# 9 Supersymmetric Theories

## 9.1 Effective Potentials and UV Behaviour

The reduction of an $n$-point loop integral to scalar boxes and triangles and to bubble graphs by repeat applications of the Veltman-Passarino (VP) reduction followed by applications of the Bern-Dunbar-Kosower (BDK) reduction rapidly becomes unwieldy.

It would be much easier if one could use the Cutkosky rules, derived form the analyticity properties and unitarity of the S-matrix, to obtain the imaginary part of any amplitude and then to reconstruct the real part.

Unfortunately, in pure QCD this is not possible because an $n$-point amplitude has a maximum if $n$ powers of the loop momentum in the numerator and $n$ propagator terms, so that repeated application of VP reduction will normally generate a tensor bubble integral, which cannot be uniquely identified from its imaginary part.

To see this, consider the effective potential, $V[A]$ as a functional of the gluon field $A$, which generate the $n$-point amplitude by

$$\mathcal{A}(p_1, \cdots p_n) \;=\; \int \prod_{i=1}^{n} d^4 p_i \, e^{i p_i \cdot x_i} \, \epsilon_i^{\mu_i} \frac{\delta^n}{\delta A(x_1) \cdots \delta A(x_n)} V[A]$$

For pure QCD we may formally write $V[A]$ at one-loop level as

$$V[A] \;=\; -\frac{1}{2} \mathrm{Tr} \ln \left( D^2 g_{\mu\nu} - g \Sigma^{\alpha\beta}_{\mu\nu} F_{\alpha\beta} \right) \;+\; \mathrm{Tr} \ln D^2,$$

where $\Sigma^{\alpha\beta}_{\mu\nu}$ are the generators of the Lorentz group in the defining representation, (the last term comes form the Fadeev-Popov ghosts in a covariant gauge)

We can expand the logarithms and we find that the leading UV behaviour of amplitudes comes from the term

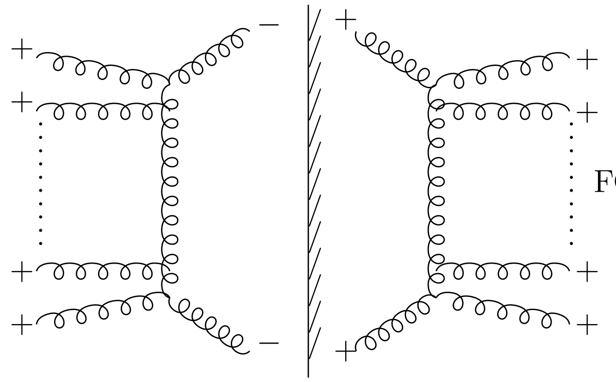
$$-\frac{1}{2} \mathrm{Tr} \ln \left( D^2 g_{\mu\nu} \right),$$

which when we expand in a power series in $A_\mu$ gives terms like

$$\sum_r g^r \left( A^\mu \partial_\mu \frac{1}{\partial^2} \right)^r$$

which leads to the terms with one power of loop momentum ($\partial^\mu$) for each propagator ($1/\partial^2$).

This means that in pure QCD we can have matrix elements at the one-loop level which are forbidden at the tree-level. For example the amplitude $\mathcal{A}(p_1^+ \cdots p_n^+)$ is allowed at one-loop even though the one-loop graph cannot be cut such that there is an allowed tree-level (e.g. an MHV amplitude) amplitude on either side of the cut.




This means that the loop amplitude has *no* imaginary part and therefore no terms involving logarithm or dilogarithm functions, but can be a rational function of the kinematic variables.

## 9.2 $\mathcal{N} = 1$ Supersymmetry

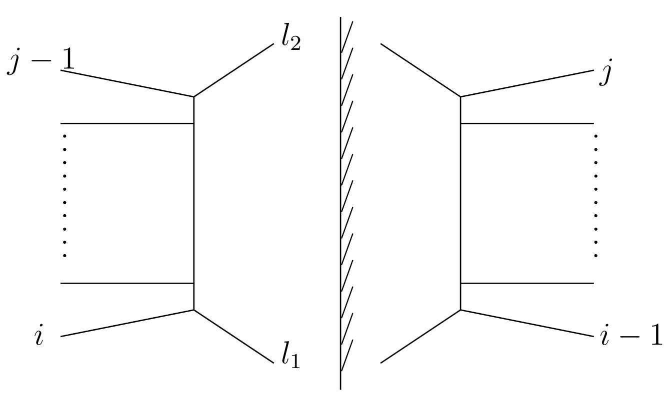
But for $\mathcal{N} = 1$ SUSY we have

$$V[A] \;=\; -\frac{1}{2}\mathrm{Tr}\ln\left(D^2 g_{\mu\nu} - g\Sigma^{\alpha\beta}_{\mu\nu}F_{\alpha\beta}\right) \;+\; \frac{1}{2}\ln\left(D^2 - \frac{g}{2}\sigma^{\alpha\beta}F_{\alpha\beta}\right) \;+\; \mathrm{Tr}\ln D^2,$$

where $\frac{1}{2}\sigma^{\alpha\beta}$ are the generators of the Lorentz group in the spinor representation. The extra term comes from the gluino loop.

In this case the contributions from $\mathrm{Tr}(D^2)$ cancel and the leading UV behaviour comes from the terms in the expansion in which there are two powers of $F_{\alpha\beta}$ (terms with only one power cancel because of the tracelessness of the generators). These terms have a maximum of $(n-2)$ powers of loop momentum. This means that successive applications of VP reduction yields scalar integrals only (albeit UV divergent bubble graphs).

This means that $\mathcal{N} = 1$ SUSY is cut-constructible. We can calculate the imaginary part of the one-loop $n$-point amplitude by summing over all two-particle cuts.



The factors on the left and right are tree-amplitudes which we have already learnt how to calculate. They will depend on the cut momenta $l_1$ and $l_2$. After adding the graph in which the internal gluon loop is replaced by a gluino loop the product of the two tree-level amplitudes can be manipulated and VP reduced until they form the sum of cut box, cut triangle and cut bubble diagrams only. We then need to sum over all possible ways of cutting two lines (Cutkosky rules). We give an example to this later.

## 9.3 $\mathcal{N} = 4$ Supersymmetry

The situation is even better in the case of $\mathcal{N} = 4$ SUSY. The one-loop effective potential in this case is

$$V[A] \;=\; -\frac{1}{2}\mathrm{Tr}\ln\left(D^2 g_{\mu\nu} - g\Sigma^{\alpha\beta}_{\mu\nu}F_{\alpha\beta}\right) \,+\, 2\ln\left(D^2 - \frac{g}{2}\sigma^{\alpha\beta}F_{\alpha\beta}\right) \,+\, \mathrm{Tr}\ln D^2 \,-\, 3\mathrm{Tr}\ln D^2,$$

where the fermion term is larger by a factor of 4 since there are four Majorana fermions for each gluon and the last term comes from the three complex scalar multiplets.

In this case, not only does the leading term with no powers of $F_{\alpha\beta}$ cancel but so do the terms with two powers. The leading UV behaviour then comes form the terms with four powers of $F_{\alpha\beta}$. Such terms give rise to one-loop amplitudes with a maximum of $(n-4)$ loop momenta in the numerator. Successive application of the VP and BDK reduction now reduces any amplitude to scalar box graphs only. These graphs are UV convergent so we arrive at the result that there is no renormalisation for $\mathcal{N} = 4$ SUSY.

There is a further simplification in the case of $\mathcal{N} = 4$ SUSY.

In general the leading colour ( as $N \to \infty$ - see later) one-loop amplitude has a part which is proportional to the tree-level amplitude and a part which is not

$$\mathcal{A}_{1-loop}(p_1 \cdots p_n) \;=\; V_n \mathcal{A}_{tree}(p_1 \cdots p_n) \,+\, F_n.$$

$V_n$ (which may be a function of the kinematic variables) contains all the UV and IR divergent parts since these are always proportional to the tree-level amplitude. However in $\mathcal{N} = 4$ SUSY we always have (again leading colour)

$$\mathcal{A}^{\mathcal{N}=4\,SUSY}_{1-loop}(p_1 \cdots p_n) \;=\; V_n \mathcal{A}^{\mathcal{N}=4\,SUSY}_{tree}(p_1 \cdots p_n).$$

## 9.4 Supersymmetric MHV amplitudes

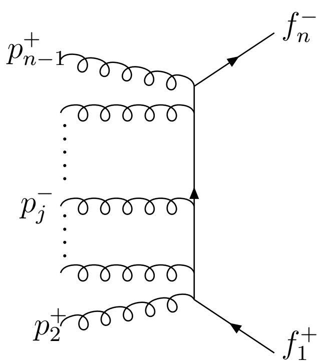
In order to calculate the 1-loop $n$-point gluon amplitudes in supersymmetric QCD, we need to calculate tree-amplitudes with one fermion or scalar line (these will be cut in the loop amplitude).

For massless fermions we identify the incoming positive helicity state as a fermion and the incoming negative helicity state (outgoing arrow) as an anti-fermion, so that an "MHV" amplitude means one of the gluons ($j$) and one of the incoming fermions has negative helicity.

Because the scalar particles are complex we need to make an analogous assignment of "helicity" for the scalars. The scalar with the ingoing arrow representing an ingoing particle is assigned positive "helicity" and an ingoing antiparticle (outgoing arrow) is assigned negative "helicity" so that once again the "MHV" amplitude with a scalar line has one negative helicity gluon ($j$).

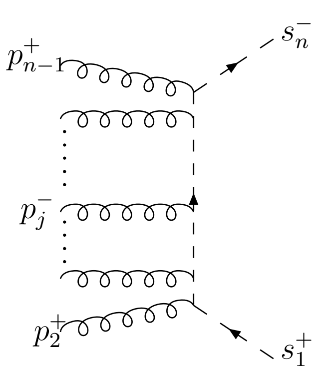
We can then use the SUSY Ward Identities to relate the MHV tree-amplitudes with a fermion ($f$) or scalar ($s$) line to the MHV vertices with only gluons. This gives

For a (colour ordered) fermion line:




$$\mathcal{A}_{tree}(f_1^+, p_2^+, \cdots p_j^- \cdots, f_n^-) \;=\; \frac{\langle p_j|p_1\rangle}{\langle p_j|p_n\rangle}\mathcal{A}_{tree}(p_1^+, p_2^+, \cdots p_j^- \cdots, p_n^-) \;=\; (-g)^{(n-2)}\frac{\langle p_j|p_n\rangle^3\langle p_j|p_1\rangle}{\langle p_1|p_2\rangle \cdots \langle p_n|p_1\rangle}$$

We have demonstrated this explicitly for the 4-point amplitude.



For a (colour-ordered) scalar line we have

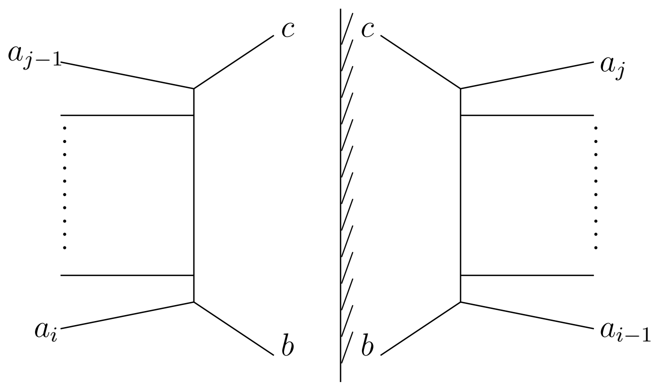
$$\mathcal{A}_{tree}(s_1^+, p_2^+, \cdots p_j^- \cdots, s_n^-) \;=\; \frac{\langle p_j|p_1\rangle^2}{\langle p_j|p_n\rangle^2}\mathcal{A}_{tree}(p_1^+, p_2^+, \cdots p_j^- \cdots, p_n^-) \;=\; (-g)^{(n-2)}\frac{\langle p_j|p_n\rangle^2\langle p_j|p_1\rangle^2}{\langle p_1|p_2\rangle\cdots\langle p_n|p_1\rangle}$$

**Exercise:**
Demonstrate this explicitly for the case of two gluons and a scalar line.

As in the case of pur gluons, the non-MHV amplitudes can be obtained from the MHV amplutdes either by using the BCF reduction or the CSW effective field-theory method, which is underwritten by modifying Mansfield's effective field theory to incorporate fermions and/or scalar particles.

## 9.5 Colour Factors

We have mentioned the "leading colour" contribution, meaning we neglected corrections of order $1/N$ for $SU(N)$. These can easily be reinstated.




Exploiting the method used to determine the colour factors for tree-level amplitudes it is straightforward to determine the colour factor associated with a particular cut graph.

$$\begin{aligned}
&\mathrm{Tr}\left(\tau^b\tau^{a_i}\cdots\tau^{a_{j-1}}\tau^c\right)\mathrm{Tr}\left(\tau^c\tau^{a_j}\cdots\tau^{a_n}\tau^b\right) \;=\\
&\qquad \mathrm{Tr}\left(\tau^{a_1}\cdots\tau^{a_{j-1}}\tau^c\tau^c\tau^{a_j}\cdots\tau^{a_n}\right) - \frac{1}{N}\mathrm{Tr}\left(\tau^{a_1}\cdots\tau^{a_{j-1}}\tau^c\right)\mathrm{Tr}\left(\tau^c\tau^{a_j}\cdots\tau^{a_n}\right) \;=\\
&\qquad \left(N-\frac{N}{2}\right)\mathrm{Tr}\left(\tau^{a_1}\cdots\tau^{a_n}\right) - \frac{1}{N}\mathrm{Tr}\left(\tau^{a_1}\cdots\tau^{a_{j-1}}\right)\mathrm{Tr}\left(\tau^{a_j}\cdots\tau^{a_n}\right)
\end{aligned}$$

The leading colour factor is $N$ times the tree-level colour factor.

In general, the one-loop amplitude in a SUSY theory can be written as

$$\mathcal{A}_{n\,1-loop} \;=\; \sum_J n_J \sum_k \sum_\sigma{}' G_{n;k}(\sigma)\mathcal{A}^J_{n;k}(\sigma),$$

where $J$ is the spin of the particles in the loop with multiplicity $n_J$,

$$G_{n;1} \;=\; N\mathrm{Tr}\left(\tau^{a_1}\cdots\tau^{a_n}\right)$$

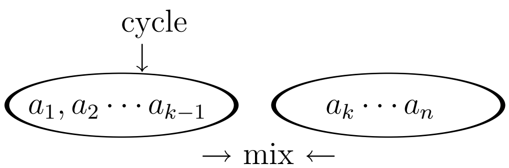
and

$$G_{n,k} \;= \mathrm{Tr}\left(\tau^{a_1}\cdots\tau^{a_{k-1}}\right)\mathrm{Tr}\left(\tau^{a_k}\cdots\tau^{a_n}\right),$$

the sum $\sum'_\sigma$ means sum over all permutations modulo permutations which leave $G_{n;k}$ invariant and $\mathcal{A}^J_{n;k}(\sigma)$ is the colour stripped amplitude with a sum over orderings, $\sigma$, of the $n$-particles.

$$\mathcal{A}^J_{n;k}(\sigma)(p_i\cdots p_n) \;=\; (-1)^{k-1}\sum_\sigma \mathcal{A}^J(\sigma)$$

where $\mathcal{A}^J(\sigma)$ is the coefficient of the leading colour contribution for permutation $\sigma$ of the external momenta and helicities.




The sum goes over all cyclic permutations of $p_1\cdots p_{k-1}$ and allowing all orderings of $p_1\cdots p_{k-1}$ amongst $p_k\cdots p_n$ but keeping $p_n$ fixed. For example if $n=5$ and $(k-1)=2$, we have the orderings

$$(1,2,3,4,5)\,,\quad(1,3,2,4,5)\,,\quad(1,3,4,2,5)\,,\quad(3,1,2,4,5)\,,\quad(3,4,1,2,5)\,,\qquad+\;\;1\leftrightarrow2$$