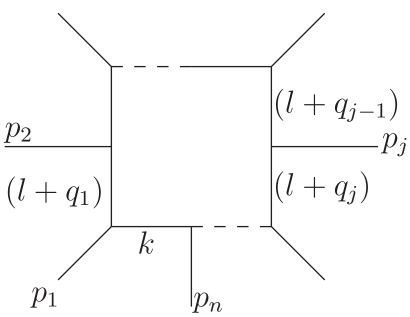
1 Veltman-Passarino Reduction

This is a method of expressing an n-point loop integral with r powers of the loop momentum l in the numerator, in terms of "scalar" s-point functions with $s = n - r, \cdots n$. "scalar" meaning an integral with *no* powers of loop-momenta ($r = 0$) in the numerator - i.e. a product of propagator denominators.

7.1 Notation



$$I_n[f(l)] \equiv -i(4\pi)^{d/2} \int \frac{d^d l}{(2\pi)^d} \frac{f(l)}{(l^2 - m_0^2)((l+q_1)^2 - m_1^2) \cdots ((l+q_{n-1})^2 - m_{n-1}^2)},$$

where

$$q_i \equiv p_1 + p_2 \cdots p_i,$$

p_i being the (incoming) external momenta. $d = 4 - 2\epsilon$ is the number of dimensions in which we perform the loop integral in order to regularise either ultraviolet (UV) or infrared/collinear (IR) divergences.

We assume that *all* external momentum are in four dimensions

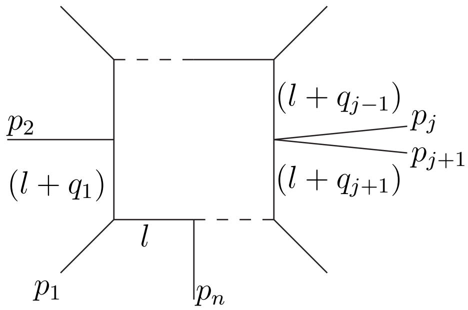
Note that masses can be included in these manipulations but add to the algebraic complexi

The case $f(l) = 1$ is what we call the "scalar integrals" - the integrals that would be obtained in a theory of scalar particles only.

We also define the integrals $I_{n-1}^{(j)}$:

$$I_{n-1}^{(j)}[f(l)] \equiv -i(4\pi)^{d/2} \int \frac{d^d l}{(2\pi)^d} f(l)$$
$$\times \frac{1}{(l^2 - m_0^2)((l+q_1)^2 - m_1^2) \cdots ((l+q_{j-1})^2 - m_{j-1}^2)((l+q_{j+1})^2 - m_{j+1}^2) \cdots ((l+q_n)^2 - m_n^2)},$$

i.e. the $n-1$-point integral obtained by "pinching out" the j^{th} propagator.



Similarly we can define integrals such as $I_{n-2}^{(j_1,j_2)}[f(l)]$, etc.

7.2 One power of loop momentum in numerator

$$f(l) \;=\; l^\mu$$

$$I_n[l^\mu] \;=\; -i(4\pi)^{d/2}\int \frac{d^dl}{(2\pi)^d}\frac{l^\mu}{(l^2-m_0^2)((l+q_1)^2-m_1^2)\cdots((l+q_n)^2-m_n^2)} \;=\; \sum_{i=1}^{n-1} C_{n;i}\,p_i^\mu\,,$$

using the fact that the vector quantity on the LHS *must* be constructed from the vectors $p_1\cdots p_{n-1}$ (by conservation of momentum $p_1+p_2+\cdots p_{n-1}=-p_n$ so that p_n is *not* independent)

Contracting both sides with p_j^μ we get

$$I_n[l\cdot p_j] \;=\; -i(4\pi)^{d/2}\int \frac{d^dl}{(2\pi)^d}\frac{l\cdot p_j}{(l^2-m_0^2)((l+q_1)^2-m_1^2)\cdots((l+q_n)^2-m_n^2)} \;=\; \sum_{i=1}^{n-1} C_{n;i}\,\Delta^{ij},$$

where $\Delta^{ij} \;=\; p_i\cdot p_j$ is the "Gram" matrix.

Since $p_j=q_j-q_{j-1}$ (with $q_0=0$) we can write the numerator of the integral as

$$l\cdot p_j \;=\; \frac{1}{2}\left(((l+q_j)^2-m_j^2)-((l+q_{j-1})^2-m_{j-1}^2)+m_j^2-m_{j-1}^2-q_j^2+q_{j-1}^2\right)$$

This is the Veltman-Passarino (VP) reduction formula.

The terms $((l+q_j)^2-m_j^2)$ and $((l+q_{j-1})^2-m_{j-1}^2$ in the numerator can be used to cancel (or "pinch") the j^{th} and $(j-1)^{th}$ propagators respectively and so we end up with a set of $n-1$ linear equations for the coefficients $C_{n;i}$.

$$\sum_{i=1}^{n-1} C_{n;i}\,\Delta^{ij} \;=\; \frac{1}{2}\left(I_{n-1}^{(j)}[1]-I_{n-1}^{(j-1)}[1]+(m_j^2-m_{j-1}^2-q_j^2+q_{j-1}^2)I_n[1]\right),$$

where the RHS is expressed in terms of scalar n-point and $(n-1)$-point integrals.

This set of linear equations is readily solved

$$C_{n;i} \;=\; \frac{1}{2}\sum_j \Delta^{-1}_{ij}\left(I^{(j)}_{n-1}[1] - I^{(j-1)}_{n-1}[1] + (m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2)I_n[1]\right)$$

However, care is needed if we are dealing with $n > 4$. The reason for this is that in this case only the first four of the external momenta p_i can be linearly independent. In fact by attempting to write the integral out as the sum of $n-1$ terms, we have over-parametrised the integral and the coefficients are not unique - the Gram matrix is not invertible as its determinant vanishes.

In such cases, it is sufficient to choose the first four external momenta only provided the Gram determinant for those four does *not* vanish.

$$det(\Delta) \;\neq\; 0.$$

In special cases, known as "exceptional momenta" this determinant may indeed vanish and we must then choose a different four.

We then have

$$I_n[l^\mu] \;=\; \sum_{i=1}^{4} C_{n;i}\, p_i^\mu,$$

with

$$C_{n;i} \;=\; \frac{1}{2}\sum_{j=1}^{4} \Delta^{-1}_{ij}\left(I^{(j)}_{n-1}[1] - I^{(j-1)}_{n-1}[1] + (m_j^2 - m_{j-1}^2 - q_j^2 + q_{j-1}^2)I_n[1]\right)$$

Example:
In this example we will consider the three-point function and (for simplicity) set the internal masses (but *not* the external square momenta) to zero, so that we have

$$-i(4\pi)^{d/2}\int\frac{d^dl}{(2\pi)^d}\frac{l^\mu}{l^2(l+q_1)^2(l+q_2)^2} \;=\; C_{3;1}\,p_1^\mu + C_{3;2}\,p_2^\mu,$$

with $q_1 = p_1$ and $q_2 = (p_1 + p_2)$.

Contracting both sides with p_1^μ gives

$$-i(4\pi)^{d/2}\int\frac{d^dl}{(2\pi)^d}\frac{l\cdot p_1}{l^2(l+q_1)^2(l+q_2)^2} \;=\; C_{3;1}\,p_1^2 + C_{3;2}\,p_1\cdot p_2,$$

Using

$$l\cdot p_1 = \frac{1}{2}\left((l+q_1)^2 - l^2 - p_1^2\right)$$

we have

$$p_1^2\, C_{3;1} + p_1\cdot p_2\, C_{3;2} \;=\; \frac{1}{2}\left(I_2^{(1)}[1] - I_2^{(0)}[1] - p_1^2 I_3[1]\right),$$

with

$$I_2^{(1)}[1] \;=\; -i(4\pi)^{d/2}\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2(l+q_2)^2}$$

and

$$I_2^{(0)}[1] \;=\; -i(4\pi)^{d/2}\int\frac{d^dk}{(2\pi)^d}\frac{1}{(l+q_1)^2(l+q_2)^2} \;=\; -i(4\pi)^{d/2}\int\frac{d^dl}{(2\pi)^d}\frac{1}{l^2(l+p_2)^2}.$$

In the last step we have performed a shift of integration variable $l \to (l-q_1)$.

Likewise, if we contract with p_2^μ we get

$$-i(4\pi)^{d/2}\int\frac{d^dl}{(2\pi)^d}\frac{l\cdot p_2}{l^2(l+q_1)^2(l+q_2)^2} \;=\; C_{3;1}\, p_1\cdot p_2 + C_{3;2}p_2^2,$$

Using

$$l\cdot p_2 = \frac{1}{2}\left((l+q_2)^2 - (l-q_1)^2 + p_1^2 - (p_1+p_2)^2\right)$$

we have

$$p_1\cdot p_2\, C_{3;1} + p_2^2\, C_{3;2} \;=\; \frac{1}{2}\left(I_2^{(2)}[1] - I_2^{(1)}[1] - (2p_1\cdot p_2 + p_2^2)I_3[1]\right),$$

The solutions are

$$\begin{aligned}
C_{3;1} \;&=\; \frac{1}{(p_1^2p_2^2-(p_1\cdot p_2)^2)}\Big(\left(p_2^2+p_1\cdot p_2\right)I_2^{(1)}[1] - p_2^2 I_2^{(0)}[1] - p_1\cdot p_2 I_2^{(2)}[1] \\
&\quad - \left(p_1^2p_2^2 - 2(p_1\cdot p_2)^2 - p_2^2(p_1\cdot p_2)\right)I_3[1]\Big),\\
C_{3;2} \;&=\; \frac{1}{(p_1^2p_2^2-(p_1\cdot p_2)^2)}\Big(\left(p_1^2+p_1\cdot p_2\right)I_2^{(1)}[1] + p_1^2 I_2^{(2)}[1] - p_1\cdot p_2 I_2^{(0)}[1] \\
&\quad - \left(p_1^2p_2^2 + p_1^2(p_1\cdot p_2)\right)I_3[1]\Big)
\end{aligned}$$

7.3 Two powers of loop momentum in numerator

For the case

$$f(l) \;=\; l^\mu k^\nu,$$

the integral is a rank-two tensor which can be formed out of the outer products of external momenta $p_i^\mu p_j^\nu$ *and* the metric $g^{\mu\nu}$. Again for $n > 5$ we only use $i,j = 1\cdots 4$.

$$I_{(n)}[l^\mu l^\nu] \;=\; -i(4\pi)^{d/2}\int\frac{d^dl}{(2\pi)^d}\frac{l^\mu l^\nu}{(l^2-m_0^2)((l+q_1)^2-m_1^2)\cdots((l+q_n)^2-m_n^2)} \;=\; C_{n:00}g^{\mu\nu} + \sum_{i,j} C_{n;ij}\, p_i^\mu p_j^\nu,$$

The first equation we can derive relating the coefficients is obtained by contracting both sides with $g^{\mu\nu}$, remembering that we need to work in general in d dimensions so that we get

$$I_{(n)}[l^2] \;=\; -i(4\pi)^{d/2}\int \frac{d^dl}{(2\pi)^d}\frac{l^2}{(l^2-m_0^2)((l+q_1)^2-m_1^2)\cdots((l+q_n)^2-m_n^2)} \;=\; C_{n:00}d+\sum_{i,j} C_{n;ij}\,\Delta^{ij}$$

Writing

$$l^2 \;=\; (l^2-m_0)^2+m_0^2,$$

we get the relation

$$I_{n-1}^{(0)}[1]+m_0^2 I_n[1] \;=\; d\,C_{n;00}+\sum_{i,j} C_{n;ij}\,\Delta^{ij}$$

Another relation is obtained by contracting both sides with $\Delta^{-1}_{lh}p_l^\mu p_h^\nu$, to obtain

$$\Delta^{-1}_{lh} I_n[p_l\cdot l\, p_h\cdot l] \;= \min(4,(n-1))C_{n;00}+\sum_{k,h} C_{n;lh}\,\Delta^{lh}$$

We note here that if $n\ \geq\ 5$ (and therefore UV finite) and the integral contains no infrared divergences, then we have two equivalent expressions for the combination of coefficients $4C_{n;00}+\sum_{i,j} C_{n;kl}\,\Delta^{kl}$. This means that the term $C_{n;00}$ is redundant (not uniquely defined) and we can set it to zero. However, in the general case we need to keep this coefficient separate.

The remaining coefficients are obtained by contracting with $p_l^\mu p_h^\nu$, to obtain

$$I_n[p_l\cdot l\, p_h\cdot l] \;= C_{n;00}\Delta^{lh}+\sum_{i,j} C_{n;ij}\,\Delta^{il}\Delta^{jh}$$

For $n\ >\ 5$ this provides 11 equations for the 10 coefficients $C_{n;ij}$ (symmetric in $\{i,j\}$) and $C_{n,00}$

The LHS is most easily handled by using the VP reduction formula once only (say for $p_h\cdot l$ and leaving the other scalar product alone.

$$p_h\cdot l \;=\; \frac{1}{2}\Big(((l+q_h)^2-m_h^2)-((l+q_{h-1})^2-m_{h-1}^2)+q_{h-1}^2-q_h^2+m_h^2-m_{h-1}^2\Big)$$

so that we get

$$\frac{1}{2}\left(I_{n-1}^{(h)}[p_k\cdot l]-I_{n-1}^{(h-1)}[p_k\cdot l]+\left(q_{h-1}^2-q_h^2+m_h^2-m_{h-1}^2\right)I_n[p_k\cdot l]\right) \;=\; C_{n;00}\Delta^{kh}+\sum_{i,j} C_{n;ij}\,\Delta^{ik}\Delta^{jh}$$

We use the results previously obtained for the case where $f(l)=p_k\cdot l$ to reduce the integrals on the LHS to scalar integrals.

Care must be taken for the case where $h = 0$ (or $(h-1) = 0$) since in that case it is the propagator denominator $l^2 - m_0^2$ that has been cancelled and we have the integral

$$I_{n-1}^{(0)}[p_l \cdot l] \;=\; -i(4\pi)^{d/2} \int \frac{d^d l}{(2\pi)^d} \frac{p_k \cdot l}{((l+q_1)^2 - m_1^2)((l+q_2)^2 - m_2^2) \cdots ((l+q_{n-1})^2 - m_{n-1}^2)}$$

To get this into the standard form we must make a shift of integration variable

$$l \;\rightarrow\; l - q_1$$

This affects the numerator also and we arrive at

$$-i(4\pi)^{d/2} \int \frac{d^d l}{(2\pi)^d} \frac{p_k \cdot l - p_k \cdot q_1}{(l^2 - m_1^2)((l+p_2)^2 - m_2^2) \cdots ((l+q_{2,n-1})^2 - m_{n-1}^2)},$$

which is the linear combination of an $r = 1$ integral and a scalar integral.

Together with the equation obtained by contracting with $g^{\mu\nu}$

$$I_{n-1}^{(0)}[1] + m_0^2 I_n[1] \;=\; d\, C_{n;00} + C_{n;lh}\, \Delta^{lh},$$

we end up with

$$(d+1-n) C_{n;00} \;=\; \frac{1}{2} I_{n-1}^{(0)}[1] + m_0^2 I_n[1] - \frac{1}{2} \sum_{h=1}^{n-1} \left(q_{h-1}^2 - q_h^2 + m_h^2 - m_{h-1}^2 \right) C_{n;h}.$$

Assuming that the $r = 1$ coefficients have already been calculated, the expression for C_{n00} can be inserted into the expressions of the other $r = 2$ coefficients, $C_{n;ij}$ and the set of $\frac{1}{2} n(n-1)$ linear equations can be solved.

Example:
Once again we consider a triangle integral with two powers of loop momentum in the numerator $n = 3,\ r = 2$, and internal masses set to zero. This has a UV divergence by power counting and will also be infrared divergent if any of the external square momenta vanish.

$$\begin{aligned} I_3[l^\mu l^\nu] \;&\equiv\; -i(4\pi)^{d/2} \int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{l^2 (l+q_1)^2 (l+q_2)^2} \\ &=\; C_{3;00} g^{\mu\nu} + C_{3;11} p_1^\mu p_1^\nu + C_{3;22} p_2^\mu p_2^\nu + C_{3;12} (p_1^\mu p_2^\nu + p_2^\mu p_1^\nu) \end{aligned}$$

We can obtain four equations for the four independent unknowns as follows:

Contract with $g^{\mu\nu}$

$$I_2^{(0)}[1] \;=\; d C_{3;00} + p_1^2 C_{3;11} + p_2^2 C_{3;22} + 2 p_1 \cdot p_2 C_{3;12} \quad (1)$$

<u>Contract with $p_1^\mu p_1^\nu$</u>

$$I_3[p_k \cdot l\, p_k \cdot l] \;=\; p_k^2 C_{3;00} + (p_k^2)^2 C_{3;11} + (p_k \cdot p_2)^2 C_{3;22} + 2p_1^2(p_1 \cdot p_2) C_{3;12}$$

Using

$$p_1 \cdot l \;=\; \frac{1}{2}\left((l+q_1)^2 - l^2 - q_1^2\right),$$

this may be written

$$\frac{1}{2}\left(I_2^{(1)}[p_1 \cdot l] - I_2^{(0)}[p_1 \cdot l] - p_1^2 I_3[p_1 \cdot l]\right) \;=\; p_1^2 C_{3;00} + (p_1^2)^2 C_{3;11} + (p_1 \cdot p_2)^2 C_{3;22} + 2p_1^2(p_1 \cdot p_2) C_{3;12} \quad (2)$$

<u>Contract with $p_2^\mu p_2^\nu$</u>

$$I_3[p_2 \cdot l\, p_2 \cdot l] \;=\; p_2^2 C_{3;00} + (p_1 \cdot p_2)^2 C_{3;11} + (p_2^2)^2 C_{3;22} + 2p_2^2 p_1 \cdot p_2 C_{3;12}$$

Using

$$p_2 \cdot l \;=\; \frac{1}{2}\left((l+q_2)^2 - (l+q_1)^2 + q_1^2 - q_2^2\right),$$

this may be written

$$\frac{1}{2}\left(I_2^{(2)}[p_2 \cdot l] - I_2^{(1)}[p_2 \cdot l] + (q_1^2 - q_2^2) I_3[p_2 \cdot l]\right) \;=\; p_2^2 C_{3;00} + (p_1 \cdot p_2)^2 C_{3;11} + (p_2^2)^2 C_{3;22} + 2p_2^2(p_1 \cdot p_2) C_{3;12} \quad (3)$$

<u>Contract with $p_1^\mu p_2^\nu$</u>

$$I_3[p_1 \cdot l\, p_2 \cdot l] \;=\; p_1 \cdot p_2 C_{3;00} + (p_1 \cdot p_2) p_1^2 C_{3;11} + (p_1 \cdot p_2) p_2^2 C_{3;22} + (p_1^2 p_2^2 + (p_1 \cdot p_2)^2) C_{3;12}$$

Using

$$p_1 \cdot l \;=\; \frac{1}{2}\left((l+q_1)^2 - l^2 - q_1^2\right),$$

this may be written

$$\frac{1}{2}\left(I_2^{(1)}[p_2 \cdot l] - I_2^{(0)}[p_2 \cdot l] - q_1^2 I_3[p_2 \cdot l]\right) \;=\; p_1 \cdot p_2 C_{3;00} + (p_1 \cdot p_2) p_1^2 C_{3;11} + (p_1 \cdot p_2) p_2^2 C_{3;22} + (p_1 \cdot p_2)^2 C_{3;12} \quad (4)$$

Alternatively we could use

$$p_2 \cdot l \;=\; \frac{1}{2}\left((l+q_2)^2 - (l+q_1)^2 + q_1^2 - q_2^2\right),$$

to obtain an equivalent equation

$$\frac{1}{2}\left(I_2^{(2)}[p_1 \cdot l] - I_2^{(1)}[p_1 \cdot l] + (q_1^2 - q_2)^2 I_3[p_1 \cdot l]\right) \;=\; p_1 \cdot p_2 C_{3;00} + (p_1 \cdot p_2) p_1^2 C_{3;11} + (p_1 \cdot p_2) p_2^2 C_{3;22} + (p_1 \cdot p_2)^2 C_{3;12} \quad (4')$$

It is useful to define the function $B(q^2)$ by

$$-i(4\pi)^{d/2}\int \frac{d^dl}{(2\pi)^d}\frac{1}{l^2(l+q)^2} \;\equiv\; B(q^2)$$

from which we can easily derive

$$-i(4\pi)^{d/2}\int \frac{d^dl}{(2\pi)^d}\frac{l^\mu}{k^2(l+q)^2} \;=\; -\frac{1}{2}B(q^2)q^\mu$$

With this definition the four equations become

$$B(p_2^2) \;=\; dC_{3;00} + p_1^2 C_{3;11} + p_2^2 C_{3;22} + 2p_1\cdot p_2 C_{3;12} \quad (1),$$

(where we have used $q_2 - q_1 = p_2$),

$$\frac{1}{4}\left(-B(q_2^2)(p_1^2 + p_1\cdot p_2) + B(p_2^2)\left(2p_1^2 + p_1\cdot p_2\right) - 2p_1^2 I_3[p_1\cdot l]\right) \;=\; p_1^2 C_{3;00} + (p_1^2)^2 C_{3;11} + (p_1\cdot p_2)^2 C_{3;22} + 2p_1^2(p_1\cdot p_2)C_{3;12} \quad (2),$$

where we have shifted the integration variable $l \to l - q_1$, and set $p_1\cdot q_2 = p_1^2 + p_1\cdot p_2$,

$$\frac{1}{4}\left(-B(p_1^2)p_2\cdot p_1 + B(q_2^2)(p_2^2 + p_1\cdot p_2) + 2(p_1^2 - q_2^2)I_3[p_2\cdot l]\right) \;=\; p_2^2 C_{3;00} + (p_1\cdot p_2)^2 C_{3;11} + (p_2^2)^2 C_{3;22} + 2p_2^2(p_1\cdot p_2)C_{3;12} \quad (3),$$

where we have written $p_2\cdot q_2 = p_2^2 + p.p_2$, and $q_1 = p_1$,

$$\frac{1}{4}\left(-B(q_2)^2(p_2^2 + p_1\cdot p_2) + B(p_2)^2(2p_1\cdot p_2 + p_2^2) - 2p_1^2 I_3[p_2\cdot l]\right) \;=\; p_1\cdot p_2 C_{3;00} + (p_1\cdot p_2)p_1^2 C_{3;11} + (p_1\cdot p_2)p_2^2 C_{3;22} + (p_1\cdot p_2)^2 C_{3;12} \quad (4)$$

or

$$\frac{1}{4}\left(-B(p_1^2)p_1^2 + B(q_2^2)(p_1^2 + p_1\cdot p_2) + 2(p_1^2 - q_2^2)I_3[p_1\cdot l]\right) \;=\; p_1\cdot p_2 C_{3;00} + (p_1\cdot p_2)p_1^2 C_{3;11} + (p_1\cdot p_2)p_2^2 C_{3;22} + (p_1\cdot p_2)^2 C_{3;12} \quad (4')$$

If we construct the combination

$$p_2^2(1) + p_1^2(2) - p_1\cdot p_2((4) + (4')),$$

divide by $p_1^2p_2^2 - (p_1\cdot p_2)^2$ and use the previously obtained relations

$$I_3[p_1\cdot l] \;=\; p_1^2\, C_{3;1} + p_1\cdot p_2\, C_{3;2}$$

$$I_3[p_2\cdot l] \;=\; p_1\cdot p_2\, C_{3;1} + p_2^2 C_{3;2},$$

we arrive at the promised result

$$2C_{3;00} + p_1^2 C_{3;11} + p_2^2 C_{3;22} + 2p_1\cdot p_2 C_{3;12} \;=\; \frac{1}{2}\left(B(p_2^2) - p_1^2 C_{3;1} - (q_2^2 - p_1^2)C_{3;2}\right)$$

which when combined with (1) gives an expression for $C_{3;00}$

$$C_{3:00} = \frac{1}{2(d-2)} \left(B(p_2^2) + p_1^2 C_{3;1} + (q_2^2 - p_1^2) C_{3;2} \right)$$

Note that $B(q^2)$ is UV divergent so that we *must* keep the number of dimensions,d, away from 4 even in the absence of infrared divergences. $C_{3;00}$ is ultraviolet divergent.

The remaining coefficients $C_{3;11}$, $C_{3;22}$, $C_{3;12}$ can now be obtained by solving the three simultaneous equations (1), (2), (3).

7.4 More powers of l

This process can be "rolled out" for any number of powers of loop momentum in the numerator, but the expressions become increasingly longer.

For three powers of loop momentum we have

$$I_n[l^\mu l^\nu l^\rho] = \sum_{i=1}^{4} C_{n;00i} g^{\{\mu\nu} p_i^{\rho\}} + \sum_{i,jl,=1}^{4} C_{n;ijl} p_i^{\{\mu} p_j^\nu p_l^{\rho\}}$$

and we need to contract with $g^{\mu\nu} p_r^\rho$ or with $p_r^\mu p_s^\nu p_t^\rho$ to obtain a set of linear equations for the coefficients $C_{n;00i}$ or $C_{n;ijk}$.

For four powers of loop momentum we have

$$I_n[l^\mu l^\nu l^\rho l^\sigma] = C_{n;0000} g^{\{\mu\nu} g^{\rho\sigma\}} \sum_{i,j=1}^{4} C_{n;00ij} g^{\{\mu\nu} p_i^\rho p_j^{\sigma\}} + \sum_{i,j,k,h=1}^{4} C_{n;ijlh} p_i^{\{\mu} p_j^\nu p_l^\rho p_h^{\sigma\}}$$

and we need to contract with $g^{\mu\nu} g^{\rho\sigma}$, $g^{\mu\nu} p_r^\rho\_s^\sigma$, and $p_r^\mu p_s^\nu p_t^\rho p_u^\sigma$ in order to project out the coefficients $C_{n;0000}$, $C_{n;00ij}$ and $C_{n;ijkh}$

Although this becomes more complicated, a contraction with a single $g^{\mu\nu}$ or with a single momentum p_i^μ, where p_i is one of the first four external momenta enables one to write any loop integral

$$I_n[l^{\mu_1} \cdots l^{\mu_r}]$$

in terms of

$$I_n[l^{\mu_1} \cdots l^{\mu_{r-1}}]$$

and

$$I_{n-1}^{(j)}[l^{\mu_1} \cdots l^{\mu_{r-1}}],$$

and the process can be iterated.

7.5 Reduction of n-point integral to $(n-1)$-point integral

A very useful application of Veltman-Passarino reduction is the reduction of a one-loop n-point scalar integral, expressing it as a sum of $(n-1)$-point and $(n-2)$-point integrals, which is possible provided $n > 4$, for which there are 4 linearly independent external momenta $p_i,\ i = 1 \cdots 4$ (the remaining external momenta can always be written as a linear combination of the first 4 external momenta).

$$I_n \;=\; \int d^4l \frac{1}{(l^2-m_0^2)((l+q_1)^2-m_1^2)\cdots((l+q_j)^2-m_j^2)\cdots((l+q_n)^2-m_n^2)}$$

with

$$q_j \;=\; \sum_{i=1}^{j} p_i.$$

If $n > 4$ then we may write

$$l^\mu \;=\; \sum_{i,j=1}^{4} p_i\cdot l(\Delta)^{-1}_{ij}p^\mu_j,$$

with

$$\Delta_{ij} \;=\; p_i\cdot p_j$$

So that

$$l^2 \;=\; \sum_{i,j=1}^{4} p_i\cdot l(\Delta)^{-1}_{ij}p_j\cdot l,$$

Now using

$$l^2 = (l^2-m_0^2)+m_0^2$$

$$p_j\cdot l \;=\; \frac{1}{2}\left[\left((l+q_j)^2-m_j^2\right)-\left((l+q_{j-1})^2-m_{j-1}^2\right)+\left(m_j^2-m_{j-1}^2+q_{j-1}^2-q_j^2\right)\right]$$

and introducing the notation

$$I^{(j)}_{(n-1)}[x] \;\equiv\; \int d^4l \frac{x}{(l^2-m_0^2)\cdots\left((l-q^2_{(j-1)}-m^2_{(j-1)}\right)\left((l-q^2_{(j+1)}-m^2_{(j+1)}\right)\cdots\left((l-q^2_{(n+1)}-m^2_{(n+1)}\right)},$$

(i.e. the n-point integral in which the j^{th} propagator is "pinched out") and we drop the argument $[x]$ if $x=1$. i.e. for scalar integrals.

$$I^{(0)}_{(n-1)}+m_0^2 I_n \;=\; \frac{1}{2}\sum_{i,j=1}^{4}(\Delta)^{-1}_{ij}\left[I^{(j)}_{(n-1)}[p_i\cdot l]-I^{((j-1))}_{(n-1)}[p_i\cdot l]+\frac{1}{2}\left(m_j^2-m_{j-1}^2+q_{j-1}^2-q_j^2\right)I_n[p_i\cdot l]\right]$$

$$= \frac{1}{2}\sum_{i,j=1}^{4}(\Delta)^{-1}_{ij}\left[I^{(j)}_{(n-1)}[p_i\cdot l] - I^{((j-1))}_{(n-1)}[p_i\cdot l]\right.$$

$$\left. +\frac{1}{2}\left(m_j^2 - m_{j-1}^2 + q_{j-1}^2 - q_j^2\right)\left(I^{(i)}_{(n-1)} - I^{((i-1))}_{(n-1)} + (m_i^2 - m_{(i-1)}^2 + q_{(i-1)}^2 - q_i^2\right) I_n\right]$$

The terms

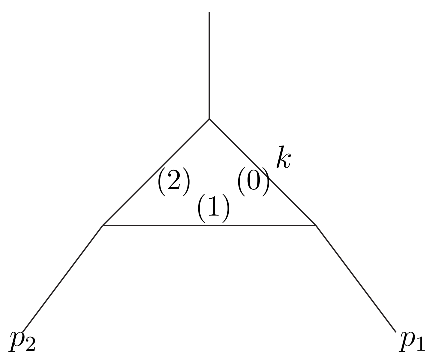
$$I^{(j)}_{(n-1)}[p_i\cdot l]$$

can be expressed in terms of

$$I^{(j)}_{(n-1)} \text{ and } I^{(j,i)}_{(n-2)}$$

using the Veltman-Passarino reduction for the case of one power of momentum in the numerator. We therefore end up with a relation between the n-point scalar integrals and the possible $(n-1)$-point and $(n-2)$-point scalar integrals, shown above.

As an example, which can be done analytically we go down to two dimensions and consider the triangle integral, with equal masses on internal lines.



$$q_1 \equiv p_1$$
$$q_2 \equiv p_1 + p_2$$

$$I_3 \equiv \int d^2l \frac{1}{(l^2-m^2)((l+q_1)^2-m^2)((l+q_2)^2-m^2)}$$

$$l^\mu = \frac{1}{q_1^2q_2^2-(q_1\cdot q_2)^2}\left[\left(q_2^2 l\cdot q_1 - q_1\cdot q_2 l\cdot q_2\right)q_1^\mu + \left(q_1^2 l\cdot q_2 - p_1\cdot q_2 l\cdot q_1\right)q_2^\mu\right]$$

$$l^2 = \frac{1}{q_1^2q_2^2-(q_1\cdot q_2)^2}\left[q_2^2(l\cdot q_1)^2 + q_1^2(l\cdot q_2)^2 - 2q_1\cdot q_2 l\cdot q_1 k\cdot q_2\right]$$

$$I_3[l^2] \equiv \int d^2l \frac{l^2}{(l^2-m^2)((l+q_1)^2-m^2)((l+q_2)^2-m^2)} = I_2^{(0)} + m^2 I_3$$

$$\begin{aligned} I_3[(l\cdot q_1)^2] &= \frac{1}{2}\left[I_2^{(1)}[l\cdot q_1] - I_2^{(0)}[l\cdot q_1] - q_1^2 I_3[l\cdot q_1]\right] \\ &= \frac{1}{2}\left[I_2^{(1)}[l\cdot q_1] - I_2^{(0)}[l\cdot q_1] - \frac{1}{2}q_1^2\left(I_2^{(1)} - I_2^{(0)}\right)\right] + \frac{1}{4}q_1^4 I_3 \end{aligned}$$

Similarly

$$\begin{aligned} I_3[(k\cdot q_2)^2] &= \frac{1}{2}\left[I_2^{(2)}[l\cdot q_2] - I_2^{(0)}[l\cdot q_2] - q_2^2 I_3[l\cdot q_2]\right] \\ &= \frac{1}{2}\left[I_2^{(2)}[l\cdot q_2] - I_2^{(0)}[l\cdot q_2] - \frac{1}{2}q_2^2\left(I_2^{(2)} - I_2^{(0)}\right)\right] + \frac{1}{4}q_2^4 I_3 \end{aligned}$$

and

$$\begin{aligned} I_3[l\cdot q_1 k\cdot q_2] &= \frac{1}{2}\left[I_2^{(1)}[l\cdot q_2] - I_2^{(0)}[l\cdot q_2] - q_1^2 I_3[l\cdot q_2]\right] \\ &= \frac{1}{2}\left[I_2^{(1)}[l\cdot q_1] - I_2^{(0)}[l\cdot q_1] - \frac{1}{2}q_1^2\left(I_2^{(2)} - I_2^{(0)}\right)\right] + \frac{1}{4}q_1^2 q_2^2 I_3 \end{aligned}$$

Finally, we use

$$I_2^{(1)}[l^\mu] \equiv \int d^2l \frac{l^\mu}{q(k^2-m^2)((l+q_2)^2-m^2)} = \frac{1}{2}\left[I_1^{(1,2)} - I_1^{(1,0)} - q_2^2 I_2^{(1)}\right] q_2^\mu = -\frac{1}{2}q_2^\mu I_2^{(2)}$$

since

$$I_1^{(1,0)} \equiv \int d^2l \frac{1}{((l+q_2)^2-m^2)} = \int d^2l \frac{1}{(l^2-m^2)} = I_1^{(1,2)}$$

and similarly

$$\begin{aligned} I_2^{(2)}[l^\mu] &= -\frac{1}{2}q_1^\mu I_2^{(2)} \\ I_2^{(0)}[l^\mu] &= -\frac{1}{2}\left(q_1^\mu + q_2^\mu\right) I_2^{(2)} \end{aligned}$$

Piecing all of this together we find that (in two dimensions)

$$\left[q_1^2 q_2^4 + q_1^4 q_2^2 - 2q_1^2 q_2^2 q_1\cdot q_2 + 4m^2(q_1\cdot q_2)^2 - 4m^2 q_1^2 q_2^2\right] I_3 =$$

$$\left[(q_1^2+q_2^2)q_1\cdot q_2 - 2(q_1\cdot q_2)^2\right] I_2^{(0)} + \left[q_1^2 q_2^2 - q_2^2 q_1\cdot q_2\right] I_2^{(1)} + \left[q_1^2 q_2^2 - q_1^2 q_1\cdot q_2\right] I_2^{(2)}$$

It therefore follows that *any* one-loop integral, I_n in four dimensions can be reduced to a sum of box-integrals, I_4, triangle-integrals, I_3, bubble-integrals, I_2, and tadpole-integrals, I_3 †

This can be achieved by successive applications of the Veltman-Passarino reduction method. However, this is very cumbersome. In the next section we show how the coefficients of these various scalar integrals can be obtained by a technique known as "cut construction". For simplicity we will explain how this works in two-dimensions. The extension of the technique to the four-dimensional case does not introduce any new principles, but simply generates somewhat more intricate algebra.

7.6 Remnant Integrals

In QCD a one-loop n-point graph has a maximum of n vertices each carrying a maximum power of one loop momentum and n propagators.

Each stage on the reduction reduces the number of loop momenta on the numerator by one and the number of denominators (propagators) by one.

This means that from an ultraviolet convergent loop integral we will generate terms which are ultraviolet divergent. For example a 5-point matrix element will contain the ultraviolet convergent integral

$$I_5[l^\mu l^\nu l^\rho l^\sigma l^\tau]$$

Applying the reduction once we will get terms of the form

$$I_4[l^\mu l^\nu l^\rho l^\sigma],$$

which by power counting in ultraviolet divergent.

Since we started with a UV convergent integral this means that the sum of all the generated UV divergent integrals will end up being finite. But care must be taken to carry out the integrals using a robust regulator (such as dimensional regularisation or dimensional reduction) in order to be sure that the remaining finite terms are correct.

The reduction process can be iterated until the only UV divergent integrals are

$$I_2[1] \;=\; -i(4\pi)^{d/2}\int \frac{d^d l}{(2\pi)^d}\frac{1}{(l^2-m_0^2)((l+q)^2-m_1^2)}$$

†We will see later that some complications can arise if the integrals are either infrared or ultraviolet divergences which need to be regularized, for example through dimensional regularization, which necessitates the inclusion pentagon integrals in five dimensions.

$$I_2[l^\mu] = -i(4\pi)^{d/2} \int \frac{d^d l}{(2\pi)^d} \frac{l^\mu}{(l^2 - m_0^2)((l+q)^2 - m_1^2)}$$

and

$$I_2[l^\mu l^\nu] = -i(4\pi)^{d/2} \int \frac{d^d l}{(2\pi)^d} \frac{l^\mu l^\nu}{(l^2 - m_0^2)((l+q)^2 - m_1^2)}.$$

All the other integrals are "scalar" (triangles, boxes, pentagons, etc).

Each of these integral has an imaginary part in some domains of the invariant square momenta q_i^2.

However, the converse statement, namely that the integrals can be reproduced from knowledge of these imaginary parts has one unfortunate exception arising from the relation (for massless internal particles)

$$I_2(l^\mu l^\nu) - \left(\frac{q^\mu q_1 \nu}{3} - \frac{g^{\mu\nu}}{12} \right) I_2[1] = \frac{1}{18} \left(g^{\mu\nu} q^2 - q^\mu q^\nu \right),$$

which is purely real. This means that precise knowledge of the imaginary part of an integral does *not* specify the coefficient of this (finite) combination of two-point integrals.