Solutions

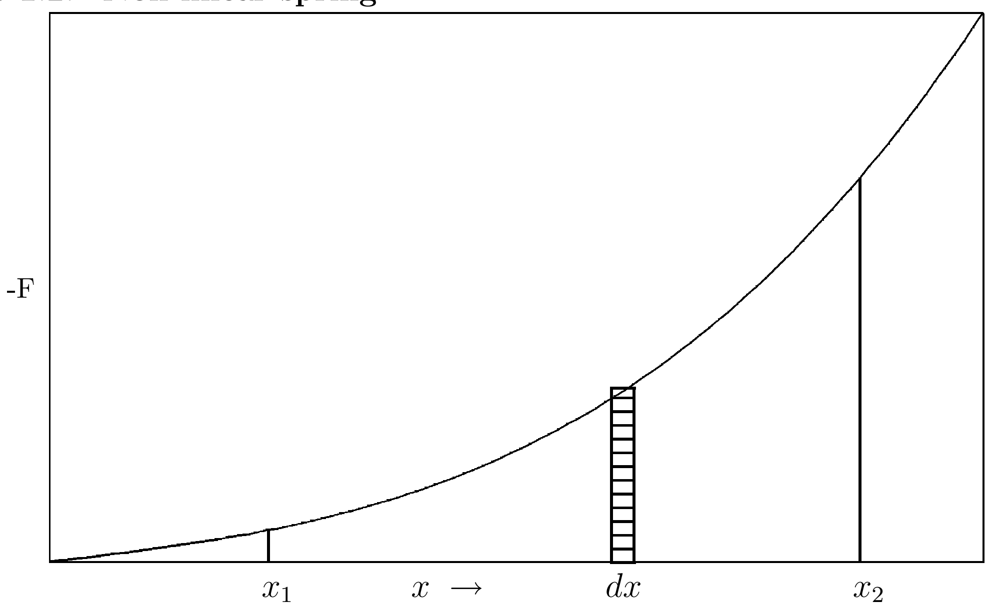
Example 1.1: Energy of an Extended spring

$$dW \;=\; k\,x\,dx$$

(positive sign because we are considering work done on the spring)

$$W \;=\; \int_{x_1}^{x_2} dW \;=\; \int_{x_1}^{x_2} k\,dx \;=\; \frac{k}{2}\left(x_2^2 - x_1^2\right)$$

Example 1.2: Non-linear spring

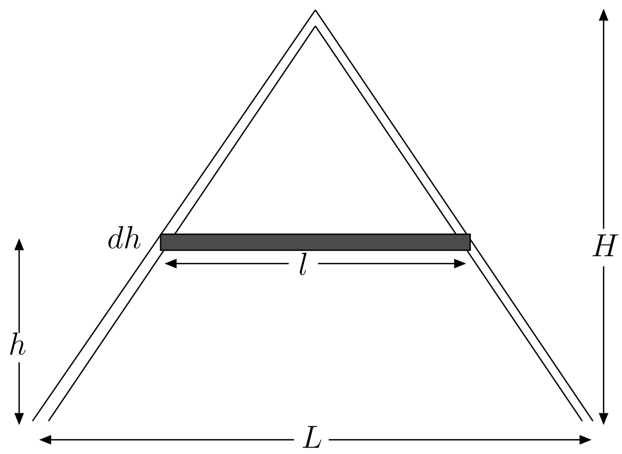


$$dW \;=\; \left(k\,x \,+ \alpha\,x^3\right) dx$$

(positive sign because we are considering work done on the spring)

$$W \;=\; \int_{x_1}^{x_2} dW \;=\; \int_{x_1}^{x_2} \left(k\,x \,+ \alpha\,x^3\right)\,dx \;=\; \left(\frac{k}{2}\left(x_2^2 - x_1^2\right) \,+\, \frac{\alpha}{4}\left(x_2^4 - x_1^4\right)\right)$$

Example 1.3: Potential energy of building structures
Consider a slice of the pyramid at a height h above the ground.



Since the length, l of the side of a slice decreases linearly with height, from $l = L$ at height $h = 0$, to $l = 0$ at $h = H$, where H and L are the height and base length of the pyramid, respectively, l is given by

$$l = L\left(1 - \frac{h}{H}\right)$$

The mass of the slice is given by

$$dM = 4\rho\, l\, d\, dh = 4\rho\, L\, d\left(1 - \frac{h}{H}\right) dh,$$

where ρ is the density and d is the depth of the bricks.

The potential energy of the slice is

$$dV = dM\, g\, h = 4\rho\, L\, d\, h\left(1 - \frac{h}{H}\right) dh$$

The total potential energy of the bricks in the pyramid is

$$V = \int_0^H dV = 4\rho\, g\, L\, d\int_0^H h\left(1 - \frac{h}{H}\right) dh = 4\rho\, g\, L\, d\, H^2\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2}{3}\rho\, g\, L\, d\, H^2$$

Inserting numbers

$$V = \frac{2}{3} \times 5000 \times 9.8 \times 20 \times 0.15 \times (15)^2 = 2.2 \times 10^7\, J.$$

Example 1.4: A non-uniformly accelerating car
(a)

$$v = \frac{dx}{dt}$$

$$a = \frac{dv}{dt}$$

For uniform acceleration starting from rest, at time t

$$v(t) = \int_0^t a\,dt' = a\,t$$

$$x(t) = \int_0^t v(t')\,dt' = \int_0^t a\,t'\,dt' = \frac{a}{2}t^2$$

(b)

$$\frac{dv}{dt} = a(t) = 2\left(1 - e^{t/8}\right)$$

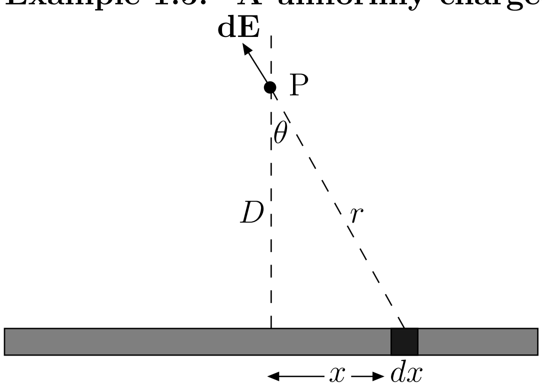
$$\frac{dx}{dt} = v(t) = \int_0^t a(t')dt' = \int_0^t 2\left(1 - e^{t'/8}\right)dt' = 2t + 16\left(e^{-t/8} - 1\right)$$

$$x(t) = \int_0^t v(t')dt' = \int_0^t \left(2t' + 16\left(e^{-t'/8} - 1\right)\right)dt' = t^2 - 16t - 128\left(e^{-t/8} - 1\right)$$

At $t = 10$

$$x(10) = 100 - 160 - 128 \times (.286 - 1) = 31\,m$$

Example 1.5: A uniformly charged rod



Charge of small segment of width dx

$$dQ = \rho dx$$

Distance r of segment at x from point P

$$r = \sqrt{D^2 + x^2}$$

Electrostatic field due to segment has a *magnitude*

$$dE = \frac{dQ}{4\pi\epsilon_0 r^2} = \frac{\rho dx}{4\pi\epsilon_0 (D^2 + x^2)}$$

But **dE** is a vector quantity, so we must consider its *direction* as well as its magnitude. By symmetry the resultant electric field only has a component in the vertical direction and so we would need to calculate the component of the electric field in the that direction, dE_V, from each segment.

$$dE_V = dE \cos\theta,$$

where θ is the angle between the vertical and the line from the segment t0 the point P. It is given by

$$\cos\theta = \frac{D}{r} = \frac{D}{\sqrt{D^2 + x^2}}$$

The *magnitude* of the total electric field is then

$$E = \int_{-l/2}^{l/2} \frac{\rho D}{4\pi\epsilon_0 (D^2 + x^2)^{3/2}} dx$$

Change of variable required

$$x = D\,y, \quad dx = D\,dy$$

Limits $y = \pm l/(2D)$, so that finally we get

$$E = \int_{-l/(2D)}^{l/(2D)} \frac{\rho}{4\pi\epsilon_0 D(1 + y^2)^{3/2}} dy = \frac{\rho}{2\pi\epsilon_0 D\sqrt{1 + (l/2D)^2}} = \frac{\rho}{\pi\epsilon_0 \sqrt{l^2 + 4D^2}}$$

Example 2.1: Average speed of a molecule in a gas
(a)

$$\int_0^\infty P(v)dv \;=\; \int_0^\infty N v^2 \exp\left(-mv^2/2kT\right) dv \;= 1$$

Change of variables required

$$v \;=\; \sqrt{\frac{2kT}{m}}x, \quad dv \;=\; \sqrt{\frac{2kT}{m}}dx$$

$$N\left(\frac{2kT}{m}\right)^{3/2}\int_0^\infty x^2 e^{-x^2}dx \;=\; N\left(\frac{2kT}{m}\right)^{3/2}\frac{\sqrt{\pi}}{4} \;=\; 1$$

$$N \;=\; \sqrt{\frac{2}{\pi}}\left(\frac{m}{kT}\right)^{3/2}$$

(b)

$$\overline{v} \;=\; \int_0^\infty vP(v)dv \;=\; \int_0^\infty N v^3 \exp\left(-mv^2/2kT\right) dv$$

Change of variables required

$$v \;=\; \sqrt{\frac{2kT}{m}}x, \quad dv \;=\; \sqrt{\frac{2kT}{m}}dx$$

$$\overline{v} \;=\; N\left(\frac{2kT}{m}\right)^{2}\int_0^\infty x^3 e^{-x^2}dx \;=\; \frac{N}{2}\left(\frac{2kT}{m}\right)^{2} \;=\; \frac{2}{\sqrt{\pi}}\sqrt{\left(\frac{2kT}{m}\right)}$$

(c)

$$\overline{v^2} \;=\; \int_0^\infty v^2 P(v)dv \;=\; \int_0^\infty N v^4 \exp\left(-mv^2/2kT\right) dv$$

Change of variables required

$$v \;=\; \sqrt{\frac{2kT}{m}}x, \quad dv \;=\; \sqrt{\frac{2kT}{m}}dx$$

$$\overline{v^2} \;=\; N\left(\frac{2kT}{m}\right)^{5/2}\int_0^\infty x^4 e^{-x^2}dx \;=\; \frac{3N\sqrt{\pi}}{8}\left(\frac{2kT}{m}\right)^{5/2} \;=\; 3\frac{kT}{m}$$

Example 2.2: Average energy of a molecule in a gas
(a)

$$\int_0^\infty P(E)dE \;=\; \int_0^\infty A\sqrt{E}e^{-E/kT}dE \;=\; 1$$

Change of variables required

$$E \;=\; kTx^2, \quad dE \;=\; 2kT\,x\,dx$$

$$2A\,(kT)^{3/2}\int_0^\infty x^2 e^{-x^2}dx \;=\; \frac{A\sqrt{\pi}}{2}\,(kT)^{3/2} \;=\; 1$$

$$A \;=\; \frac{2}{\sqrt{\pi}}\,(kT)^{-3/2}$$

(b)

$$\overline{E} \;=\; \int_0^\infty EP(E)dE \;=\; \int_0^\infty AE^{3/2}e^{-E/kT}dE$$

Change of variables required

$$E \;=\; kTx^2, \quad dE \;=\; 2kT\,x\,dx$$

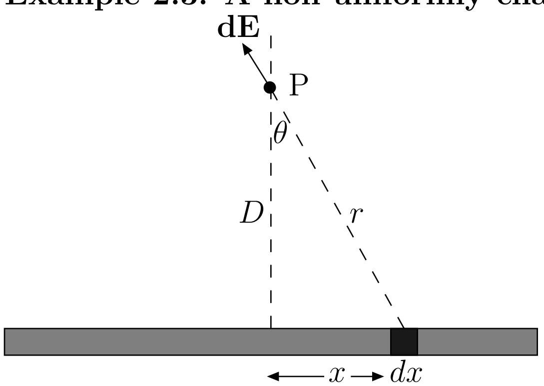
$$\overline{E} \;=\; =\; 2A(kT)^{5/2}\int_0^\infty x^4 e^{-x^2}dx = 2A(kT)^{5/2}\frac{3\sqrt{\pi}}{8} \;=\; \frac{3}{2}kT$$

(c)

$$\overline{E} \;=\; \frac{1}{2}m\overline{v^2}.$$

This is expected as the energy, E, of a molecule in a non-interacting gas is $E = \frac{1}{2}mv^2$.

Example 2.3: A non-uniformly charged rod



Charge of small segment of width dx

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Distance r of segment at x from point P

$$r \;=\; \sqrt{D^2 + x^2}$$

Electrostatic field due to segment has a *magnitude*

$$dE \;=\; \frac{dQ}{4\pi\epsilon_0 r} \;=\; \frac{\rho dx}{4\pi\epsilon_0(D^2+x^2)}$$

But **dE** is a vector quantity, so we must consider its *direction* as well as its magnitude. By symmetry the resultant electric field only has a component in the vertical direction and so we would need to calculate the component of the electric field in the that direction, dE_V, from each segment.

$$dE_V \;=\; dE\cos\theta,$$

where θ is the angle between the vertical and the line from the segment t0 the point P. It is given by

$$\cos\theta \;=\; \frac{D}{r} \;=\; \frac{D}{\sqrt{D^2+x^2}}$$

The *magnitude* of the total electric field is then

$$E \;=\; \int_{-l/2}^{l/2} \frac{\alpha|x|D}{4\pi\epsilon_0(D^2+x^2)^{3/2}}dx$$

Split the integral into two ranges

$$-l/2 \;<\; x \;<\; 0, \quad \text{where} \quad |x| = -x$$

$$0 \;<\; x \;<\; l/2, \quad \text{where} \quad |x| = +x$$

$$E \;=\; -\int_{-l/2}^{0} \frac{\alpha x D}{4\pi\epsilon_0(D^2+x^2)^{3/2}}\,dx \;+\; \int_{0}^{l/2} \frac{\alpha x}{4\pi\epsilon_0(D^2+x^2)^{3/2}}\,dx$$

In the first integral change variables $x \to -x$, $dx \to -dx$ and reverse the limits. The first integral becomes identical to the second integral and we get

$$E \;=\; 2\int_{0}^{l/2} \frac{\alpha x}{4\pi\epsilon_0(D^2+x^2)^{3/2}}\,dx$$

Change variables to, $y \;=\; D^2+x^2$, $\;xdx \;=\; \frac{1}{2}dy$, limits $D^2 < y < (l/2)^2 + D^2$) to get

$$E \;=\; \frac{2\alpha}{4\pi\epsilon_0}\int_{D^2}^{(l/2)^2+D^2} \frac{1}{y^{3/2}}dy \;=\; -\left.\frac{\alpha}{\pi\epsilon_0\sqrt{y}}\right|_{D^2}^{(l/2)^2+D^2} \;=\; \frac{\alpha}{\pi\epsilon_0}\left(\frac{1}{D} - \frac{4}{\sqrt{l^2+4D^2}}\right)$$

Example 2.4: A particle wavefunction

$$P(x) \;=\; \frac{2}{L}\sin^2\left(\frac{\pi\,x}{L}\right), \quad 0 \;<\; x \;<\; L$$

(zero otherwise).

$$\overline{x} \;=\; \frac{2}{L}\int_0^L x\sin^2\left(\frac{\pi\,x}{L}\right)dx.$$

Change variables to θ where

$$x \;=\; \frac{L}{\pi}\theta, \quad dx \;=\; \frac{L}{\pi}d\theta, \quad \text{limits } 0 \;<\; \theta \;<\; \pi$$

$$\overline{x} \;=\; 2\frac{L}{\pi^2}\int_0^{\pi}\theta\sin^2\theta d\theta \;=\; \frac{2L}{\pi^2}\frac{\pi^2}{4} \;=\; \frac{L}{2}$$

Example 2.5: Volume and mass of a sphere
(a)

$$S(r) \;=\; 4\pi r^2$$

$$V \;=\; \int_0^R S(r)\,dr \;=\; 4\pi \int_0^R r^2 2dr \;=\; \frac{4\pi}{3} R^3$$

(b) Mass of a shell of radius r and thickness dr is

$$dM \;=\; 4\pi r^2 \rho(r) dr \;=\; 4\pi r^2 \rho_0 e^{-\alpha r} dr$$

Total mass

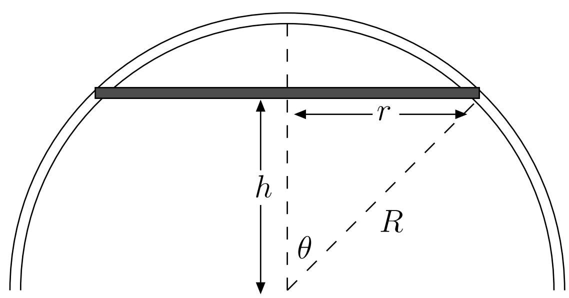
$$M \;=\; \int_0^R dM \;=\; 4\pi\rho_0 \int_0^R r^2 e^{-\alpha r} dr$$

Change variables,

$$r \;=\; \frac{x}{\alpha}, \quad dr \;=\; \frac{dx}{\alpha}$$

$$M \;=\; \frac{4\pi\rho_0}{\alpha^3} \int_0^{\alpha R} x^2 e^{-x} dx$$

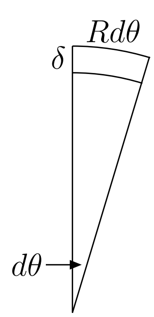
Example 2.6: Potential energy of building structures
Consider a slice of the dome, whose circumference makes and angle θ with the vertical.



The radius of this slice is

$$r \;=\; R\sin\theta,$$

where R is the radius of the dome.



The mass of the slice between angles θ and $\theta + d\theta$ is

$$dM \;=\; 2\pi\,\rho\,r\,\delta\,R\sin\theta \;=\; 2\pi\,\rho\,\delta R^2 \sin\theta d\theta,$$

where ρ is the density and δ is the thickness of the dome.

The height of the slice is

$$h \;=\; R\cos\theta$$

Therefore the potential energy of the slice is

$$dV \;=\; dM\,g\,h \;=\; 2\pi\,\rho\,\delta\,g\,R^3 \sin\theta\,\cos\theta\,d\theta$$

The total potential energy of the material of the dome is

$$V \;=\; \int_0^{\pi/2} dV \;=\; 2\pi\,\rho\,\delta\,g\,R^3 \int_0^{\pi/2} \sin\theta\,\cos\theta\,d\theta \;=\; 2\pi\,\rho\,\delta\,g\,R^3 \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} \;=\; \pi\rho\,\delta\,g\,R^3$$

Inserting numbers

$$V \;=\; \pi \times 3000 \times 0.1 \times 9.8 \times (30)^3 \;=\; 2.5 \times 10^8\,J$$

Example 2.7: Compressing a perfect gas

For an adiabatic change in volume

$$P\,V^\gamma \text{ is constant}$$

If gas is initially at pressure P_2 and occupies volume V_2 then when the gas occupies volume V, the pressure is

$$P(V) \;=\; P_2 V_2^\gamma V^{-\gamma}$$

Work done *on* a gas at pressure P when its volume changes by an infinitesimal amount dV

$$dW \;=\; -P\,dV$$

Total work done is

$$W \;=\; -\int_{V_2}^{V_1} P(V)\,dV \;=\; -\int_{V_2}^{V_1} P_2 V_2^\gamma V^{-\gamma} dV \;=\; \frac{P_2 V_2}{(\gamma-1)}\left(\left(\frac{V_2}{V_1}\right)^{\gamma-1} - 1\right)$$