2 Maximum Helicity Violating Amplitudes

The solution to the exercise in the previous section is that for quark-antiquark to two gluons with the same helicity

$$\mathcal{A}(p_1^-, p_2^+, p_3^+, p_4^+) \;=\; 0 \;!!$$

The reason for this is that there are fewer than two particles with negative helicity and in such cases the amplitude (for massless particles) is *always* zero.

For the time being, we will restrict our discussion to the pure gluon case and consider the colour-ordered scattering amplitude for n gluons

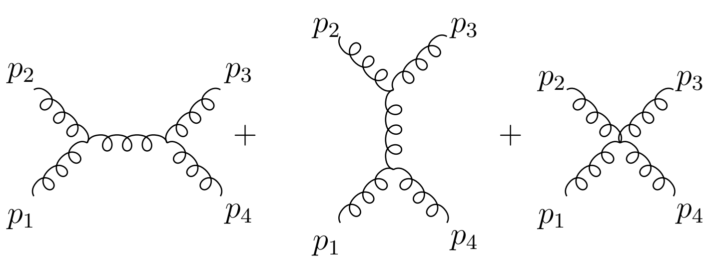
$$\tilde{\mathcal{A}}(1, 2, 3, \cdots n),$$

where the ˜ indicates a colour ordered amplitude and the arguments $(1, 2\cdots)$ represent both the incoming momentum of the particles and the incoming helicities.

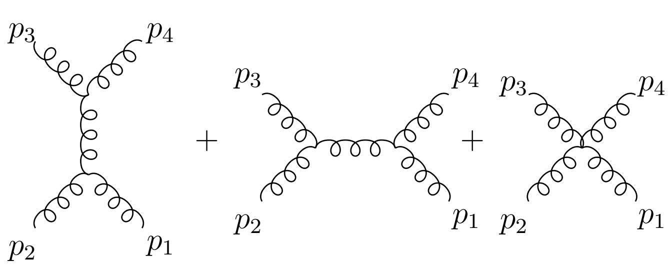
These amplitudes possess a cyclic symmetry

$$\tilde{\mathcal{A}}(1, 2, 3, \cdots n) \;= \tilde{\mathcal{A}}(2, 3, \cdots n, 1) \;\;=\;\; \cdots,$$

which becomes clear of draw all the graphs contributing to a particular amplitude. For example, in the four-gluon case



is the same as



There is also a reflection symmetry

$$\tilde{\mathcal{A}}(n \cdots 3, 1, 2) \;=\; (-1)^n \tilde{\mathcal{A}}(1, 2, 3, \cdots n).$$

Thus, if we assume that we have a graph with only triple-gluon vertices and as we run the particles backwards, each vertex has the order of the momenta reversed and therefore from the colour ordered Feynman rule, we get a minus for each vertex. A four-gluon vertex replaces two three-gluon vertices, so the sign change is not affected.

Now suppose we want to calculate the amplitude for the scattering of n incoming gluons, all of which have positive incoming helicity.

$$\tilde{\mathcal{A}}(1^+, 2^+, \cdots n^+)$$

Each of the contributing diagram has at most (n-2) vertices and n polarisation vectors. This means that each term will contain at least one scalar produce of the form $\epsilon^+(p_i, n_i) \cdot \epsilon^+(p_j, n_j)$. If we choose all of the gluons to have the same auxiliary vector ($n_i = n_j$) then , as we have seen, all these scalar products vanish and so the amplitude vanishes.

Now suppose that one of the helicities is negative. Because of th cyclic symmetry of the amplitude we can choose this to be particle n,

$$\tilde{\mathcal{A}}(1^+, 2^+, \cdots n^-)$$

Now we can choose $n_i = p_n$ for $(i \neq n)$, and this us

$$\epsilon^+(p_i, p_n) \cdot \epsilon^+(p_j, p_n) \;=\; 0 \quad (i, j, \neq n)$$

and

$$\epsilon^+(p_i, p_n) \cdot \epsilon^-(p_n, n_n) \;=\; 0,$$

so that again all possible scalar products vanish and the amplitude again vanishes.

Thus the minimum number of incoming negative helicities is two (with the other helicities positive). For the scattering of incoming and outgoing gluons this is interpreted as the fact that at least two of the helicities of the outgoing gluons must be the same as two of the helicities of the incoming gluons, and hence this is called the "maximal helicity violating amplitude MHV".

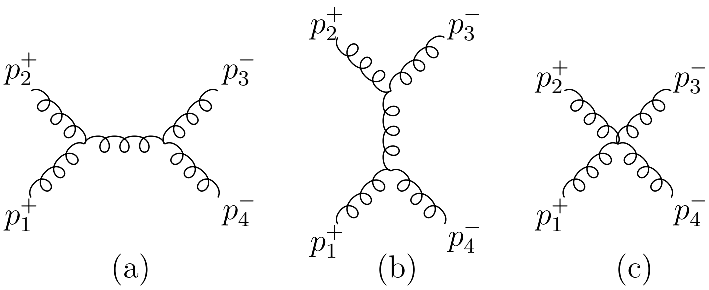
Note that under the parity operation

$$\langle p\ q \rangle \leftrightarrow [p|q],$$

so that we can also determine the amplitude for the case where all but two of the incoming helicities are negative by interchanging all of these scalar products. Since $\langle p\ q \rangle = -[p|q]^*$, this means that the parity flipped amplitude is the complex conjugate of the un-flipped amplitude - up to an overall sign - (a consequence of the CP invariance of QCD)

2.1 Four Gluon Amplitude

Let us now consider the four-gluon amplitude with helicities $(+,+,-,-)$



If we choose the auxiliary vectors for the four gluons as follows

$$\begin{aligned} n_1 &= n_2 = p_4 \\ n_3 &= n_4 = p_1, \end{aligned}$$

then the only non-vanishing scalar product of polarisation vectors is $\epsilon_2^+ \cdot \epsilon_3^-$.

Graph (c) vanishes because that involves two such scalar products and one of them must be zero. Furthermore, although graph (b) has a term proportional to $\epsilon_2^+ \cdot \epsilon_3^-$, the lower vertex will always give $\epsilon_1^+ \cdot p_4$ or $\epsilon_4^- \cdot p_1$ which both vanish.

Therefore the only contributing graph (for this choice of polarisation vectors) is graph (a) and the only non-vanishing term is

$$\begin{aligned} \tilde{\mathcal{A}}(p_1^+, p_2^+, p_3^-, p_4^-) &= \frac{ig^2}{4p_1 \cdot p_2} \epsilon_2^+ \cdot \epsilon_3^- \left(-2p_3 \cdot \epsilon_4^-\right) \left(-2p_2 \cdot \epsilon_1^+\right) \\ \epsilon_1^{+\mu} &= \frac{1}{\sqrt{2}} \frac{\langle p_4|\gamma^\mu|p_1]}{\langle p_1|p_4\rangle} \\ \epsilon_2^{+\mu} &= \frac{1}{\sqrt{2}} \frac{\langle p_4|\gamma^\mu|p_2]}{\langle p_2|p_4\rangle} \\ \epsilon_3^{-\mu} &= \frac{1}{\sqrt{2}} \frac{[p_1|\gamma^\mu|p_3\rangle}{[p_3|p_1]} \\ \epsilon_4^{-\mu} &= \frac{1}{\sqrt{2}} \frac{[p_1|\gamma^\mu|p_4\rangle}{[p_4|p_1]} \end{aligned}$$

So (using the Fierz identity and $\frac{1}{2}(1+\gamma^5)\gamma \cdot p = p\rangle[p)$

$$\epsilon_2^+ \cdot \epsilon_3^- = \frac{[p_1|p_2]\langle p_4|p_3\rangle}{\langle p_2|p_4\rangle[op_3|p_1]}$$

$$
\begin{aligned}
p_2 \cdot \epsilon_1^+ &= \frac{1}{\sqrt{2}} \frac{\langle p_4|p_2\rangle [p_2|p_1]}{\langle p_1|p_4\rangle} \\
p_3 \cdot \epsilon_4^- &= \frac{1}{\sqrt{2}} \frac{[p_1|p_3]\langle p_3|p_4\rangle}{[p_4|p_1]} \\
2p_1 \cdot p_2 &= \langle p_1|p_2\rangle [p_2|p_1]
\end{aligned}
$$

so we have

$$
\begin{aligned}
\tilde{\mathcal{A}}(p_1^+, p_2^+, p_3^-, p_4^-) &= ig^2 \frac{[p_1|p_2]\langle p_4|p_3\rangle\langle p_3|p_4\rangle [p_1|p_3][p_2|p_1]\langle p_4|p_2\rangle}{\langle p_1|p_2\rangle [p_2|p_1]\langle p_2|p_4\rangle [p_3|p_1]\langle p_1|p_4\rangle [p_4|p_1]} \\
&= ig^2 \frac{\langle p_3|p_4\rangle^2 [p_1|p_2]}{\langle p_1|p_2\rangle\langle p_4|p_1\rangle [p_1|p_4]},
\end{aligned}
$$

where we have cancelled some terms. Now multiplying numerator and denominator by $\langle p_2|p_3\rangle$ and by $\langle p_3|p_4\rangle$ we may rewrite this as

$$
\tilde{\mathcal{A}}(p_1^+, p_2^+, p_3^-, p_4^-) = ig^2 \frac{\langle p_3|p_4\rangle^3}{\langle p_1|p_2\rangle\langle p_2|p_3\rangle\langle p_4|p_1\rangle} \times \left\{ \frac{[p_1|p_2]\langle p_2|p_3\rangle}{[p_1|p_4]\langle p_3|p_4\rangle} \right\}
$$

The term in $\{\}$ may be written

$$
-\frac{[p_1|\gamma \cdot p_2|p_3\rangle}{[p_1|\gamma \cdot p_4|p_3\rangle}
$$

But since $p_4 = -(p_1 + p_2 + p_3)$ and $[p_1|\gamma \cdot p_1 = \gamma \cdot p_3|p_3\rangle = 0$ we have

$$
[p_1|\gamma \cdot p_2|p_3\rangle = -[p_1|\gamma \cdot p_4|p_3\rangle
$$

so the term in $\{\}$ is unity and we have finally

$$
\tilde{\mathcal{A}}(p_1^+, p_2^+, p_3^-, p_4^-) = ig^2 \frac{\langle p_3|p_4\rangle^3}{\langle p_1|p_2\rangle\langle p_2|p_3\rangle\langle p_4|p_1\rangle} = ig^2 \frac{\langle p_3|p_4\rangle^4}{\langle p_1|p_2\rangle\langle p_2|p_3\rangle\langle p_3|p_4\rangle\langle p_4|p_1\rangle}
$$

Exercise:
Show that

$$
\tilde{\mathcal{A}}(p_1^+, p_2^-, p_3^+, p_4^-) = ig^2 \frac{\langle p_2|p_4\rangle^4}{\langle p_1|p_2\rangle\langle p_2|p_3\rangle\langle p_3|p_4\rangle\langle p_4|p_1\rangle}
$$

Using the cyclic symmetry and the reflection symmetry properties, this gives us *all* the possible coloured ordered amplitudes for four gluons.

2.2 Parke-Taylor Formula

Parke and Taylor proved the remarkable result that this can be extended to the MHV amplitude for any number of gluons

$$\tilde{\mathcal{A}}(p_1^+, p_2^+, \cdots p_i^- \cdots p_j^- \cdots p_n^+) \;=\; i(-g)^{(n-2)} \frac{\langle p_i|p_j\rangle^4}{\langle p_1|p_2\rangle\langle p_2|p_3\rangle \cdots \langle p_{n-1}|p_n\rangle}$$

We have demonstrated this for the case of four gluons and in the next section, we will establish that it is valid for any number of gluons, by induction.

This means that we have also done five gluons, since we either have three positive helicities and two negative for which we can use the above formula, or three negative helicities and two positive for which we take the complex conjugate. But for six gluons there is a possible configuration with three positive and three negative helicities for which we need to be able to go beyond MHV. We will see how to do this in the next section.

2.3 Three Gluon Amplitude

There is a small complication:

In order to go beyond MHV we will need to construct sub-amplitudes in which all of the particles are "on-shell".

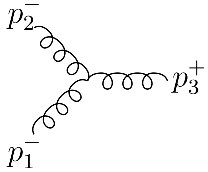
For the three gluon sub-amplitude we might initially think that this vanishes, since we can only have one helicity which can differ from the other two and we have just shown that in such cases the amplitude vanishes.

In fact, we cannot have a three-gluon on-shell amplitude for which p_1^2, p_2^2 and p_3^2 all vanish since by conservation of momentum this would require that *all* scalar products $p_i \cdot p_j$, vanish.

But we *can* achieve this, if we use the trick of extending the spinors so that the three momenta are *complex*. The point here is that

$$-2p_1 \cdot p_2 \;=\; \langle p_1|p_2\rangle[p_1|p_2],$$

and for this to vanish in the case of real momenta *both* $\langle p_1|p_2\rangle$, and $[p_1|p_2]$ must vanish, since these are complex conjugates. However if the momenta are complex then these quantities are not complex conjugates of each other and it is sufficient for one *or* the other to vanish.



Therefore, in order to calculate $\tilde{\mathcal{A}}(p_1^-, p_2^-, p_3^+)$, we make the spinor transformations on the negative helicity states

$$|p_1] \to |\hat{p}_1] \;=\; |p_1] - \frac{[p_2|q][p_3|p_1] + [p_3|q][p_1|p_2] - [p_1|q][p_2|p_3]}{2[p_2|q][p_3|q]},$$

where q is any light-like vector. Transformations of $|p_2]$ and $|p_3]$ are obtained from cyclic permutations. The right-helicity spinors are not transformed so that

$$|\hat{p}_1\rangle \;=\; |p_1\rangle, \quad \text{etc.}$$

In this way

$$[\hat{p}_1|\hat{p}_2] \;=\; [\hat{p}_2|\hat{p}_3] \;=\; [\hat{p}_3|\hat{p}_1] \;=\; 0$$

but

$$\langle\hat{p}_1|\hat{p}_2\rangle \;\neq\; \langle\hat{p}_2|\hat{p}_3\rangle \;\neq\; \langle\hat{p}_3|\hat{p}_1\rangle \;\neq\; 0$$

Now let the auxiliary vectors be such that

$$n_1 \;=\; n_2 \;\equiv\; n, \quad n_3 \;=\; \hat{p}_1,$$

so that $\epsilon_{\hat{1}}^- \cdot \epsilon_{\hat{2}}^- = 0$ and $\epsilon_{\hat{1}}^- \cdot \epsilon_{\hat{3}}^+ = 0$.

The only surviving term in the amplitude is

$$\hat{\mathcal{A}}(\hat{p}_1^-, \hat{p}_2^-, \hat{p}_3^+) \;=\; i\frac{g}{\sqrt{2}}\epsilon_{\hat{2}}^- \cdot \epsilon_{\hat{3}}^+ \, 2\hat{p}_2 \cdot \epsilon_{\hat{1}}$$

$$\begin{aligned}
\epsilon_{\hat{2}}^- \cdot \epsilon_{\hat{3}}^+ &= \frac{[n|\hat{p}_3]\langle\hat{p}_1|\hat{p}_2\rangle}{[\hat{p}_1|n]\langle\hat{p}_1|\hat{p}_3\rangle} \\
\hat{p}_2 \cdot \epsilon_{\hat{1}} &= \frac{[n|\hat{p}_2]\langle\hat{p}_2|\hat{p}_1\rangle}{[\hat{p}_2|n]},
\end{aligned}$$

so that

$$\hat{\mathcal{A}}(\hat{p}_1^-, \hat{p}_2^-, \hat{p}_3^+) \;=\; ig\frac{[n|\hat{p}_3]\langle\hat{p}_1|\hat{p}_2\rangle[n|\hat{p}_2]\langle\hat{p}_2|\hat{p}_1\rangle}{[\hat{p}_1|n][\hat{p}_2|n]\langle\hat{p}_1|\hat{p}_3\rangle}$$

Multiplying numerator and denominator by $\langle\hat{p}_1|\hat{p}_2\rangle^2\langle\hat{p}_2|\hat{p}_3\rangle$, we may write this as

$$\hat{\mathcal{A}}(\hat{p}_1^-, \hat{p}_2^-, \hat{p}_3^+) \;=\; -ig\frac{\langle\hat{p}_1|\hat{p}_2\rangle^4}{\langle\hat{p}_1|\hat{p}_2\rangle\;\langle\hat{p}_2|\hat{p}_3\rangle\langle\hat{p}_3|\hat{p}_1\rangle} \times \left\{\frac{[n|\hat{p}_2][n|\hat{p}_3]\langle\hat{p}_2|\hat{p}_3\rangle}{[\hat{p}_1|n][\hat{p}_2|n]\langle\hat{p}_1|\hat{p}_2\rangle}\right\}$$

The term inside {} may be written as

$$\frac{-[n|\gamma \cdot \hat{p}_3 \gamma \cdot \hat{p}_2|n]}{[n|\gamma \cdot \hat{p}_1 \gamma \cdot \hat{p}_2|n]}.$$

But

$$\gamma \cdot \hat{p}_3 = -\gamma \cdot \hat{p}_2 - \gamma \cdot \hat{p}_1$$

and $\gamma \cdot \hat{p}_2 \gamma \cdot \hat{p}_2 = 0$, so the term inside $\{\}$ is unity and we are left with

$$\hat{\mathcal{A}}(\hat{p}_1^-, \hat{p}_2^-, \hat{p}_3^+) \;=\; -ig \frac{\langle \hat{p}_1 | \hat{p}_2 \rangle^4}{\langle \hat{p}_1 | \hat{p}_2 \rangle \; \langle \hat{p}_2 | \hat{p}_3 \rangle \langle \hat{p}_3 | \hat{p}_1 \rangle},$$

which is the expression we would expect from the Parke-Taylor formula.