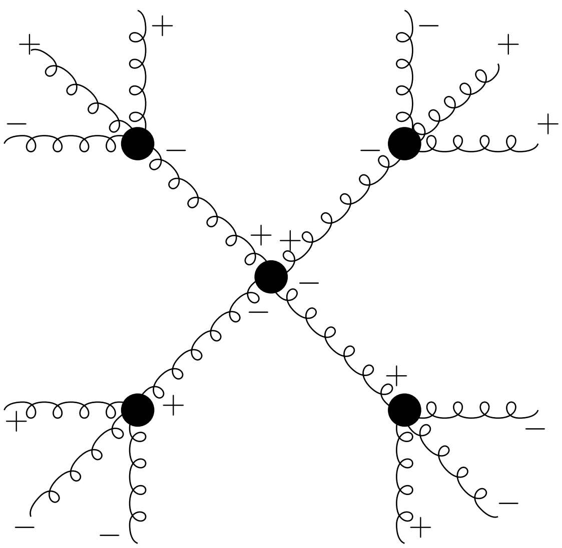
4 Cachazo, Svrcek, Witten (CSW) Algorithm

A somewhat simpler algorithm was devised by Cachazo, Svrcek, Witten. Here we consider "sewing together" MHV amplitudes which are treated as vertices, in order to draw effective Feynman graphs for any amplitude that we wish to calculate.

Note that we really mean MHV vertices and *not* $\overline{\text{MHV}}$ vertices. Thus we include the triple-gluon vertex $(-,-,+)$ but *not* the $(+,+,-)$ vertex.

A typical diagram may look like



where we have indicated that internal gluon propagators have *opposite* helicities at the ends. Propagators have the usual tree-level form

$$P^{+-}(Q) \;=\; -\frac{i}{Q^2}$$

Each "vertex" had exactly two gluons with negative helicity.

The difficulty is that we need to extend one or more legs of each vertex so that the internal lines may be off-shell.

We do this by defining the light-like vector q^μ which has the same components as the off-shell momentum Q^μ except for the component in a particular light-like direction n^μ

$$q^\mu \;=\; Q^\mu - \frac{Q^2}{2q.n} n^\mu$$

The spinor $|q\rangle$ can be written as

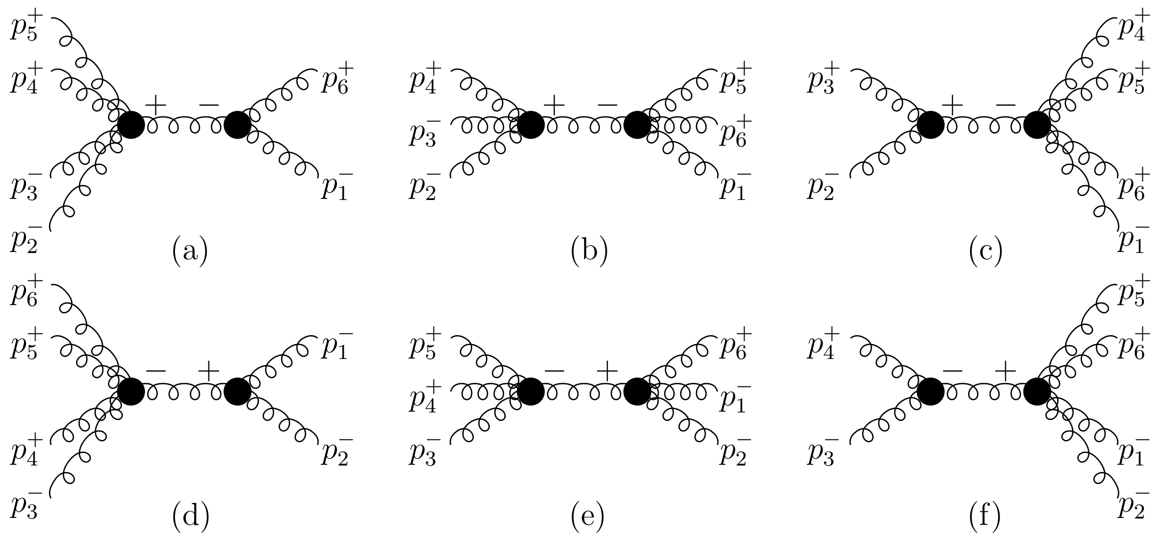
$$|q\rangle \;=\; \frac{\gamma \cdot Q}{[q|n]}|n],$$

as can be seen by left multiplying both sides by $\gamma \cdot Q$, and using $\gamma \cdot q|q\rangle = 0$.

Thus all we need to do is to draw all possible graphs involving MHV vertices and gluon propagators between them and use as the vertices the MHV amplitudes with the spinor for each off-shell gluon of momentum Q replaced by $|q\rangle$.

As an example we re-visit the six-point NMHV amplitude $\tilde{\mathcal{A}}(p_1^-, p_2^-, p_3^-, p_4^+, p_5^+, p_6^+)$

There are six possible graphs which are:



We will concentrate on graph (a), for which the internal propagator has momentum $Q_{6,1}$ and we replace the right hand spinor by $\alpha\gamma \cdot (p_1 + p_6)|n]$, where $\alpha^{-1} = \langle q_{6,1}|n\rangle$, but this factor will cancel out, so we leave it as α.

The MHV vertex on the left is

$$-ig^3 \frac{\langle p_2|p_3\rangle^3}{\langle p_3|p_4\rangle\langle p_4|p_5\rangle\langle p_5|q_{6,1}\rangle\langle q_{6,1}|p_2\rangle}$$

The MHV vertex on the right is

$$-ig \frac{\langle 1|q_{6,1}\rangle^3}{\langle p_6|p_1\rangle\langle q_{6,1}|p_6\rangle}$$

and the propagator is

$$\frac{-i}{\langle p_1|p_6\rangle[p_6|p_1]}$$

We can simplify this by making a choice of the vector n. If we choose it to be equal to p_3

(we run into zero over zero problems if we select p_1, p_2 or p_6) then

$$|q_{6,1}\rangle \;=\; \alpha(|p_1\rangle[p_1|p_3] + |p_6\rangle[p_6|p_3])$$

and the contribution from graph (a) to this amplitude simplifies to

$$-ig^4 \frac{\langle p_2|p_3\rangle^3[p_6|p_3]^3}{\langle p_3|p_4\rangle\langle p_4|p_5\rangle[p_6|p_1][p_3|p_1]\left(\langle p_5|p_1\rangle[p_1|p_3] + \langle p_5|p_6\rangle[p_6|p_3]\right)\left(\langle p_1|p_2\rangle[p_1|p_3] + \langle p_6|p_2\rangle[p_6|p_3]\right)}$$

CSW demonstrated the equivalence of this method with the BCF reduction in several cases. They also demonstrated that all the soft- and collinear singularities of amplitudes were correctly reproduced, and that the result was always independent of the choice of vector $\overline{n}$. In their own words this was at best a "heuristic proof" and the demonstration that the method was valid was finally established by an effective field theory derived by Mansfield.

5 Effective Field Theory

We begin by discussing QCD in the light-cone axial gauge and using light-cone quantisation. This means that "time" is along light-like direction n^+, and we choose the gauge $A_- = 0$, leaving components A_+, $\mathbf{A}$ and $\overline{\mathbf{A}}$, where the latter two are the transverse components treated as complex (holomorphic and anti-holomorphic) degrees of freedom (colour indices have been suppressed).

It is convenient to define A_L (longitudinal) which is the superposition of $A_+, \mathbf{A}$ and $\overline{\mathbf{A}}$,.

$$A_L \;\equiv\; A_+ - \frac{1}{\partial_-}\left(\overline{\boldsymbol{\partial}}\mathbf{A} + (\boldsymbol{\partial}\overline{\mathbf{A}}\right)$$

In terms of the variables $A_L, \mathbf{A}$ and $\overline{\mathbf{A}}$, we obtain a Lagrangian density in which A_L has *no* "time" dependence - i.e. no terms in $\partial_+ A_L$. This means that A_L has *no* canonical momentum and so it is not a quantised degree of freedom. As such it can be eliminated form the Lagrangian using its (linear) equation of motion

$$\partial_-^2 A_L - g\,(\partial_- \mathbf{A})\,\overline{\mathbf{A}} - g\left(\partial_- \overline{\mathbf{A}}\right)\mathbf{A} \;= 0$$

After this elimination we obtain an effective QCD Lagrangian (with non-local terms) which we may write as

$$\mathcal{L} \;=\; \mathcal{L}^{+-} + \mathcal{L}^{++-} + \mathcal{L}^{--+} + \mathcal{L}^{--++}$$

with

$$\begin{aligned}
\mathcal{L}^{+-} \;&=\; \mathbf{A}(\partial_+\partial_- - \boldsymbol{\partial}\overline{\boldsymbol{\partial}})\overline{\mathbf{A}} \\
\mathcal{L}^{++-} \;&=\; -g\left(\left(\overline{\boldsymbol{\partial}}\frac{1}{\partial_-}\mathbf{A}\right)\mathbf{A}\partial_-\overline{\mathbf{A}}\right) \\
\mathcal{L}^{--+} \;&=\; -g\left(\left(\boldsymbol{\partial}\frac{1}{\partial_-}\overline{\mathbf{A}}\right)\overline{\mathbf{A}}\partial_-\mathbf{A}\right) \\
\mathcal{L}^{--++} \;&=\; -g^2\left(\overline{\mathbf{A}}\partial_-\mathbf{A}\right)\frac{1}{\partial_-^2}\left(\mathbf{A}\partial_-\overline{\mathbf{A}}\right)
\end{aligned}$$

The momentum conjugate to $\mathbf{A}$ is $\partial_-\overline{\mathbf{A}}$ (and vice-versa) so the equal "time" commutation relations are

$$\left[\partial_-\overline{\mathbf{A}}(x^+, x^-, \mathbf{x}), \mathbf{A}(x^+, y^-, \mathbf{y})\right] \;=\; i\delta(x^- - y^-)\delta^2(\mathbf{x} - \mathbf{y})$$

This effective Lagrangian (which is none other than QCD in light-cone gauge and light-cone quantisation) is not ready to be applied to an algorithm using only MHV vertices because of the term.

We can, however, perform a canonical transformation of the transverse field variables. This transformation can be non-local and non-polynomial as long as the canonical quantisation relations are maintained. Thus we define a holomorphic field $\mathbf{B}$

$$\mathbf{B}(x^+, x^0, \mathbf{x}) \;=\; \sum_n \int dy_1^- d^2\mathbf{y}_1 \cdots dy_n^- d^2\mathbf{y}_n \Gamma_n(y_1^-, \mathbf{y}_1 \cdots y_n^-, \mathbf{y}_n)\mathbf{A}(x^+, y_1^-, \mathbf{y}_1) \cdots \mathbf{A}(x^+, y_n^-, \mathbf{y}_n)$$

The antiholomorphic field $\overline{\mathbf{B}}$ is then determined by the requirement that the transformation should be canonical, which leads to

$$\partial_- \overline{\mathbf{A}}(x^+, y^- \mathbf{y}) \;=\; \int dx^- d^2\mathbf{x} \frac{\delta \mathbf{B}(x^+, x^-, \mathbf{x})}{\delta \mathbf{A}(x^+, y^-, \mathbf{y})} \partial_- \overline{\mathbf{B}}(x^+, x^-, \mathbf{x})$$

The coefficients Γ_n are chosen so that the above term $\mathcal{L}^{++-}$ is absorbed into the quadratic term in terms of $\mathbf{B}$ and $\overline{\mathbf{B}}$, i.e.

$$\mathcal{L}^{+-} + \mathcal{L}^{++-} \;=\; \mathbf{B}(\partial_+\partial_- - \boldsymbol{\partial}\overline{\boldsymbol{\partial}})\overline{\mathbf{B}}$$

We now *never* have any terms with more than two powers of $\overline{\mathbf{B}}$, but we get an infinite series of interaction terms of the form

$$g^{n-2} \int dx_1^- d^2\mathbf{x}_1 \cdots dx_n^- d^2\mathbf{x}_n V_n(y_1^-, \mathbf{y}_1, \cdots y_n^-, \mathbf{y}_n) \overline{\mathbf{B}}(x^+, y_1^- \mathbf{y}_1) \overline{\mathbf{B}}(x^+, y_2^-, \mathbf{y}_2) \mathbf{B}(x^+, y_3^-, \mathbf{y}_3) \\ \times \cdots \mathbf{B}(x^+, y_n^-, \mathbf{y}_n),$$

which contain two factors of $\overline{\mathbf{B}}$ corresponding to negative helicity gluons and many $\mathbf{B}$ factors corresponding to positive helicity gluons. Ettle and Morris have demonstrated explicitly that these "vertices", V generate the MHV amplitudes.

The transverse components of the gluon polarisation vectors in this gauge are given by

$$\boldsymbol{\epsilon}_- \;=\; \overline{\boldsymbol{\epsilon}}_+ \;=\; \frac{[n^-|\boldsymbol{\gamma}|p\rangle}{\sqrt{2}[n^-|p]},$$

where $\boldsymbol{\gamma}$, $\overline{\boldsymbol{\gamma}}$ are the transverse components of γ^μ. It turns out that this is just a phase, independent of the momentum, p, of the gluon. This means that for an off-shell gluon with momentum Q we may write this as

$$\boldsymbol{\epsilon}_- \;=\; \overline{\boldsymbol{\epsilon}}_+ \;=\; \frac{[n^-|\boldsymbol{\gamma}|q\rangle}{\sqrt{2}[n^-|q]},$$

where $|q\rangle$ can be *any* spinor. The CSW prescription is a special case.

In order to quantise the theory we need to expand the fields $\mathbf{B}$ and $\overline{\mathbf{B}}$ is terms of creation and annihilation operators for and right- and left helicity gluons. The field $\mathbf{B}$ destroys right-handed positive "energy" gluons and creates left-handed negative "energy" gluons, so we Fourier expand this as

$$\mathbf{B}(x^+, x^-, \mathbf{x}) = \frac{1}{(2\pi)^3} \int dp_+ d^2\mathbf{p} \left(a_+(p_+, \mathbf{p}, x^+) e^{-i(p_+ x^- - \mathbf{x}\mathbf{p}^* - \mathbf{x}^* \mathbf{p})} + a_-^\dagger(p_+, \mathbf{p}, x^+) e^{i(p_+ x^- - \mathbf{x}\mathbf{p}^* - \mathbf{x}^* \mathbf{p})} \right)$$

and the expansion for $\overline{\mathbf{B}}$ is obtained by interchanging the creation/annihilation operators for positive and negative helicity.

The canonical quantisation relations are equivalent to the equal "time" commutation relation of the creation and annihilation operators

$$\left[a_\pm(p_+, \mathbf{p}, x^+), a_\pm^\dagger(q_+, \mathbf{q}, x^+) \right] = i\delta(p_+ - q_+)\delta^2(\mathbf{p} - \mathbf{q}).$$

For a free-field obeying the free-field equations of motion, the x^+ dependence of the creation and annihilation operators is given by

$$a_\pm(p_+, \mathbf{p}, x^+) \sim e^{-ip_- x^+}$$

$$a_\pm^\dagger(p_+, \mathbf{p}, x^+) \sim e^{ip_- x^+},$$

where

$$p_- = \frac{\mathbf{p}\,\overline{\mathbf{p}}}{p_+}.$$

To linear order in $\mathbf{B}$ the canconical transformation is such that $\mathbf{A} = \mathbf{B}$, so that for on-shell particles the field $\mathbf{B}$ is interpreted as creating or annihilating transverse gluons in the same way that $\mathbf{A}$, but the interacting fields differ.

For an interacting field, e.g a field representing an off-shell internal gluon, the x^+ dependence of the creation and annihilation operators are different, but the canonical commutation relations are the same. The quantisation therefore remains valid if we associate with the off-shell vector Q a light-like vector q, whose components are identical with the exception of the q^- component (i.e. the component conjugate to the direction selected to be the "time direction".