

Example 1.1: Energy of an Extended Spring

The restoring force on a spring with extension x is given by

$$F = -kx,$$

where k is the spring constant.

How much work is needed to stretch the spring from extension x_1 to an extension x_2

If we move a force F through a distance x (between x_1 and x_2), the work done is $F(x_2 - x_1)$. But in this case, the force is varying with the extension, so this would give the wrong answer.

On the other hand, if we consider a spring whose extension is x and stretch it further by an infinitesimal amount dx , we can neglect the change in the force over that infinitesimal extension (the error can be shown to be $\mathcal{O}(dx^2)$).

Draw a diagram, and write down the infinitesimal work done, dW , for this infinitesimal process.

The total work done on the spring when it is extended from $x = x_1$ to $x = x_2$ is then

$$W \approx \sum dW,$$

where the sum goes over all the infinitesimal steps needed to stretch the spring from $x = x_1$ to $x = x_2$.

The approximation becomes an equality when we take the limit, where $dx \rightarrow 0$ and the number of steps, N , required to stretch the spring goes to infinity in such a way that the product Ndx remains fixed and is equal to $(x_2 - x_1)$.

What does this sum become? Express your answer in terms of an integral and perform the integration.

Example 1.2: Non-linear Extended Spring

A non-linear spring has a force $F(x)$ which depends on the extension according to

$$F(x) = -kx - \alpha x^3$$

Sketch the graph of $-F(x)$ against x and draw a small rectangle at position x of width dx and height $-F(x)$.

How much work needs to be done in this case to stretch the spring from x_1 to x_2 ?

Example 1.3: Potential energy of building structures

A hollow pyramid has a square base with side of length 20 m and a height of 15 m. The perimeter decreases linearly with height, vanishing at the top of the pyramid. The pyramid is made of small bricks of depth 15 cm. and density 5000 kg m^{-3} . Write down the expression for the length of the square perimeter of the pyramid at height h .

Hence, estimate the total potential energy of the bricks that make up the pyramid.

Example 1.4: A non-uniformly accelerating car

For motion in one direction, write down the relation (in terms of differentials) between acceleration, a , velocity, v and distance travelled, x .

Hence show by integrating twice that an object which starts from rest at time $t = 0$ and accelerates uniformly with acceleration, a , covers a distance

$$x = \frac{1}{2}at^2$$

in time t .

A careful driver presses her foot on the accelerator very slowly so that the acceleration of the car as a function of time, is given by

$$a = 2(1 - e^{-t/8}) \text{ m s}^{-2}$$

How far does the car travel in the first 10 seconds (starting from rest)?

Example 1.5: A uniformly charged rod

A horizontal rod of length l has a uniform charge density ρ C/m. What is the electric field at a point P vertically above the centre of the rod at a distance D from the centre?

Draw the diagram.

Consider a thin segment of the rod at a distance x from the centre of width dx . Calculate (using Pythagoras' theorem) the distance of this segment from the point P and hence the electric field due to the segment (*remember that electric field is a vector*).

Now sum over all the segments and determine the total electric field at this point. Express your answer in terms of a definite integral. (Why can we simply sum the contributions from each segment in this case - what would we have to do if we were to calculate the electric field?)

You can't do the integral? Never mind - the objective is to get the answer in terms of an integral. If you can't do the integral you can look it up in a table of integrals or use an Algebraic Computer package such as MAPLE or MATHEMATICA. Many integrals cannot be done analytically and have to be calculated numerically.

In this case the necessary integral is given in the list of integrals. In order to cast your result into terms of this integral, you need to make a simple change of variable from x to y . You must remember that dx needs to be converted into dy and the limits $-a$ and a must be expressed in terms of D and l . Calculate the integral.

Averages and Probabilities:

Another important use of integration is the calculation of the average value of a continuous quantity, given a probability distribution.

If I throw a balanced dice, the probability of it landing on any one number is $1/6$. This means that if I throw it a large number of times, I expect the average value over all the throws to be

$$\bar{n} = \frac{1}{6} \times 1 + \frac{1}{6} \times 2 + \cdots = \frac{1}{6} \sum_{n=1}^6 n = 3.5.$$

On the other hand if I have a loaded dice for which the probability of it landing on the number n is $P(n)$ then the average becomes

$$\bar{n} = \sum_{n=1}^6 n P(n).$$

We can also have die with more than 6 sides (say N sides), in which case this average becomes

$$\bar{n} = \sum_{n=1}^N n P(n).$$

In the limit $N \rightarrow \infty$, and some continuous variable x rather than the number n , we talk of a probability density $P(x)$, which means that the probability of the variable x lying in the region x to $x + dx$ is

$$P(x)dx.$$

The average value of x is then given by

$$\bar{x} = \int x P(x) dx,$$

where the integral goes over the entire range that the variable x can take.

Example 2.1: Average speed of a molecule in a gas

The probability of a molecule of mass m in a non-interacting gas at temperature T (K) having a magnitude of velocity (speed) between v and $v + dv$ is given by the Maxwell-Boltzmann distribution

$$P(v)dv = Nv^2 \exp(-mv^2/2kT) dv,$$

where k is the Boltzmann constant, and N is a constant.

We can calculate the constant N from the fact that the total probability of the molecule having a speed between $v = 0$ and $v = \infty$ is one (it must have *some* speed), so that

$$\int_0^\infty P(v)dv = 1$$

(a) What is N ?

(b) Now calculate the average speed.

Can't do the integral?

$$\int_0^\infty x^3 e^{-x^2} dx = \frac{1}{2}$$

(again you will need a change of variable).

(c) What is the average value, $\overline{v^2}$, of the *square* of the speed?

Example 2.2: Average energy of a molecule in a gas

We can arrive at the same result as part (c) above from the probability distribution in terms of the energy, E , of a molecule. This is given by

$$P(E)dE = A\sqrt{E}\exp(-E/kT)dE$$

(a) What is the constant A ?

You don't need any new integrals. By making a suitable change of variables you can cast the integral into one that has previously been given.

(b) Now calculate the average energy $\overline{E}$.

(c) What is the relation between $\overline{E}$ and $\overline{v^2}$? Is this what you would expect?

(**Hint:** What is the relation between the energy of a molecule in a non-interacting gas and its speed?)

More Practice

For each problem remember:

- Draw a diagram
- Think about the physics
- For numerical questions, do not put in numbers until the very end. Define symbols for all the required quantities and state clearly in your presentation of the solution, what these symbols mean.

Example 2.3: A non-uniformly charged rod

A charged horizontal rod of length l has a non-uniform charge density given by

$$\rho = \alpha|x|$$

where x is the distance from the centre of the rod, i.e. the charge density is zero at the centre of the rod and increases linearly up to $\alpha l/2$ at the ends.

What is the electric field at the point P vertically above the centre of the rod, at a distance D from the rod?

N.B. To deal with $|x|$ (as opposed to x) you will need to split the integral into two pieces.

Example 2.4: A particle wavefunction

The wavefunction for a particle moving in an infinite one-dimensional potential well of width L , in its lowest energy level, is given by

$$\begin{aligned}\Psi(x) &= \sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{L}\right), \quad \text{for } 0 < x < L \\ &= 0, \quad \text{for } x < 0 \text{ or } x > L\end{aligned}$$

The probability, $P(x)dx$, of finding the particle between x and $x + dx$ is related to the wavefunction by

$$P(x) = |\Psi(x)|^2.$$

Find the average value of x for this particle.

Example 2.5: Volume and mass of a sphere

(a) The surface area of a sphere of radius r is given by

$$S = 4\pi r^2.$$

Consider a solid sphere as consisting of thin shells of radius r and thickness dr (like an onion) with r ranging from $r = 0$ to $r = R$. Calculate the volume of a sphere of radius R .

(b) A sphere of radius R is composed of infinitesimally thin shells of decreasing density, such that the density at a distance r from the centre is given by

$$\rho = \rho_0 e^{-\alpha r}.$$

What is the mass of the sphere? Express your answer in terms of an integral of the form

$$\int_0^a f(x)dx,$$

where $f(x)$ is a function of x *only* (i.e. perform the necessary change of variables).

Example 2.6: Potential energy of building structures

A (hollow) hemispherical dome of radius 30 m is made from a material of density 3000 kg m^{-3} . The thickness of the dome is 10 cm. Calculate the radius and height of the slice of the dome for which the line from the centre of the sphere and the circumference of the slice makes an angle between θ and $\theta + d\theta$ with the vertical. Hence calculate the total potential energy of the material of the dome.

$$(g = 9.8 \text{ m s}^{-2})$$

Example 2.7: Compressing a perfect gas

How much work is done when a perfect gas at initial pressure P_2 is compressed adiabatically from a volume V_2 to a volume V_1 ?

$$P_1 V_1^\gamma = P_2 V_2^\gamma$$

LIST OF INTEGRALS

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$$\int_{-a}^a \frac{dy}{(1+y^2)^{3/2}} = \frac{2a}{\sqrt{1+a^2}}$$

•

$$\int_0^\infty x^2 e^{-x^2} dx = \frac{\sqrt{\pi}}{4}.$$

•

$$\int_0^\infty x^4 e^{-x^2} dx = \frac{3\sqrt{\pi}}{8}$$

•

$$\int_0^a \frac{x}{(1+x^2)^{3/2}} dx = 1 - \frac{1}{\sqrt{1+a^2}}.$$

•

$$\int_0^\pi \theta \sin^2 \theta d\theta = \frac{\pi^2}{4}$$