{E_{q_1}}{2E_{q_1}}dE_{q_1},$$

leaving (after integration over ϕ)

$$\frac{d\sigma}{dt} = \frac{1}{2\pi}\int\frac{dE_{q_1}}{4|\mathbf{p_1}|}\delta(s-2\sqrt{s}E_{q_1}+m_3^2-m_4^2)\frac{|\mathcal{M}|^2}{F}$$

Performing the integration over E_{q_1} to absorb the remaining delta-function an inserting the expression for the flux, F, we have

$$\frac{d\sigma}{dt} = \frac{1}{16\pi|\mathbf{p_1}|\sqrt{s}}\frac{|\mathcal{M}|^2}{2\lambda^{1/2}(s,m_1^2,m_2^2)}$$

But

$$|\mathbf{p_1}| = \frac{\lambda^{1/2}(s,m_1^2,m_2^2)}{2\sqrt{s}},$$

and so we finally end up with

$$\frac{d\sigma}{dt} = \frac{1}{16\pi\lambda(s,m_1^2,m_2^2)}|\mathcal{M}|^2$$

Note that $\lambda^{1/2}$ is only real if $s > (m_1+m_2)^2$, which is the physical threshold for the scattering to occur.

In the ϕ^3 case (with equal masses) that we have been considering we therefore have

$$\frac{d\sigma}{dt} = \frac{g^4}{16\pi s(s-4m^2)}\left(\frac{1}{(s-m^2)}+\frac{1}{(t-m^2)}+\frac{1}{(3m^2-s-t)}\right)^2.$$

(Note that we have used $u = 4m^2-s-t$).

The integration over t needed to calculate the total cross-section is often very messy. The limits on t are obtained in terms of $\cos\theta = \pm 1$ giving

$$t_{min} = m_1^2+m_3^2-2E_{p_1}E_{q_1}-2|\mathbf{p_1}||\mathbf{q_1}|$$

$$t_{max} = m_1^2+m_3^2-2E_{p_1}E_{q_1}+2|\mathbf{p_1}||\mathbf{q_1}|$$

In this case where all the masses are equal, the energies of the particles are equal and so are the magnitude of their three-momenta (in the centre-of-mass frame) and this simplifies to

$$t_{min} = -(s-4m^2)$$

$$t_{max} = 0$$

Furthermore, we can obtain the differential cross-section with respect to the centre-of-mass scattering angle, θ by

$$\frac{d\sigma}{d\cos\theta} = 2|\mathbf{p_1}||\mathbf{q_1}|\frac{d\sigma}{dt}$$

Again, if all the masses are equal this simplifies to

$$\frac{d\sigma}{d\cos\theta} = \frac{(s-4m^2)}{2}\frac{d\sigma}{dt}$$

Sometimes differential cross-sections are quoted in terms of $\frac{d\sigma}{d\Omega}$ where Ω is the solid angle. This is what is measured directly as a detector will subtend a given element of solid angle $d\Omega$. This is simply obtained by *not* performing the integration over the azimuthal angle ϕ, i.e.

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{d\sigma}{d\cos\theta}.$$

again this quantity is frame dependent and different in a collider experiment from a fixed-target experiment.