Approximation in Physics Problems

Remember: Approximation is meant to make life easier for you!

1 Approximation vs. Estimation

The object of *estimation* is to make educated guesses in the absence of information, to 'get a feel' for the order of magnitude of a required result, to have a first stab at a problem which is complicated to solve in detail.

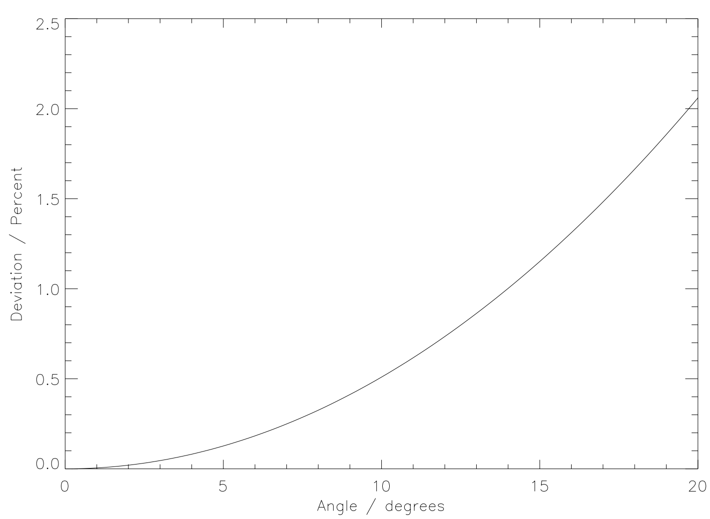
The idea behind *approximation* is to simplify complicated mathematical expressions. So, approximation is often used as a tool in estimation, but also in full-blown calculations.

2 Examples

Below are only a few examples where you can use approximations. Many more applications could be found.

2.1 Small angle approximation

In many situations you will come across the approximations $\sin(\alpha) \sim \alpha$ and $\cos(\alpha) \sim 1$ for 'small' α. What does small mean in this context? The graphic shows the error made in percent by setting $\sin(\alpha) = \alpha$ as a function of α:



So, if your measurement error for α is of order 1%, then you are not introducing an additional error for, say, $\alpha < 10°$. **Important:** In 'final' calculations, approximation should not introduce an error larger than the measurement error.

Exercise: Two stars are both 57 lightyears away from us. Their positions on the sky are separated by an angle $\alpha = 1^\circ \pm 0.1^\circ$.

(a) Calculate the physical distance between the two stars in lightyears using the correct formula involving $\tan(\alpha)$.

(b) Calculate the same distance using a small angle approximation.

(c) Determine the error introduced by the small angle approximation in (b) and compare to the measurement error.

2.2 Binomial expansion

The binomial expansion is given by $(1+x)^n \sim 1+nx,$ for $x \ll 1$. Why is this useful?

Often you come across questions like ‘By how much does the energy stored in a spring, stretched to a length x, increase (in percent) when I extend it by a further, small Δx?’

The energy stored in the spring is $E = kx^2/2$, with $k = \text{const.}$. The question above asks for $\Delta E/E$ where $E + \Delta E = k(x+\Delta x)^2/2$. So, we first have to calculate E and then $E+\Delta E$ and then subtract E from the second result and then divide this by E ...

... or may be not. Consider the following:

$$\begin{aligned} E+\Delta E &= \frac{k}{2}(x+\Delta x)^2 \\ &= \frac{k}{2}x\left(1+\frac{\Delta x}{x}\right)^2 \\ &= E\left(1+\frac{\Delta x}{x}\right)^2 \end{aligned}$$

Since $\Delta x/x$ is small, we use the binomial expansion ...

$$E+\Delta E \sim E\left(1+2\frac{\Delta x}{x}\right) = E + 2E\frac{\Delta x}{x} \tag{1}$$

... and we finally get:

$$\frac{\Delta E}{E} \sim 2\frac{\Delta x}{x}. \tag{2}$$

Much simpler!

Exercise: By what percentage does the gravitational force on a climber change on top of Mt. Everest (8850 m above average sealevel) compared to its value at the bottom? (Radius of Earth at average sealevel: 6378.137 km) Does the gravitational force increase or decrease?

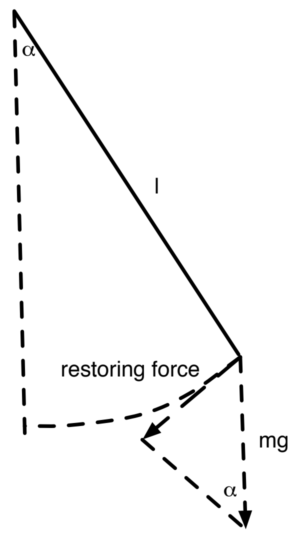
How large is the error arising from the approximation made by using the binomial expansion?

2.3 Taylor expansion

Every mathematical function $f(x)$ that can be differentiated ‘many times’ at the point $x = 0$ is given by

$$f(x) = \sum_{i=0}^{\infty} \frac{\mathrm{d}^i}{\mathrm{d}x^i} f(x)\bigg|_{x=0} \frac{x^i}{i!}. \tag{3}$$

In practice, often only a few or even only one differentiation are needed if $x \ll 1$, because the higher terms decay as $x^i!$



The Taylor expansion is *very* powerful! Consider the simple pendulum above. The restoring force is $mg\sin(\alpha)$ and so the correct equation of motion for the pendulum is:

$$\begin{aligned} m\frac{\mathrm{d}^2(l\alpha)}{\mathrm{d}t^2} + mg\sin(\alpha) &= 0 \\ \frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} + \frac{g}{l}\sin(\alpha) &= 0, \end{aligned}$$

...but we are all very used to using

$$\frac{\mathrm{d}^2\alpha}{\mathrm{d}t^2} + \frac{g}{l}\alpha = 0 \tag{4}$$

instead. We can simply accept that this follows from the small angle approximation above. Or, we can show that this simplification, and therefore the small angle approximation, follow from the Tyalor expansion of $\sin(\alpha)$:

$$\sin(\alpha) = \sum_{i=0}^{\infty} \frac{\mathrm{d}^i}{\mathrm{d}\alpha^i}\sin(\alpha)\bigg|_{\alpha=0}\frac{\alpha^i}{i!} \sim \sum_{i=0}^{i=1} \frac{\mathrm{d}^i}{\mathrm{d}\alpha^i}\sin(\alpha)\bigg|_{\alpha=0}\frac{\alpha^i}{i!} = \sin(\alpha)|_{\alpha=0} + \cos(\alpha)|_{\alpha=0}\,\alpha = \alpha. \tag{5}$$

Exercise: Show that the small angle approximation is still correct even for $i = 2$.

Show that the binomial expansion is another special case of the Taylor expansion.

The kinetic energy of a particle with rest mass m travelling at velocity v is given by $K = (\gamma - 1)mc^2$, where the Lorentz factor $\gamma = 1/\sqrt{1 - v^2/c^2}$. Using the Taylor expansion of γ show that for $v \ll c$ this reduces to the non-relativistic form $K = mv^2/2$.