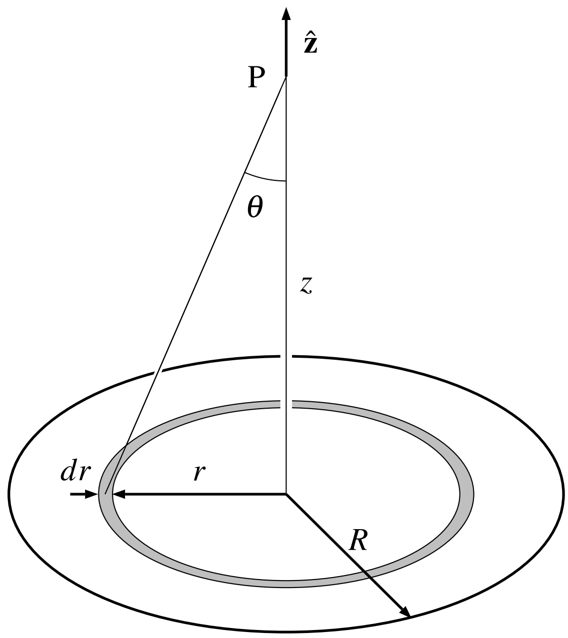
Example: Electric Field Due to a Charged Disk A circular disk of radius R has a uniform surface charge density σ. What is the electric field at a point P, at distance z from the disk along its central axis?



Because of the rotational symmetry about the central axis, $\mathbf{E}$ will lie along the axis, $\mathbf{E} = E_z\hat{\mathbf{z}}$. Moreover, all points in a thin ring of width dr at radius r in the disk are at the same distance from P and will give the same contribution to E_z. Hence we can write the contribution from the ring as

$$dE_z = \frac{2\pi r\,dr\,\sigma}{4\pi\varepsilon_0(r^2+z^2)}\cos\theta.$$

Here, $2\pi r\,dr\,\sigma$ is the charge on the thin ring. The angle θ is marked on the figure: the $\cos\theta$ factor is used to pick up the electric field component along the axis. Using $\cos\theta = z/(r^2+z^2)^{1/2}$, we find,

$$dE_z = \frac{\sigma z}{2\varepsilon_0}\frac{r\,dr}{(r^2+z^2)^{3/2}}.$$

To get the total field we integrate over the whole disk:

$$E_z = \frac{\sigma z}{2\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}} = \frac{\sigma z}{2\varepsilon_0}\left[\frac{-1}{(r^2+z^2)^{1/2}}\right]_0^R = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{z}{(R^2+z^2)^{1/2}}\right].$$

The field is plotted below as a function of z. Let's check two limiting cases. First, let $z\to 0$ (or let $R\to\infty$ with z fixed). This corresponds to a disk of infinite size, so we find the field of an infinite plane sheet of charge with uniform surface charge density. As $z\to 0$, the last term in square brackets in E_z drops out, so

$$E_z \overset{z\to 0}{\rightarrow} \frac{\sigma}{2\varepsilon_0}.$$

This is the correct result for an infinite plane of charge. The other limit is to let $z\to\infty$. As we get further and further from the disk it looks more and more like a single point charge. We reexpress E_z as

$$E_z = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{1}{(1+R^2/z^2)^{1/2}}\right].$$

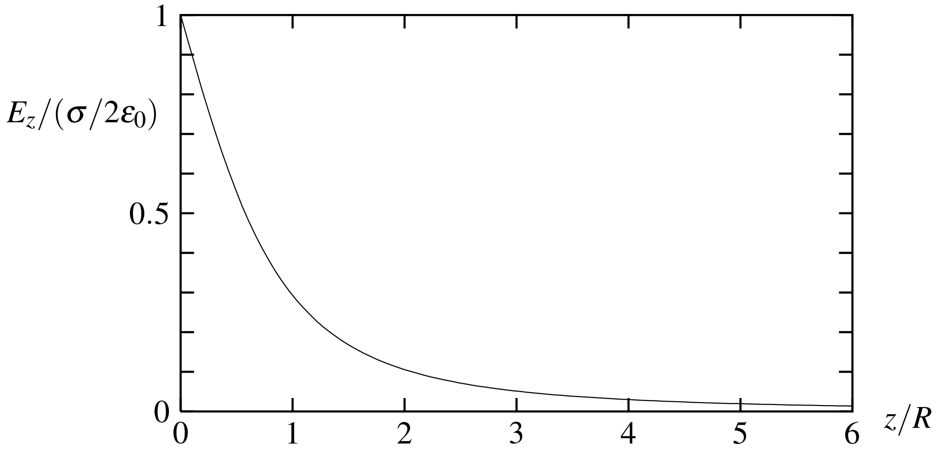
The quantity R^2/z^2 gets smaller and smaller as z increases, so we have to do a binomial expansion of the last term in the square brackets,

$$\frac{1}{(1+R^2/z^2)^{1/2}} = 1 - \frac{1}{2}\frac{R^2}{z^2} + \cdots .$$

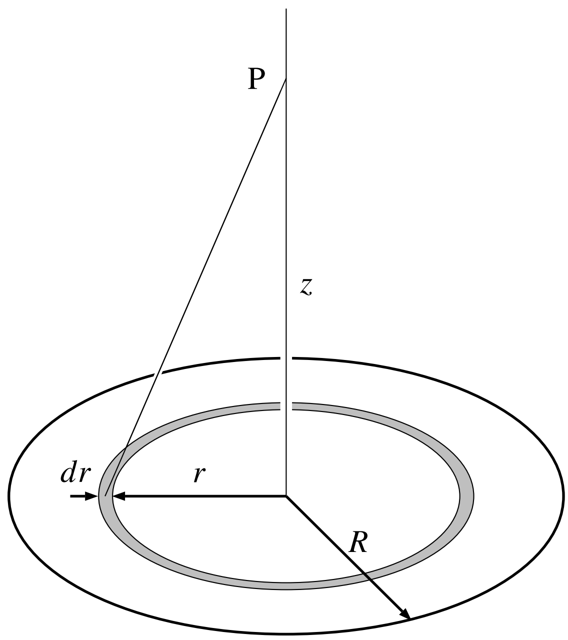
Dropping higher order terms, we find

$$E_z \overset{z\to\infty}{\to} \frac{\sigma R^2}{4\varepsilon_0 z^2} = \frac{Q}{4\pi\varepsilon_0 z^2},$$

where $Q = \pi R^2 \sigma$ is the total charge on the disk. This is just the expected result for the field of a point charge Q.



Example: Electric Potential Due to a Charged Disk A circular disk of radius R has a uniform surface charge density σ. What is the electric potential at a point P, at distance z from the disk along its central axis?



All points in a thin ring of width dr at radius r in the disk are at the same distance from P and will give the same contribution to the potential V. The charge in the ring is

$$dq = 2\pi r\, dr\, \sigma.$$

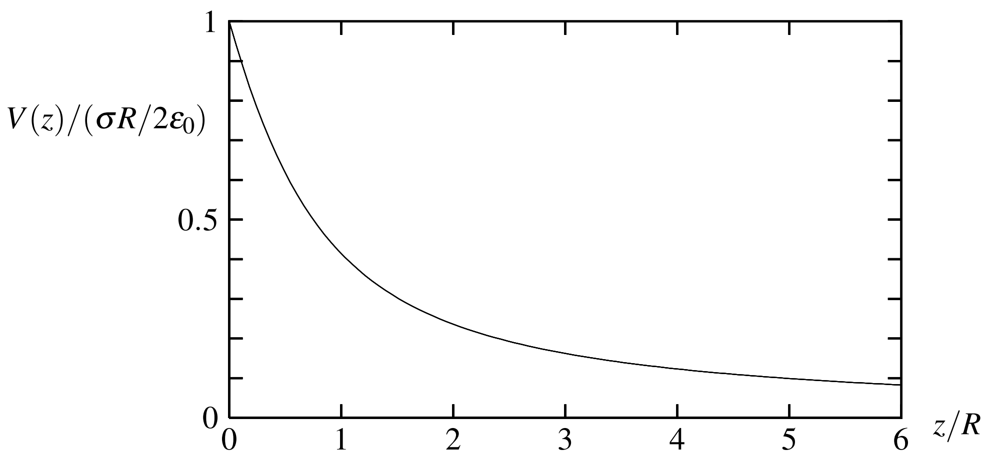
Its contribution to the potential is (using the result for a point charge)

$$dV = \frac{dq}{4\pi\varepsilon_0 (r^2+z^2)^{1/2}} = \frac{\sigma}{2\varepsilon_0}\frac{r\,dr}{(r^2+z^2)^{1/2}}.$$

Now we integrate over the whole disk:

$$V(z) = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{r\,dr}{(r^2+z^2)^{1/2}} = \frac{\sigma}{2\varepsilon_0}\left[(r^2+z^2)^{1/2}\right]_0^R = \frac{\sigma}{2\varepsilon_0}\left[(R^2+z^2)^{1/2} - z\right].$$

The potential is plotted below as a function of z.



Let's check the limiting case where we are very far from the disk. As we get further and further away, it looks more and more like a single point charge, so we expect the potential to look like that of a point charge. Rewrite V as,

$$V(z) = \frac{\sigma z}{2\varepsilon_0}\left[(1+R^2/z^2)^{1/2} - 1\right].$$

The quantity R^2/z^2 gets smaller and smaller as z increases, so we have to do a binomial expansion,

$$(1+R^2/z^2)^{1/2} = 1 + \frac{1}{2}\frac{R^2}{z^2} + \cdots.$$

Dropping higher order terms, we find

$$V(z) \overset{z\to\infty}{\to} \frac{\sigma z}{2\varepsilon_0}\frac{R^2}{2z^2} = \frac{\sigma R^2}{4\varepsilon_0 z} = \frac{Q}{4\pi\varepsilon_0 z},$$

where $Q = \pi R^2 \sigma$ is the total charge on the disk. This is just the expected result for the potential of a point charge Q.

In an earlier example we calculated the electric field on axis. You should check that $-dV/dz$ gives the same field we found before.