MODERN METHODS IN PETURBATIVE QCD

1 Tree-Level Helicity Amplitudes

1.1 Clearing the air !!

We are discussing high-energy QCD and for the moment without heavy flavours. We can therefore set all masses to zero and consider a gauge theory with massless gluons interacting with massless fermions (massless scalars can also be added to the theory with ease).

This means that all particles can come in two possible helicities. In the helicity amplitude approach, we calculate amplitudes for a given assignment of particle helicities. This is in contrast to the conventional Feynman rule approach in which helicities are summed over by taking traces - and indeed only after these traces have been performed do we get a result for the square matrix elements, in terms of kinematic variables (scalar products of momenta). If individual helicities are required then a helicity projection operator is inserted into the trace.

In this approach individual helicity amplitudes are calculated directly, and the results are obtained in terms of scalar products of momenta and phases, which can also be determined from the kinematics.

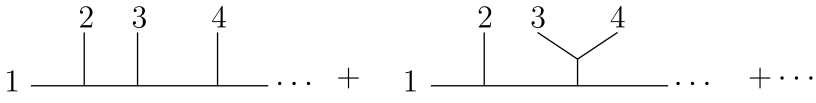
The convention used is that *all* particles are considered to be incoming, and are assigned a momentum and a helicity. If a particle is in reality outgoing, the helicity must be flipped as well as the momentum.

FORGET ABOUT COLOUR FOR THE MOMENT!!!

Later we will discuss a relatively straightforward algorithm for accounting for colour factors in helicity amplitudes. For the moment we just strip these off.

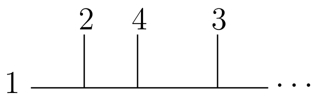
A consequence of doing this is that for any helicity amplitude, we can order the momenta/helicities of the external particles and we consider *planar* graphs only, i.e. we drop graphs in which any two lines are crossed over.

Thus we look at graphs of the form



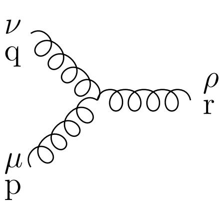
(a) (b)

but not

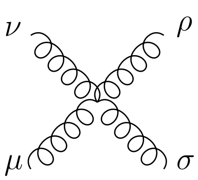


Now it may appear that the ordering of graph(b) is ambiguous since it would be equally valid to draw it with particles 3 and 4 reversed. It turns out that these two configurations carry different colour factors and this matter is dealt with when the colour factors are restored. The upshot is that these planar graphs are associated with a particular colour factor and for this reason they are known as "colour ordered" graphs.

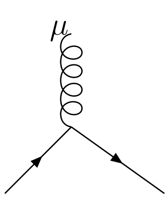
1.2 Colour Ordered Feynman Rules



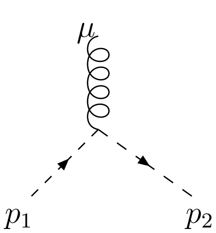
$\frac{1}{\sqrt{2}} ig\left(g_{\mu\nu}(p-q)_\rho + g_{\nu\rho}(q-r)_\mu + g_{\rho\mu}(r-p)_\nu\right)$



$ig^2 g_{\mu\rho} g_{\nu\sigma} - \frac{1}{2} ig^2 \left(g_{\mu\nu} g_{\rho\sigma} + g_{\mu\sigma} g_{\nu\rho}\right)$



-i$\frac{1}{\sqrt{2}} g\gamma^\mu$



i$\frac{1}{\sqrt{2}} g(p_1 + p_2)^\mu$

1.3 Mapping light-like vectors to chiral spinors

Any light-like vector, p (momentum of a massless particle) can be mapped into a positive helicity spinor

$$|p\rangle \equiv \frac{(1+\gamma^5)}{2} u(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^+(p) \\ \chi^+(p) \end{pmatrix}$$

where

$$\chi^+(p) = \sqrt{2p} \left(\begin{array}{c} \cos\left(\frac{\theta}{2}\right) e^{-i\phi} \\ \sin\left(\frac{\theta}{2}\right) \end{array} \right)$$

(Here we are using the Dirac rep. for the γ-matrices, rather than dotted or undotted notation),
OR into a negative helicity spinor

$$|p] \equiv \frac{(1-\gamma^5)}{2} u(p) = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^-(p) \\ -\chi^-(p) \end{pmatrix}$$

where

$$\chi^-(p) = \sqrt{2p} \left(\begin{array}{c} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{array} \right), \quad \chi^-(p)_\alpha = \epsilon_{\alpha\beta} \chi^{+*}(p)_\beta$$

The conjugate spinors are defined as

$$\langle p| \equiv \bar{u}(p) \frac{(1+\gamma^5)}{2} = u^\dagger(p) \frac{(1-\gamma^5)}{2} \gamma^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^-(p) \\ \chi^-(p) \end{pmatrix}$$

and

$$[p| \equiv \bar{u}(p) \frac{(1-\gamma^5)}{2} = u^\dagger(p) \frac{(1+\gamma^5)}{2} \gamma^0 = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi^+(p) \\ -\chi^+(p) \end{pmatrix}$$

Thus we have the amplitudes:

$$\langle p|q\rangle = -\langle q|p\rangle = \epsilon_{\alpha\beta} \chi^+(p)_\alpha \chi^+(q)_\beta = \sqrt{2p \cdot q}\, e^{i\eta(p,q)}$$

$$[p|q] = -[q|p] = \epsilon_{\alpha\beta} \chi^-(p)_\alpha \chi^-(q)_\beta = -\sqrt{2p \cdot q}\, e^{-i\eta(p,q)}$$

(signs are a nightmare !!)
where the phase $\eta(p,q)$ is given by

$$e^{2i\eta} = \frac{(p_- q_T - q_- p_T)}{(p_- q_T^* - q_- p_T^*)}$$

$p_- = p_0 - p_z,\ p_T = p_x + ip_y,\ q_- = q_0 - q_z,\ q_T = q_x + iq_y$. We also have

$$[p|q\rangle = \langle p|q] = 0$$

Furthermore
$$\langle p|q\rangle[p|q] = -2p\cdot q$$

Note that these two maps are *non-linear*. The spinor $|p\rangle + |q\rangle$ is another massless spinor, but it can't be used to represent the vector $p+q$, which is not light-like (massless).

On the other hand we have the linear map
$$\begin{aligned} |p\rangle[p| &= \frac{(1+\gamma^5)}{2}\gamma\cdot p \\ |p]\langle p| &= \frac{(1-\gamma^5)}{2}\gamma\cdot p \end{aligned}$$
and so we *can* write
$$\frac{(1+\gamma^5)}{2}\gamma\cdot(p+q) = |p\rangle[p| + |q\rangle[q|$$
or
$$\frac{(1-\gamma^5)}{2}\gamma\cdot(p+q) = |p]\langle p| + |q]\langle q|$$

1.4 Useful Identities

1. $$\langle p|\gamma^\mu|q] = [q|\gamma^\mu|p\rangle = \epsilon_{\alpha\beta}\sigma^\mu_{\beta\gamma}\chi^+(p)_\alpha\chi^-(q)_\gamma \quad (\sigma^\mu = (1,\sigma^i))$$
2. $$\langle p|\gamma^\mu|p] = 2p^\mu$$
   This follows by contracting with any light-like vector, q, to obtain
   $$\langle p|\gamma\cdot q|p\rangle = [p|q]\langle q|p] = 2p\cdot q$$
3. Fierz identity
   $$\langle p|\gamma^\mu|q]\langle k|\gamma_\mu|l] = 2\langle p|k\rangle[q|l]$$
   This follows from the relation
   $$g_{\mu\nu}\sigma^\mu_{\alpha\beta}\sigma^\nu_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}$$
4. Schouten identity
   $$\langle p|q\rangle\langle k|l\rangle = \langle p|k\rangle\langle q|l\rangle + \langle p|l\rangle\langle k|q\rangle,$$
   which follows from the relation
   $$\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} = \epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\gamma\beta}$$
   (Jacobi identity)

1.5 Gluon Polarization

The polarisation vector for a gluon with a given helicity, $\epsilon_\mu^\lambda(k)$ and momentum k is not unique as a result of gauge invariance. External gluons will be described in light-cone axial gauge, and will depend on an external light-like vector n^μ, such that

$$n \cdot \epsilon \;=\; 0$$

Since p and n are light-like we can express ϵ in terms of massless spinors with momenta k and n as

$$\epsilon_\mu^+(k,n) \;=\; \frac{\langle n|\gamma_\mu|k]}{\sqrt{2}\langle n|k\rangle}$$

and

$$\epsilon_\mu^-(k,n) \;=\; \frac{[n|\gamma_\mu|k\rangle}{\sqrt{2}[k|n]}$$

When an external gluon is attached to a fermion line we get (a sum of) terms of the form

$$\langle p|\epsilon(k,n)\cdot\gamma|q]$$

Using the Fierz identity we get

$$\sqrt{2}\frac{[k|q]\langle n|p\rangle}{\langle n|k\rangle]}$$

for a positive helicity spinor, and

$$\sqrt{2}\frac{[n|q]\langle k|p\rangle}{[k|n]}$$

for a negative helicity spinor. So if we choose $n = p$ for a positive helicity and $n = q$ for negative helicity then this particular term vanishes and the result can be simplified. (Note that $\langle p|p\rangle = [q|q] = 0$)

A judicious choice of the auxiliary vectors n can simplify the calculations of helicity amplitudes enormously.

NOTE: It is *NOT* a requirement of gauge invariance that the auxiliary vector should be the same for all external gluons. We can choose the most convenient gluon for each gluon. However, when more than one Feynman graph contributes to a given helicity amplitude, it is *essential* to use the same assignment of auxiliary vectors to all graphs.

Furthermore, it is permitted to use a different gauge for the internal gluons from the external ones. So we will use the light-like axial gauge(s) for the external gluons and Feynman gauge for all internal gluons. Again this must be done consistently for all graphs contributing to a given amplitude.

If we *do* choose to assign the same auxiliary vector n to two gluons the Fierz identity leads to the following useful relations.

$$\epsilon^+(k_1,n)\cdot\epsilon^+(k_2,n) \;=\; \epsilon^-(k_1,n)\cdot\epsilon^-(k_2,n) \;=0,$$

which again can be use to simplify calculations.

Moreover of we have the scalar product of two gluons with opposite helicity

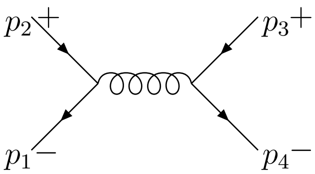
$$\epsilon^{+}(k_1, n_1) \cdot \epsilon^{-}(k_2, n_2)$$

then this vanishes if either $n_1 = k_2$ or $n_2 = k_1$.

1.6 Simple Examples

We are now in a position to calculate *any* helicity matrix element in terms of the overlap amplitudes of the form $\langle p|q\rangle$ or $[p|q]$.

Example 1:
Quark-antiquark to quark-antiquark



Applying the colour-ordered Feynman rules the amplitude is simply

$$\mathcal{A}(p_1^-, p_2^+, p_3^+, p_4^-) \;=\; i\frac{g^2}{2s}[p_1|\gamma^\mu|p_2\rangle[p_4|\gamma_\mu|p_3\rangle$$

Using the Fierz identity this is

$$\mathcal{A}(p_1^-, p_2^+, p_3^+, p_4^-) \;=\; i\frac{g^2}{s}[p_1|p_4]\langle p_2|p_3\rangle$$

Note that $[p_1|p_4]$ and $\langle p_2|p_3\rangle$ are both equal to $\sqrt{t}$ up to a phase (recall that all momenta are incoming), then the amplitude is

$$\mathcal{A}(p_1^-, p_2^+, p_3^+, p_4^-) \;= \eta\frac{g^2 t}{s}$$

where η is a phase, which doesn't matter on this case since there is only one graph.

If we calculate the amplitude for the helicities of the final state fermions reversed we have

$$\mathcal{A}(p_1^-, p_2^+, p_3^-, p_4^+) \;=\; i\frac{g^2}{2s}[p_1|\gamma^\mu|p_2\rangle[p_3|\gamma_\mu|p_4\rangle,$$

giving

$$\mathcal{A}(p_1^-, p_2^+, p_3^-, p_4^+) \;= \eta'\frac{g^2 u}{s}$$

Now if we square these amplitudes and sum, we get

$$g^4 \frac{(t^2+u^2)}{s^2}$$

which (up to a colour factor) is precisely what you get from taking traces of the square Feynman graph.

Example 2:

Quark antiquark to two gluons with opposite helicity

We do *not* include the graph in which the outgoing gluons are attached in the other order as this is not a planar graph (it carries a different colour factor).

We make the following choice of auxiliary vectors for gluons (3 & 4)

$$|n_4] = |p_1]$$

$$|n_3\rangle = |p_2\rangle$$

Recall that we *may* write

$$\gamma \cdot (p_1 + p_4) = |p_1\rangle[p_1| + |p_4\rangle[p_4|$$

we see that graph (a) always gives us a term of the form

$$[p_1|\gamma^\mu|q\rangle\epsilon^-_\mu(4) = \frac{[p_1|\gamma^\mu|q\rangle[n_4|\gamma_\mu|p_4\rangle}{\sqrt{2}[p_4|n_4]} = \sqrt{2}\frac{[p_1|n_4]\langle q|p_4\rangle}{[p_4|n_4]},$$

so the choice $|n_4] = |p_1]$ gives zero and this graph vanishes.

Two of the three terms from graph (b) also vanish. One of them again gives a term proportional to $[p_1|\gamma^\mu|q\rangle\epsilon^-_\mu(4)$ and another gives a term proportional to $[q|\gamma^\mu|p_2\rangle\epsilon^+_\mu(3)$ and we can use the same argument to show that this is proportional to $\langle n_3|p_2\rangle$, which vanishes if we choose $|n_3\rangle = |p_2\rangle$.

We are left with a term

$$\mathcal{A}(p_1^-, p_2^+, p_3^+, p_4^-) = \frac{ig^2}{2s}\epsilon^+(3)\cdot\epsilon^-(4)[p_1|\gamma\cdot(p_3-p_4)|p_2\rangle$$

Using momentum conservation and the Dirac equations

$$[p_1|\gamma\cdot p_1 \;=\; \gamma\cdot p_2|p_2\rangle = 0$$

we have

$$[p_1|\gamma\cdot(p_3-p_4)|p_2\rangle \;=\; -2[p_1|p_4]\langle p_4|p_2\rangle$$

and using the Fierz identity

$$\epsilon^+(3)\cdot\epsilon^-(4) \;=\; \frac{[p_1|p_3]\langle p_2|p_4\rangle}{[p_4|p_1]\langle p_2|p_3\rangle}$$

So the amplitude is

$$-ig^2\frac{\langle p_2|p_4\rangle^2[p_1|p_3]}{\langle p_3|p_4\rangle[p_3|p_4]\langle p_2|p_3\rangle}$$

where we have written

$$s \;=\; -\langle p_3|p_4\rangle[p_3|p_4]$$

We can rewrite this as

$$\mathcal{A}(p_1^-,p_2^+,p_3^+,p_4^-) \;=\; -ig^2\frac{\langle p_2|p_4\rangle^3}{\langle p_1|p_2\rangle\langle p_2|p_3\rangle\langle p_3|p_4\rangle}\times\left\{\frac{[p_1|p_3]\langle p_1|p_2\rangle}{[p_3|p_4]\langle p_2|p_4\rangle}\right\}$$

The expression inside $\{\cdots\}$ is actually -1. It can be written as

$$\frac{[p_3|\gamma\cdot p_1|p_2\rangle}{[p_3|\gamma\cdot p_4|p_2\rangle}$$

Again using momentum conservation and the Dirac equation we have

$$[p_3|\gamma\cdot p_1|p_2\rangle \;=\; -[p_3|\gamma\cdot p_4|p_2\rangle$$

So the amplitude is simply

$$ig^2\frac{\langle p_2|p_4\rangle^3}{\langle p_1|p_2\rangle\langle p_2|p_3\rangle\langle p_3|p_4\rangle}$$

Exercise:
Calculate the amplitude for quark-antiquark scattering into two gluons of the same helicity. [Hint: Choose the auxiliary vectors for the external gluons both to be equal to one of the fermion momenta.]

1.7 Spinor Transformations

Before considering some more examples, we look at a rather peculiar transformation of spinors, which we will need in future manipulation, and to which we will return in due course.

$$\begin{aligned} |p\rangle &\rightarrow |\hat{p}(z)\rangle = |p\rangle + z|n > \\ |p] &\rightarrow |\hat{p}(z)] = |p] \end{aligned}$$

where n^μ is a light-like vector.

Here we have performed a transformation on the right-helicity spinor but *not* on the left-helicity spinor (we could also have done this the other way around). This means that we lose the relation between the two spinors because we now have

$$\chi^+_\alpha(\hat{p}(z)) \neq \epsilon_{\alpha\beta}\chi^{-*}_\beta(\hat{p}(z)),$$

which is possible *provded we allow the components of the momentum $\hat{p}(z)$ to be complex.*

For a complex momentum the normalized spinors are

$$\chi^+(\hat{p}(z)) = \frac{\sqrt{2\hat{p}(z)}}{\sqrt{|\sin\left(\frac{\theta}{2}\right)|^2 + |\cos\left(\frac{\theta}{2}\right)|^2 \exp(2\Im m(\phi))}} \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) e^{-i\phi} \\ \sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

and

$$\chi^-(\hat{p}(z)) = \frac{\sqrt{2\hat{p}(z)}}{\sqrt{|\sin\left(\frac{\theta}{2}\right)|^2 + |\cos\left(\frac{\theta}{2}\right)|^2 \exp(-2\Im m(\phi))}} \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) e^{i\phi} \end{pmatrix}$$

Clearly if θ and ϕ are real the denominator is equal to one in each case, but if not, then we can find complex values which will equate these spinors to the defined spinors $|\hat{p}(z)\rangle$ and $\hat{p}(z)]$ above.

Example: Start with a spinor with momentum magnitude p making an angle of 60^0 with the z-axis (in the $z-x$ plane)

$$\chi^+(p) = \sqrt{2p}\begin{pmatrix} \frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix}$$

$$\chi^-(p) = \sqrt{2p}\begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

Now perform the transformation with

$$z = \sqrt{2p}\frac{(2-\sqrt{3})}{\sqrt{2}}$$

and

$$\chi^+(n) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

so that we have

$$\chi^+(\hat{p}) = \sqrt{2|p|}\begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

$$\chi^{-}(\hat{p}) = \sqrt{2|p|} \begin{pmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

These are the right- and left- spinors for momentum $\hat{p}$ with components

$$\hat{p} = |p| \left(1, \frac{2}{\sqrt{3}}, \frac{i}{\sqrt{3}}, 0\right)$$

Note that although the components are complex, the vector is still light-like,

$$\hat{p}(z) \cdot \hat{p}(z) = 0$$

This is expected because we are still mapping the complex vector $\hat{p}(z)$ onto chiral spinors. We have

$$\frac{(1+\gamma^5)}{2} \gamma \cdot p = |\hat{p}(z)\rangle [\hat{p}(z)| = |p\rangle [p| + z|n\rangle [p|$$