ADVANCED QUANTUM FIELD THEORY

Syllabus

- Non-Abelian gauge theories
- Higher order perturbative corrections in ϕ^3 theory
- Renormalization
- Renormalization in QED
- The renormalization group - $\beta-$functions
- Infrared and collinear singularities
- Causality, unitarity and dispersion relations.
- Anomalies

1 Non-Abelian Gauge Theories

1.1 QED as an Abelian Gauge Theory

Gauge transformations
Consider the Lagrangian density for a free Dirac field ψ:

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi \tag{1.1}$$

Now this Lagrangian density is invariant under a phase transformation of the fermion field

$$\psi \rightarrow e^{i\omega}\psi,$$

since the conjugate field $\overline{\psi}$ transforms as

$$\overline{\psi} \rightarrow e^{-i\omega}\overline{\psi}.$$

The set of all such phase transformations is called the "group $U(1)$" and it is said to be "Abelian" which means that any two elements of the group commute. This just means that

$$e^{i\omega_1}e^{i\omega_2} = e^{i\omega_2}e^{i\omega_1}.$$

For the purposes of these lectures it will usually be sufficient to consider infinitesimal group transformations, i.e. we assume that the parameter ω is sufficiently small that we can expand in ω and neglect all but the linear term. Thus we write

$$e^{i\omega} = 1 + i\,\omega + \mathcal{O}(\omega^2).$$

Under such infinitesimal phase transformations the field ψ changes by $\delta\psi$, where

$$\delta\psi = i\,\omega\,\psi,$$

and the conjugate field $\overline{\psi}$ by $\delta\overline{\psi}$, where

$$\delta\overline{\psi} = -\,i\,\omega\,\overline{\psi},$$

such that the Lagrangian density remains unchanged (to order ω).

Now suppose that we wish to allow the parameter ω to depend on space-time. In that case, infinitesimal transformations we have

$$\delta\psi(x) = i\,\omega(x)\,\psi(x), \tag{1.2}$$

$$\delta\overline{\psi}(x) \;=\; -\,i\,\omega(x)\,\overline{\psi}(x). \tag{1.3}$$

Such local (i.e. space-time dependent) transformations are called "gauge transformations". Note now that the Lagrangian density (1.1) is *no longer* invariant under these transformations, because of the partial derivative that is interposed between $\overline{\psi}$ and ψ, which will act on the space-time dependent parameter $\omega(x)$, such that the Lagrangian density changes by an amount $\delta\mathcal{L}$, where

$$\delta\mathcal{L} \;=\; -\overline{\psi}(x)\,\gamma^\mu\,(\partial_\mu\omega(x))\,\psi(x). \tag{1.4}$$

It turns out that we can repair the damage if we assume that the fermion field interacts with a vector field A_μ, called a "gauge field", with an interaction term

$$-\,e\,\overline{\psi}\,\gamma^\mu A_\mu\psi$$

added to the Lagrangian density which now becomes

$$\mathcal{L} \;=\; \overline{\psi}\;(i\gamma^\mu\,(\partial_\mu + i\,e\,A_\mu) - m)\,\psi. \tag{1.5}$$

In order for this to work we must also assume that apart the fermion field transforming under a gauge transformation according to (1.2, 1.3), the gauge field, A_μ, also changes by δA_μ where

$$\delta A_\mu(x) \;=\; -\frac{1}{e}\partial_\mu\,\omega(x). \tag{1.6}$$

This change exactly cancels with eq.(1.4), so that once this interaction term has been added the gauge invariance is restored.

We recognize eq.(1.5) as being the fermionic part of the Lagrangian density for QED, where e is the electric charge of the fermion and A_μ is the photon field.

In order to have a proper Quantum Field Theory, in which we can expand the photon field, A_μ, in terms of creation and annihlation operators for photons, we need a kinetic term for the field, A_μ, i.e. a term which is quadratic in the derivative of the field. We need to ensure that in introducing such a term we do not spoil the invariance under gauge transformations. This is achieved by defining the field strength, $F_{\mu\nu}$ as

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{1.7}$$

It is easy to see that under the gauge transformation (1.6) each of the two terms on the R.H.S. of eq.(1.7) changes, but the changes cancel out. Thus we may add to the Lagrangian any term which depends on $F_{\mu\nu}$ (and which is Lorentz invariant - so we must contract all Lorentz indices). Such a term is $aF_{\mu\nu}F^{\mu\nu}$, which gives the desired term which is quadratic in the derivative of the field A_μ, and furthermore if we choose the constant a to be $-\frac{1}{4}$ then the Lagrange equations of motion match exactly the (relativistic formulation) of Maxwell's equations. †

†The determination of this constant a is the *only* place that a match to QED has been used. The rest of the Lagrangian density is obtained purely from the requirement of local $U(1)$ invariance.

We have thus arrived at the Lagrangian density for QED, but from the viewpoint of demanding invariance under $U(1)$ gauge transformations rather than starting with Maxwell's equations and formulating the equivalent Quantum Field Theory.

The Lagrangian density is:

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,\left(i\gamma^\mu\left(\partial_\mu + i\,e\,A_\mu\right) - m\right)\psi. \tag{1.8}$$

Note that we are *not* allowed to add a mass term for the photon. A term such as $M^2 A_\mu A^\mu$ added to the Lagrangian density is not invariant under gauge transformations, but would give us a transformation

$$\delta\mathcal{L} \;=\; -\frac{2M^2}{e}A^\mu(x)\partial_\mu\omega(x).$$

Thus the masslessness of the photon can be understood in terms of the requirement that the Lagrangian be gauge invariant.

Covariant derivatives

It is useful to introduce the concept of a "covariant derivative". This is not essential for Abelian gauge theories, but will be an invaluable tool when we extend these ideas to non-Abelian gauge theories.

The covariant derivative D_μ is defined to be

$$D_\mu \;=\; \partial_\mu + i\,e\,A_\mu. \tag{1.9}$$

This has the property that given the transformations of the fermion field (1.2) and the gauge field (1.6) the quantity $D_\mu\psi$ is tansforming the same way (covariantly) as ψ under gauge transformation:

$$\delta D_\mu\psi = i\omega(x)D_\mu\psi$$

We may thus rewrite the QED Lagrangian density as

$$\mathcal{L} \;=\; -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\,\left(i\gamma^\mu D_\mu - m\right)\psi. \tag{1.10}$$

Furthermore the field strength $F_{\mu\nu}$ can be expressed in terms of the commmutator of two covariant derivatives, i.e.

$$F_{\mu\nu} \;=\; -\frac{i}{e}\left[D_\mu, D_\nu\right] \;=\; -\frac{i}{e}[\partial_\mu,\partial_\nu]+[\partial_\mu,A_\nu]+[A_\mu,\partial_\nu]+i\,e\,[A_\mu,A_\nu] \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \tag{1.11}$$

1.2 Non-Abelian gauge transformations

We now move on to apply the ideas of the previous lecture to the case where the transformations are "non-Abelian", i.e. different elements of the group do not commute with each other. As an example we take use isospin, although this can easily be extended to other Lie groups.

The fermion field, ψ_i, now carries an index i, which takes the value 1 if the fermion is a $u-$type quark and 2 of the fermion is a $d-$type quark. The conjugate field is written $\overline{\psi}^i$.

The Lagrangian density for a free isodoublet is

$$\mathcal{L} \ = \ \overline{\psi}^i \left(i\gamma^\mu \partial_\mu - m\right) \psi_i, \tag{1.12}$$

where the index i is summed over 1 and 2. Eq.(1.12) is therefore shorthand for

$$\mathcal{L} \ = \ \overline{u} \left(i\gamma^\mu \partial_\mu - m\right) u \ + \ \overline{d} \left(i\gamma^\mu \partial_\mu - m\right) d, \tag{1.13}$$

where $u, \ d$ are fermion fields for the $u-$quark and $d-$quark respectively.

A general isospin rotation requires three parameters $\omega^a, \ a = 1 \cdots 3$ (in the same way that a rotation is specified by three parameters which indicate the angle of the rotation and the axis about which the rotation is performed). Under such an isospin transformation the field ψ_i transforms as

$$\psi_i \to \left(e^{i\omega^a \mathbf{T}^a}\right)_i^j \psi_j,$$

where $\mathbf{T}^a, \ a = 1 \cdots 3$ are the generators of isospin transformations in the isospin one-half representation. As in the case of the generators of rotations for a spin one-half particle these are $\frac{1}{2}$ times by the Pauli spin matrices, i.e.

$$\mathbf{T}^1 \ = \ \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \mathbf{T}^2 \ = \ \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \mathbf{T}^3 \ = \ \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.14}$$

and obey the commutation relations

$$\left[\mathbf{T}^a, \mathbf{T}^b\right] \ = \ i\,\epsilon_{abc} \mathbf{T}^c. \tag{1.15}$$

This means that two such isospin transformations do *not* commute.

$$\left(e^{i\omega_1^a \mathbf{T}^a}\right)_i^k \left(e^{i\omega_2^b \mathbf{T}^b}\right)_k^j \psi_j, \ \neq \ \left(e^{i\omega_2^b \mathbf{T}^b}\right)_i^k \left(e^{i\omega_1^a \mathbf{T}^a}\right)_k^j \psi_j.$$

Groups of such transformations are called "non-Abelian groups".

Once again, it is convenient to consider only infinitesimal transformations under which the field ψ_i changes by an infinitesimal amount $\delta\psi_i$, where

$$\delta\psi_i \ = \ i\,\omega^a \left(\mathbf{T}^a\right)_i^j \psi_j, \tag{1.16}$$

and the conjugate field $\overline{\psi}^i$ † changes by $\delta\overline{\psi}^i$, where

$$\delta\overline{\psi}^i \;=\; -i\,\omega^a\overline{\psi}^j\,(\mathbf{T}^a)^i_j\,. \tag{1.17}$$

We see that these two changes cancel each other out in the Lagrangian (1.12), provided that the parameters, ω_a are constant.

If we allow these parameters to depend on space-time, $\omega^a(x)$, then the Lagrangian density changes by $\delta\mathcal{L}$ under this "non Abelian gauge transformation", where

$$\delta\mathcal{L} \;=\; -\,\overline{\psi}^i\,(\mathbf{T}^a)^j_i\,\gamma^\mu\,(\partial_\mu\omega^a(x))\,\psi_j.$$

1.3 Non-Abelian Gauge Fields

The symmetry can once again be restored by introducing interactions with vector (spin-one) gauge bosons. In this case we need three such gauge bosons, A^a_μ - one for each generator of SU(2). Under an infinitesimal gauge transformation these gauge bosons transform as

$$\delta A^a_\mu(x) \;=\; \epsilon_{abc}A^b_\mu(x)\,\omega^c(x) - \frac{1}{g}\partial_\mu\omega^a(x). \tag{1.18}$$

The first term on the R.H.S. of eq.(1.18) is the transformation that one would expect if the gauge bosons transformed as a usual triplet (isospin one), and this is indeed the case for constant ω^a. The second term, (which is non-linear in the field A^a_μ) is an extra term which needs to be added for the case of space-time dependent ω^a.

For non-infinitesimal gauge transformations we can write the gauge transformation as †

$$\mathbf{A}_\mu \;\rightarrow\; \mathbf{U}\,\mathbf{A}_\mu\mathbf{U}^{-1} + \frac{1}{g}\mathbf{U}\partial_\mu\mathbf{U}^{-1},$$

where

$$\mathbf{U} \;=\; e^{i\boldsymbol{\omega}}.$$

The interaction with these gauge bosons is again encoded by replacing the ordinary partial derivative in the Lagrangian density (1.12) with a covariant derivative, which in this case is a 2×2 matrix defined by

$$\mathbf{D}_\mu \;=\; \left(\partial_\mu\mathbf{I} + i\,g\,\mathbf{T}^a A^a_\mu\right), \tag{1.19}$$

where $\mathbf{I}$ is the unit matrix.

†Note that the conjugate field has a superscript i because strictly it transforms as the $\bar{2}$ representation of SU(2). For SU(2) these two representations are identical, but this will not be the case when we consider other groups.

†The boldface indicates a matrix valued quantity. Thus for example $\boldsymbol{\omega}$ means $\mathbf{T}^{\boldsymbol{a}}\boldsymbol{\omega}^{\boldsymbol{a}}$, etc.

The Lagrangian density thus becomes

$$\mathcal{L} \;=\; \overline{\psi}^{\,i}\,\left(i\gamma^\mu \mathbf{D}_\mu - m\mathbf{I}\right)_i^j \psi_j, \tag{1.20}$$

The quantity $\mathbf{D}_\mu\psi$ is not invariant under gauge transformations, but using eqs.(1.16), (1.18) and the commutation relations (1.15) we obtain the change of $\mathbf{D}_\mu\psi$ under an infinitesimal gauge transformation to be

$$\delta\left(\mathbf{D}_\mu\psi\right) \;=\; i\,\omega^a \mathbf{T}^a \mathbf{D}_\mu\psi, \tag{1.21}$$

which, together with eq.(1.17), tell us that the new Lagrangian density (1.20) is invariant under local isospin transformations (“ SU(2) gauge transformations”).

We can express the transformation rule for $\mathbf{D}_\mu\psi$ in terms of a transformation rule for the matrix $\mathbf{D}_\mu$ as

$$\delta\mathbf{D}_\mu \;=\; i\,\left[\omega^a\mathbf{T}^a, \mathbf{D}_\mu\right] \tag{1.22}$$

The kinetic term for the gauge bosons is again constructed from the field strengths $F^a_{\mu\nu}$ which are defined from the commutator of two covariant derivatives:

$$\mathbf{F}_{\mu\nu} \;= -\frac{i}{g}\,\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right]. \tag{1.23}$$

where the matrix $\mathbf{F}_{\mu\nu}$ is given by

$$\mathbf{F}_{\mu\nu} \;= \mathbf{T}^a F^a_{\mu\nu},$$

This gives us

$$F^a_{\mu\nu} \;=\; \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\,\epsilon_{abc}\,A^b_\mu A^c_\nu \tag{1.24}$$

From (1.21) we obtain the change in $\mathbf{F}_{\mu\nu}$ under an infinitesimal gauge transformation as

$$\delta\mathbf{F}_{\mu\nu} \;=\; i\omega^a\left[\mathbf{T}^a, \mathbf{F}_{\mu\nu}\right] \tag{1.25}$$

which leads to

$$\delta F^a_{\mu\nu} \;=\; \epsilon_{abc}\,F^b_{\mu\nu}\,\omega^c. \tag{1.26}$$

The gauge invariant term which contains the kinetic term for the gauge bosons is therefore

$$-\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu},$$

where a summation over the isospin index a is implied.

In sharp contrast with the Abelian case, this term does not only contain the terms which are quadratic in the derivatives of the gauge boson fields, but also the terms

$$g\,\epsilon_{abc}(\partial_\mu A^a_\nu)A^b_\mu A^c_\nu - \frac{1}{4}g^2\epsilon_{abc}\epsilon_{ade}A^b_\mu A^c_\nu A^d_\mu A^e_\nu.$$

This means that there is a very important difference between Abelian and non-Abelian gauge theories. For non-Abelian gauge theories the gauge bosons interact with each other via both three-point and four-point interaction terms. The three point interaction term contains a derivative, which means that the Feynman rule for the three-point vertex involves the momenta of the particles going into the vertex. We shall write down the Feynman rules in detail later.

Once again, a mass term for the gauge bosons is forbidden, since a term proportional to $A^a_\mu A^{a\,\mu}$ is *not* invariant under gauge transformations.

The Lagrangian for a General non-Abelian Gauge Theory
Consider a gauge group, $\mathcal{G}$ of "dimension" N, whose N generators, $\mathbf{T}^a$, obey the commutation relations

$$\left[\mathbf{T}^a, \mathbf{T}^b\right] = i f_{abc} \mathbf{T}^c, \tag{1.27}$$

where f_{abc} are called the "structure constants" of the group (they are antisymmetric in the indices a, b, c).

The Lagrangian density for a gauge theory with this group, with a fermion multiplet ψ_i is given (in Feynman gauge) by

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu} + i\overline{\psi}\left(\gamma^\mu \mathbf{D}_\mu - m\mathbf{I}\right)\psi \tag{1.28}$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, f_{abc} A^b_\mu A^c_\nu, \tag{1.29}$$

$$\mathbf{D}_\mu = \partial_\mu \mathbf{I} + i\, g\, \mathbf{T}^a A^a_\mu \tag{1.30}$$

Under an infinitesimal gauge transformation, the N gauge bosons, A^a_μ change by an amount that contains a term which is not linear in A^a_μ:

$$\delta A^a_\mu(x) = f_{abc} A^b_\mu(x) \omega^c(x) - \frac{1}{g} \partial_\mu \omega^a(x), \tag{1.31}$$

whereas the field strengths $F^a_{\mu\nu}$ transform by a change

$$\delta F^a_{\mu\nu}(x) = f_{abc}\, F^b_{\mu\nu}(x)\, \omega^c. \tag{1.32}$$

In other words they transform as the "adjoint" representation of the group (which has as many components as there are generators). This means that the quantity $F^a_{\mu\nu} F^{a\,\mu\nu}$ (summation over a implied) is invariant under gauge transformations.

1.4 Gauge Fixing

As in the case of QED, there is a problem in determining the propagator of the gauge field, which necessitates a "gauge choice".

The quadratic part of the action can be written as

$$S^{quad} \;=\; \int d^4x \frac{i}{2} A^a_\mu(x) \mathcal{O}^{\mu\nu}_{ab} A^b_\nu(x),$$

where

$$\mathcal{O}^{\mu\nu}_{ab} \;=\; \delta_{ab}\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)$$

The propagator should normally be the inverse of the operator $\mathcal{O}^{\mu\nu}$, but in this case the operator does possess an inverse as can be seen from the fact that it has a zero mode $\partial_\nu\omega^b(x)$, i.e.

$$\mathcal{O}^{\mu\nu}\partial_\nu\omega^b(x) \;=\; 0$$

.

What has actually happened is that the path integral over-counts the space of functions of the gauge field $A^a_\mu(x)$, because a function

$$\mathbf{A}^\omega_\mu \;=\; \mathbf{A}^{\mathbf{0}}_\mu + \frac{1}{g}\left[\mathbf{D}_\mu, \boldsymbol{\omega}\right]$$

results in precisely the same action as $\mathbf{A}^{\mathbf{0}}_\mu$, and therefore contributes the same weight to the path integral.

We can write the measure of the path integral as

$$\mathcal{D}\left[\mathbf{A}_\mu\right] \;=\; \mathcal{D}\left[\mathbf{A}^{\mathbf{0}}_\mu\right] \mathcal{D}\left[\boldsymbol{\omega}\right].$$

In the path integral we should drop the integration over the function $\boldsymbol{\omega}$. We can do this by inserting the functional $\delta-$function, $\delta[\boldsymbol{\omega}]$ into the path integral so that the generating function (in the absence of sources) becomes

$$\mathcal{Z} \;=\; \int \mathcal{D}\left[\mathbf{A}_\mu\right] \delta\left[\boldsymbol{\omega}\right] e^{iS}$$

$\mathbf{A}^{\mathbf{0}}_\mu$ is selected by imposing some gauge condition, such as a condition on the divergence $\partial\cdot\mathbf{A}$, e.g.

$$\partial\cdot\mathbf{A} \;=\; \mathbf{f}(x),$$

for some arbitrary function $\mathbf{f}(x)$.

We now wish to substitute the $\delta-$function $\delta\left[\boldsymbol{\omega}\right]$ for the $\delta-$function $\delta\left[\partial\cdot\mathbf{A}-\mathbf{f}\right]$, which we can do as long as we remember there will be a (functional determinant) $J^{F.P.}$, so that we have

$$\mathcal{Z} \;=\; \int \mathcal{D}\left[\mathbf{A}_\mu\right] \delta\left[\partial\cdot\mathbf{A}-\mathbf{f}\right]\, J^{F.P.} e^{iS}.$$

Finally we can (up to an overall constant) "smear" over all functions $f(x)$, by writing

$$\mathcal{Z} \;=\; \int \mathcal{D}\left[\mathbf{f}\right]\, e^{i\int d^4x \frac{1}{2(1-\xi)}\mathbf{f}^2(x)} \int \mathcal{D}\left[\mathbf{A}_\mu\right] \delta\left[\partial\cdot\mathbf{A}-\mathbf{f}\right]\, J^{F.P.} e^{iS}.$$

Now we can perform the integration over the function $\mathbf{f}$, picking up the δ-function. The upshot is that a term $\frac{1}{2(1-\xi)}(\partial \cdot \mathbf{A})^2$ appears in the effective Lagrangian density, so that the quadratic part of the action becomes

$$S_{G.F}^{quad} = \int d^4x \frac{1}{2} A_\mu^a(x) \mathcal{O}_{ab\,(G.F.)}^{\mu\nu} A_\nu^b(x),$$

where

$$\mathcal{O}_{ab\,(G.F.)}^{\mu\nu} = i\delta_{ab}\left(g^{\mu\nu}\partial^2 + \frac{\xi}{(1-\xi)}\partial^\mu\partial^\nu\right)$$

This gauge-fixed quadratic part *does* have an inverse, namely (in momentum space)

$$-\delta_{ab}\left(g_{\mu\nu} - \xi\frac{p_\mu p_\nu}{p^2}\right)\frac{i}{p^2}.$$

One of the simplest choices of ξ is $\xi = 0$, which is called the"Feynman gauge". In this gauge the propagator for a gauge boson with momentum p is

$$-i\delta_{ab}\frac{g^{\mu\nu}}{p^2}$$

Now we return to the Jacobian factor $J^{F.P.}$. This is the determinant of the derivative of the gauge fixing condition with respect to the gauge parameter $\boldsymbol{\omega}$. For infinitesimal gauge transformations

$$\partial \cdot \mathbf{A} \ \to \partial \cdot \mathbf{A} + \partial \cdot \mathbf{D}\boldsymbol{\omega}$$

so that

$$J^{F.P.} = \det\left[\partial \cdot \mathbf{D}\right].$$

We can use a trick to calculate this determinant

$$\det\left[\partial \cdot \mathbf{D}\right] = \int \mathcal{D}\left[\boldsymbol{\zeta}\right]\mathcal{D}\left[\boldsymbol{\eta}\right] e^{i\int d^4x(\boldsymbol{\zeta}\partial\cdot\mathbf{D}\boldsymbol{\eta})},$$

where the functions $\boldsymbol{\zeta(x)}$ $(= \mathbf{T}^a\zeta^a(x))$ and $\boldsymbol{\eta(x)}$ $(= \mathbf{T}^a\eta^a(x))$, are Grassmanian quantities, i.e. they are anti-commuting objects. They are known as "Faddeev-Popov ghosts".

These are *not* to be interpreted as physical scalar particles which could in principle be observed experimentally, but merely as part of the gauge-fixing programme. For this reason they are referred to as 'ghosts". Furthermore they have two peculiarities

1. They only occur inside loops. This is because they are not really particles and cannot occur in initial or final states, but are introduced to clean up a difficulty that arises in the gauge-fixing mechanism.

2. They behave like fermions even though they are scalars (spin zero). This means that we need to count a minus sign for each loop of Faddeev-Popov ghosts in any Feynman diagram.

Having introduced these fields, we add to the effective action a term

$$S^{F.P.} = \int d^4x \left(\zeta^a(x)\partial \cdot \mathbf{D}_{ab}\eta^b(x)\right).$$

Writing this out gives

$$S^{F.P.} = \int d^4x \left(\zeta^a\partial^2\eta^a + g f_{abc}\partial^\mu\zeta^a A^b_\mu\eta^c\right),$$

(where we have exploited integration by parts). From this we see that the fields propagate into each other with the same propagator as a massless scalar field.

$$\delta_{ab}\frac{i}{p^2},$$

and they interact with the gauge field with a vertex that is linear in the momentum of the field ζ.

Thus, for example, the Feynman diagrams which contribute to the one-loop corrections to the gauge boson propagator are

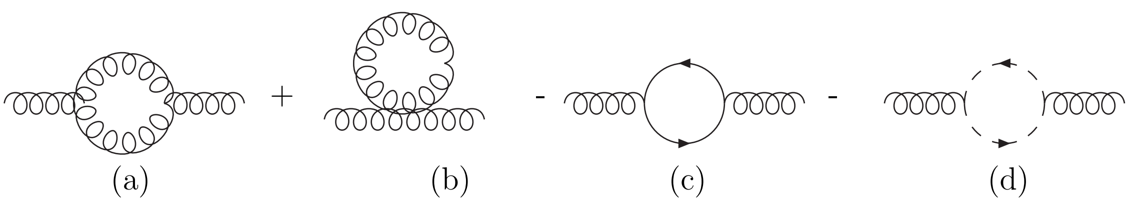


Diagram (a) involves the three-point interaction between the gauge bosons, diagram (b) involves the four-point interaction between the gauge bosons, diagram (c) involves a loop of fermions, and diagram (d) is the extra diagram involving the Faddeev-Popov ghosts. Note that both diagrams (c) and (d) have a minus sign in front of them because both the fermions and Faddeev-Popov ghosts obey Fermi statistics.

It is, in principle, possible to make a different sort of gauge choice in which the Faddeev-Popov ghosts do not interact or do not propagate. The price one pays is that the gauge condition is no longer Lorentz invariant which is very inconvenient.

- **Axial gauge**
  The axial gauge is the gauge in which a particular component of the gauge-boson field is set to zero.

$$n \cdot A^a \;=\; 0, \quad \text{for all } a$$

Under an infinitiesimal gauge transformation this becomes

$$n \cdot A^a + n \cdot (D)_{ab}\omega^b \quad = n \cdot A^a + n \cdot \partial\,\omega^a - g f_{abc} n \cdot A^b \omega^c.$$

But the last term vanishes since $n \cdot A^b = 0$. Thus the Faddeev-Popov action is simply

$$S^{F.P.} \;=\; \zeta^a n \cdot \partial \eta^a,$$

and there is no interaction between the Faddeev-Popov ghosts and the gauge boson.

On the other hand the propagator, which vanishes when contracted with n^ν is

$$-i\delta_{ab}\left(g^{\mu\nu} - \frac{(n^\mu p^\nu + n^\nu p^\mu)}{n \cdot p} + n \cdot n \frac{p^\mu p^\nu}{(n \cdot p)^2}\right)\frac{1}{p^2}$$

- **Coulomb gauge**
  In the Coulomb gauge we impose the vanishing of the space-like part of the divergence of the gauge boson field only
  $$\underline{\nabla} \cdot \underline{A}^a \;= 0.$$
  In this case the Faddeev-Popov action is
  $$S^{F.P.} \;=\; \int d^4x \left(\zeta^a \nabla^2 \eta^a + g f_{abc} \zeta^a \underline{\nabla} \cdot \underline{A}^b \eta^c.\right)$$
  The Faddeev-Popov fields interact but they do *not* propogate in time -they give rise to a Coulomb background field only.

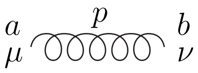
  The gauge-boson propagator, in this gauge, is
  $$i\delta_{ab}\left(\delta_{ij} - \frac{p_i p_j}{|\underline{p}|^2}\right)\frac{1}{p^2}, \;\; (i,j = 1\cdots 3)$$
  and the time-component A_0^a does not propagate.

1.5 Feynman Rules

The Feynman rules for a non-abelian gauge theory are given by:

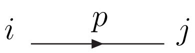
Propagators:

Gluon (Feynman gauge)



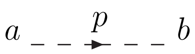
$-i\,\delta_{ab} g_{\mu\nu}/p^2$

Fermion



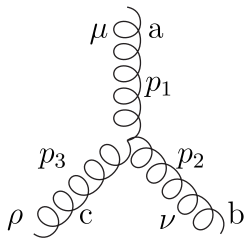
$i\,\delta_{ij}(\gamma^\mu p_\mu + m)/(p^2 - m^2)$

Faddeev-Popov ghost

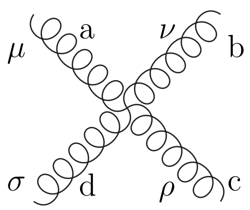


$i\,\delta_{ab}/p^2$

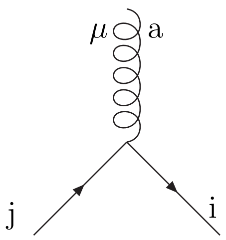
Vertices:
(all momenta are flowing into the vertex).



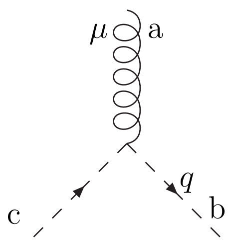
$$-g\, f_{abc}\left(g_{\mu\nu}\,(p_1 - p_2)_\rho + g_{\nu\rho}\,(p_2 - p_3)_\mu + g_{\rho\mu}\,(p_3 - p_1)_\nu\right)$$



$$-i\, g^2 f_{eab} f_{ecd}\,(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})$$
$$-i\, g^2 f_{eac} f_{ebd}\,(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\sigma} g_{\nu\rho})$$
$$-i\, g^2 f_{ead} f_{ebc}\,(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma})$$



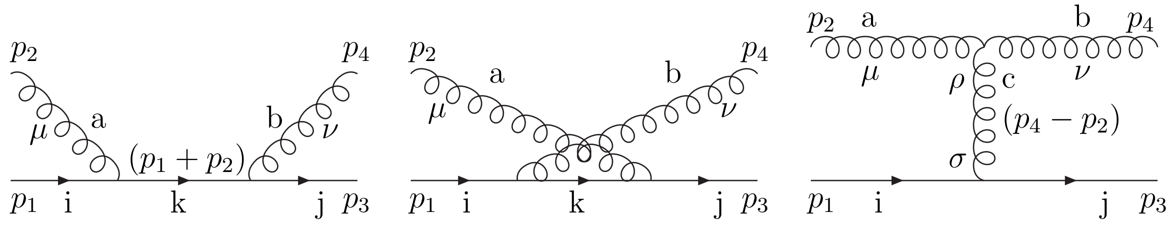
$$-i\, g\, \gamma^\mu\, (T^a)_{ij}$$



$$g\, f_{abc}\, q_\mu$$

1.6 An Example:

As an example of the application of these Feynman rules, we consider the process of Compton scattering, but this time for the scattering of non-Abelian gauge-bosons and fermions, rather than photons. We need to calculate the amplitude for a gauge-boson of momentum p_2 and isospin a to scatter fermion of momentum p_1 and isospin i producing a fermion of momentum p_3 and isospin j and a gauge-boson of momentum p_4. In addition to the two Feynman diagrams one gets in the QED case there is a third diagram involving the self-interaction of the gauge bosons.



(a) (b) (c)

We will assume that the fermions are massless (i.e. that we are sufficiently high energies that we may neglect their masses), and work in terms of the Mandelstam variables

$$
\begin{aligned}
s &= (p_1+p_2)^2 = (p_3+p_4)^2 \\
t &= (p_1-p_3)^2 = (p_2-p_4)^2 \\
u &= (p_1-p_4)^2 = (p_2-p_3)^2
\end{aligned}
$$

The polarisations are accounted for by contracting the amplitude obtained for the above diagrams with the polarisation vectors $\epsilon^\mu(\lambda_2)$ and $\epsilon^\nu(\lambda_4)$. Each diagram consists of a two vertices and a propagator and so their contributions can be read off from the Feynman rules.

For diagram (a) we get

$$
\begin{aligned}
&\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\,g\,\gamma^\nu(\mathbf{T}^b)^k_j\right)\left(i\,\frac{\gamma\cdot(p_2+p_2)}{s}\right)\left(-i\,g\,\gamma^\mu(\mathbf{T}^a)^i_k\right)u_i(p_1) \\
&= -i\,\frac{g^2}{s}\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}\,(p_3)\left(\gamma^\nu\gamma\cdot(p_1+p_2)\gamma^\mu\right)\left(\mathbf{T}^b\mathbf{T}^a\right)u(p_1).
\end{aligned}
$$

For diagram (b) we get

$$
\begin{aligned}
&\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\,g\,\gamma^\mu(\mathbf{T}^a)^k_j\right)\left(i\,\frac{\gamma\cdot(p_1-p_4)}{u}\right)\left(-i\,g\,\gamma^\nu(\mathbf{T}^b)^i_k\right)u_i(p_1) \\
&= -i\,\frac{g^2}{u}\epsilon_\mu(\lambda_2)\epsilon_\nu(\lambda_4)\overline{u}\,(p_3)\left(\gamma^\nu\gamma\cdot(p_1-p_4)\gamma^\mu\right)\left(\mathbf{T}^a\mathbf{T}^b\right)u(p_1).
\end{aligned}
$$

Note here that the order of the **T** matrices is the other way around from diagram (a).

Diagram (c) involves the three-point gauge-boson self-coupling. Since the Feynman rule for this vertex is given with incoming momenta, it is useful to replace the outgoing gauge-boson momentum p_4 by $-p_4$ and understand this to be an incoming momentum. Note that the internal gauge-boson line carries momentum $p_4 - p_2$ coming into the vertex - the three incoming momenta that are to be substituted into the Feynman rule for the vertex are therefore p_2, $-p_4$, $p_4 - p_2$. The vertex thus becomes

$$-g\, f_{abc}\left(g_{\mu\nu}(p_2+p_4)_\rho + g_{\rho\nu}(p_2-2p_4)_\mu + g_{\mu\rho}(p_4-2p_2)_\nu\right)$$

and the diagram gives

$$\begin{aligned} & \epsilon^\mu(\lambda_2)\epsilon^\nu(\lambda_4)\overline{u}^j(p_3)\left(-i\,g\,\gamma_\sigma(\mathbf{T}^c)^i_j\right)u_i(p_1)\left(-i\frac{g^{\rho\sigma}}{t}\right) \\ & \times\ (-g\,f_{abc})\left(g_{\mu\nu}(p_2+p_4)_\rho + g_{\rho\nu}(p_2-2p_4)_\mu + g_{\mu\rho}(p_4-2p_2)_\nu\right) \\ = \ & -i\frac{g^2}{t}\epsilon^\mu(\lambda_2)\epsilon^\nu(\lambda_4)\overline{u}\,(p_3)\left[\mathbf{T}^a,\mathbf{T}^b\right]\gamma^\rho u(p_1)\left(g_{\mu\nu}\,(p_2+p_4)_\rho - 2(p_4)_\mu g_{\nu\rho} - 2(p_2)_\nu g_{\mu\rho}\right), \end{aligned}$$

where in the last step we have used that the polarisation vectors are transverse so that $p_2 \cdot \epsilon(\lambda_2) = 0$ and $p_4 \cdot \epsilon(\lambda_4) = 0$ and the commutation relations (1.27).

2 Loop corrections in ϕ^3 Theory

Consider the Lagrangian density for a scalar particle of mass m with cubic self-interaction with coupling constant λ

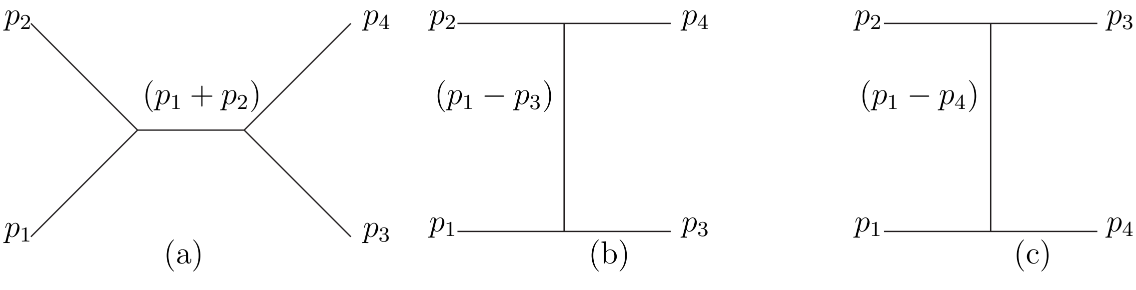
$$\mathcal{L} \;=\; \frac{1}{2}\left(\partial_\mu \phi\right)^2 - m^2\phi^2 - \frac{\lambda}{3!}\phi^3$$

We wish to calculate the scattering amplitude for two particles of momenta, p_1 and p_2 into two particles with momenta p_3 and p_4.

The Feynman rules for the n^{th} order perturbative contribution are:

1. Draw all the possible Feynman graphs with n vertices.
2. Write a factor of $1/\sqrt{Z}$ for each external line (this will be explained later).
3. Write a factor of
$$\frac{i}{k^2 - m^2 + i\epsilon}$$
for each internal propagator with momentum k (we take the limit $\epsilon \to 0$, but we need to keep this term to guarantee the proper time-ordering).
4. Write a factor of $i\lambda$ at each vertex.
5. Introduce an energy-momentum conserving $\delta-$function, $(2\pi)^4\delta^4(k_1 + k_2 + k_3)$ for a vertex between particles with momenta k_1, k_2 and k_3.
6. Integrate over $d^4k_i/(2\pi)^4$ for each internal line of momentum k_i.

At order λ^2 we just have the three tree diagrams



In each diagram, the integration over the internal particle momentum is "soaked up" by one of the energy-momentum conserving $\delta-$functions and we are left with one overall delta function

$$(2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4),$$

which multiplies the entire amplitude.

For example, the contribution from tree-graph (a) is

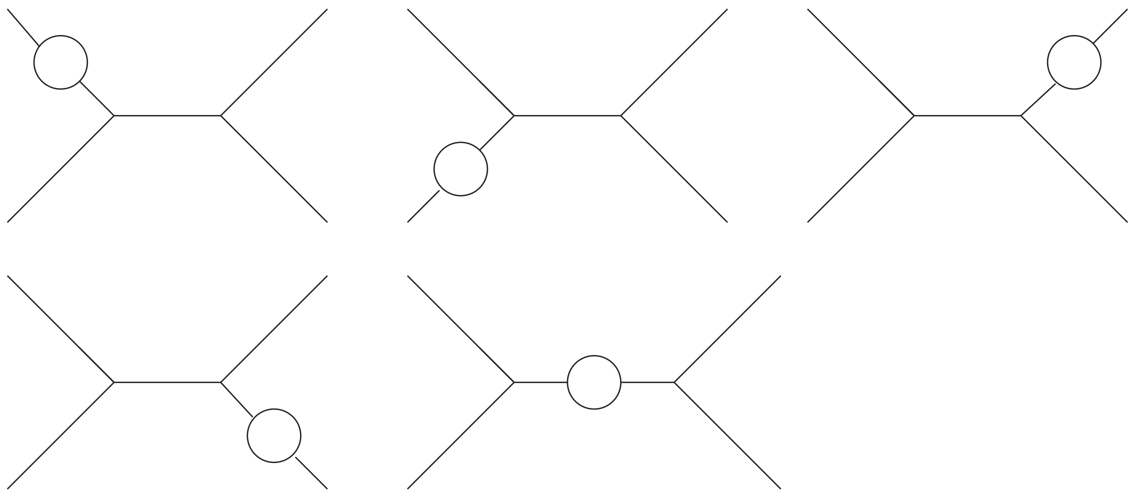
$$-i\frac{\lambda^2}{(s-m^2)}, \tag{2.1}$$

where we have suppressed the overall energy-momentum conserving $\delta-$function and used $s = (p_1 + p_2)^2$.

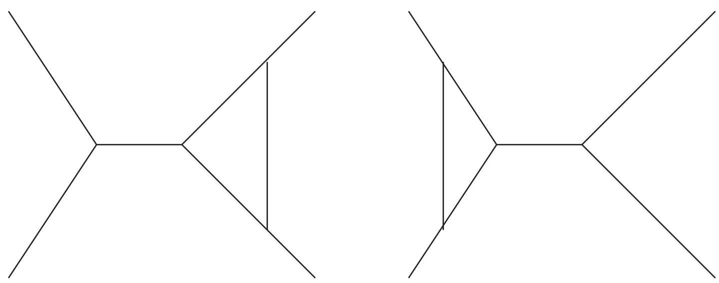
At the next order λ^4 we have graphs which contain one "loop" of internal particles and we will indeed need to integrate over an internal momentum.

For the corrections to the tree-graph (a), we have the following types of one-loop Feynman graphs

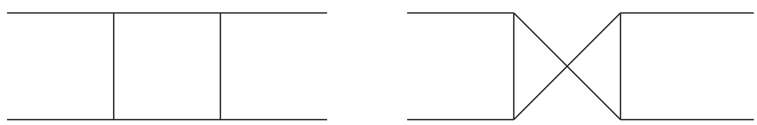
- **Self-energy corrections:**



- **Vertex corrections:**



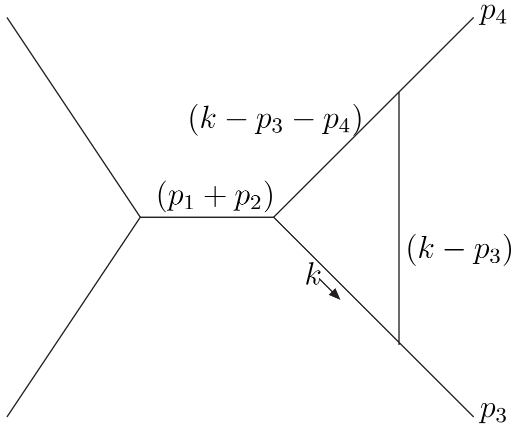
- **Box graphs:** These "box graphs" are generic one-loop graphs and cannot be associated with specific tree-level graphs (unlike the vertex or self-energy correction graphs)



Note that each of these graphs has three more internal lines than the tree-level graph and two more vertices. There is therefore a remaining integral over one of the internal momenta.

2.1 Vertex Corrections:

We will concentrate first on one of the vertex graphs



We have implemented the energy-momentum conserving δ-functions, by ensuring that momentum is conserved at each vertex. There is a remaining internal momentum l over which we need to integrate.

The contribution to the scattering amplitude from this term is

$$\frac{\lambda^4}{(p_1+p_2)^2-m^2+i\epsilon}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\epsilon)\left((k-p_3)^2-m^2+i\epsilon\right)\left((k-p_3-p_4)^2-m^2+i\epsilon\right)} \tag{2.2}$$

We have suppressed the overall energy-momentum conserving $\delta-$function and also the factor $1/Z^2$, since Z has a perturbation expansion and is unity at leading order, i.e.

$$Z=1+\mathcal{O}(\lambda^2),$$

so we do not need it to this order in perturbation theory.

Using the on-shell condition $p_3^2=m^2$ and $(p_1+p_2)^2=(p_3+p_4)^2=s$ we may write this as

$$\frac{\lambda^4}{(s-m^2+i\epsilon)}\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2+i\epsilon)\,(k^2-2k\cdot p_3+i\epsilon)\,(k^2-2k\cdot(p_3+p_4)+s-m^2+i\epsilon)} \tag{2.3}$$

The integration over k is implemented using the following steps:

- **Feynman parametrize:**
Here we use the relation

$$\frac{1}{a_1\, a_2 \cdots a_n} \;=\; (n-1)! \int_0^1 d\alpha_1 d\alpha_2 \cdots d\alpha_n \frac{\delta(1-\sum_i \alpha_i)}{(a_1\alpha_1 + a_2\alpha_2 \cdots + a_n\alpha_n)^n} \tag{2.4}$$

Using this, the integral in eq.(2.3) may be written

$$2\int \frac{d^4k}{(2\pi)^4} d\alpha d\beta d\gamma \frac{\delta(1-\alpha-\beta-\gamma)}{(k^2 - m^2 - 2k\cdot(p_3(\alpha+\beta)+p_4\beta) + s\beta + m^2\alpha)^3}, \tag{2.5}$$

where we have used $\alpha+\beta+\gamma=1$ in the k^2 and m^2 terms.

- **Shift integration variable:**

$$k^\mu \to k^\mu + p_3^\mu(\alpha+\beta) + p_4^\mu\beta$$

The integral now becomes

$$2\int \frac{d^4k}{(2\pi)^4} d\alpha\beta d\gamma \frac{\delta(1-\alpha-\beta-\gamma)}{(k^2 - A^2 + i\epsilon)^3}, \tag{2.6}$$

where

$$A^2 = -s\beta - m^2\alpha + (p_3(\alpha+\beta)+p_4\beta)^2 = -s\beta(1-\alpha-\beta) + m^2(1-\alpha(1-\alpha))$$

(we have used $(p_3+p_4)^2 = s$ and $p_3^2 = p_4^2 = m^2$)

- **Integration over k:**
This is most easily achieved by rotating k^0 to ik^4 and performing the integral in Euclidean space.

$$2\int \frac{k^3 dk d\Omega}{(2\pi)^4} d\alpha d\beta d\gamma \frac{\delta(1-\alpha-\beta-\gamma)}{(k^2 + A^2 + i\epsilon)^3}, \tag{2.7}$$

The integration over Ω gives $2\pi^2$, the area of a three-dimensional spherical surface, and

$$\int \frac{k^3 dk}{(k^2+A^2)^n} \;=\; \frac{(n-3)!}{2(n-1)!(A^2)^{(n-2)}},$$

(provided $n > 2$).

We end up with

$$\frac{i}{16\pi^2}\int_0^1 d\alpha d\beta d\gamma \frac{\delta(1-\alpha-\beta-\gamma)}{(s(1-\alpha-\beta)\beta - m^2(1-\alpha(1-\alpha)) + i\epsilon)} \tag{2.8}$$

The integration over γ can be done trivially to give

$$\frac{i}{16\pi^2}\int_0^1 d\alpha d\beta \frac{\theta(1-\alpha-\beta)}{(s(1-\alpha-\beta)\beta - m^2(1-\alpha(1-\alpha)) + i\epsilon)} \tag{2.9}$$

We will leave the result in terms of this integral - whose exact value would be different in the more realistic cases where the masses of the internal particles were not the same. We note, however, that in general the integral has an imaginary part arising form the fact that

$$\Im m \left(\frac{1}{(s(1-\alpha-\alpha)\beta - m^2(1-\alpha(1-\alpha)) + i\epsilon)} \right) = \\ -\pi\delta\left(s(1-\alpha-\beta)\beta - m^2(1-\alpha(1-\alpha))\right)$$

We write the contribution from this graph to the scattering amplitude as

$$-i\frac{\lambda^4}{(s-m^2)}\Delta F(s) \tag{2.10}$$

What this means is that the right-most coupling λ is replaced by an effective coupling, which depends on the square momentum s coming into the vertex.

$$\lambda \;\rightarrow\; \lambda\left(1+\lambda^2\Delta F(s)\right). \tag{2.11}$$

This means that the coupling is not really constant, but depends on the momenta coming into the vertex.

We now have to give a definition of the coupling in terms of some measurement, which we call the "renormalized coupling constant". There is some arbitrariness in this definition and we call this arbitrariness "renormalization scheme dependence".

The coupling parameter that we started off with is called the "bare coupling" and is written λ_0. t is *not* directly measurable.

Thus, for example, we could define the renormalized coupling to be the coupling in which all momenta coming into the vertex are on shell, i.e. we set ($s = m^2$) and obtain the renormalized coupling as

$$\lambda_R \;=\; \frac{\lambda_0}{Z_1}, \tag{2.12}$$

where (to order λ^2)

$$Z_1 \;=\; \left(1-\lambda^2\Delta F(m^2)\right). \tag{2.13}$$

This renormalized coupling can be measured experimentally, and we wish to express the scattering amplitude in terms of this physically measured coupling. To do this we subtract off a "counterterm" corresponding to the conversion of the expression (2.10) into an expression in terms of this renormalized coupling, i.e. adding the contribution (2.10) to

$$-i\frac{\lambda_R^2}{(s-m^2)}\left(1+\lambda_R^2\left(\Delta F(s)-\Delta F(m^2)\right)\right). \tag{2.14}$$

(The replacement of λ by λ_R in the correction term does not affect the result at this order in perturbation theory).

The renormalization scheme we have chosen here is called the "on-shell" scheme since it defines the renormalized coupling as the value of the three-point coupling when all three particles are on-shell.

We could have chosen to define λ_R at the coupling at some value $s = \mu^2$, so that eq.(2.13) becomes

$$Z_1 \;=\; \left(1 - \lambda^2 \Delta F(\mu^2)\right). \tag{2.15}$$

and eq.(2.14) becomes

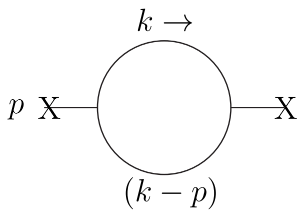
$$-i\frac{\lambda_R^2(\mu^2)}{(s-m^2)}\left(1+\lambda_R^2\left(\Delta F(s) - \Delta F(\mu^2)\right)\right). \tag{2.16}$$

The numerical values of eqs.(2.14 and 2.16) are identical (up to order λ^4) - the explicit μ dependence appearing in eq.(2.16) being compensated by the μ^2 dependence of $\lambda_R(\mu^2)$.

We need not have chosen any directly measurable way to define λ_R. For example we could have defined $\lambda_R(\mu)$ as the coupling of the interaction in which all particles are off-shell with square momentum μ^2. This is often done and it is called the "MOM" scheme. In this scheme the subtraction would again be different and we would get a different expression for the contribution to the scattering amplitude in terms of $\lambda_R^{MOM}(\mu^2)$, but the numerical value would again be the same once we had inserted the corresponding value of the renormalized coupling.

2.2 Self-energy Corrections:

Now we look at the "self-energy" graphs. These are the ones in which the loop has one incoming and one outgoing line (sometimes also called the "two-point function").

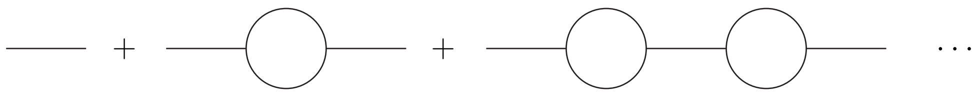


The crosses on the external lines indicate that they have been "truncated" - i.e. the external line propagators are not included in the calculation of the graph.

Define the "self-energy" function, $\Sigma(p^2)$ such that the contribution from the self-energy diagram is $-i\Sigma(p^2)$. This is also a function of the particle mass, m and the coupling λ_R.

Putting back the external propagators, gives (suppressing the $i\epsilon$)

$$\frac{i\Sigma(p^2)}{(p^2 - m^2)^2}$$



It looks as though this has a double pole at $p^2 = m^2$, but if we sum over all the possible numbers of self-energy insertions (including no insertions) we get a geometric series who sum is

$$\frac{i}{(p^2 - m^2 - \Sigma(p^2))} \tag{2.17}$$

and this is how the propagator is modified by the self-energy insertions.

Expand $\Sigma(p^2)$ about $p^2 = m_R^2$ as

$$\Sigma(p^2) \;=\; \frac{1}{Z}\Big((m_R^2 - m^2) + (Z-1)(p^2 - m^2) + \Sigma_R(p^2)\Big) \tag{2.18}$$

The quantity $\Sigma_R(p^2)$ vanishes quadratically as $p^2 \to m_R^2$. (The factor Z in the denominator is unnecessary to this order but would be required for higher order calculations). Inserting this into the expression for the corrected propagator, (2.17) gives

$$\frac{iZ}{(p^2 - m_R^2 - \Sigma_R(p^2))}$$

We see that the pole has moved to m_R. This "renormalized mass" is therefore the physical mass and the parameter, m, used in the Lagrangian is the bare mass and is henceforth written as m_0. Note that in general $\Sigma(p^2)$ will be complex. For a resonance of an unstable particles, the imaginary part of m_R is the half-width $\Gamma/2$ of the resonance..

In the same way, the field ϕ, which appears in the Lagrangian are "bare fields",

$$\phi_B \;=\; \sqrt{Z}\phi_R$$

These are interacting fields which tend asymptotically (in time) to free in or out free-fields.

$$\phi_B(\mathbf{x}, t) \;\stackrel{t\to\pm\infty}{\to}\; \phi_{in(out)}(\mathbf{x}, t)$$

It is the propagator of these free fields ϕ_{in} or ϕ_{out} which behave like $i/(p^2 - m_R^2)$

The upshot of this is twofold:

1. The LSZ reduction for an S-matrix element in terms of Green functions (i.e. vacuum expectation values of time-ordered products of fields) should have a factor of $1/\sqrt{Z}$ for each external line. This is the origin of the factor in th Feynman rules mentioned previously.

2. The renormalization of the coupling constant has a factor of $\sqrt{Z}$ for each line coming into the vertex, i.e. a factor of $Z^{3/2}$. So that Eq.(2.12) becomes

$$\lambda_R \;=\; \frac{Z^{3/2}}{Z_1}\lambda_0, \tag{2.19}$$

For a self-energy insertion on an internal line the factor of Z is absorbed because a factor of $\sqrt{Z}$ is absorbed into the renormalization of the coupling a either end of the internal propagator. For a self-energy insertion on an external line a factor of $\sqrt{Z}$ is absorbed into the renormalization of the coupling where the external line is attached to the rest of the graph and a factor of $\sqrt{Z}$ cancels against the factor $1/\sqrt{Z}$ in the Feynman rule obtained from the more careful derivation of the LSZ reduction.

Calculation of m_R and Z:
Applying the Feynman rules to the self-energy diagram, we have

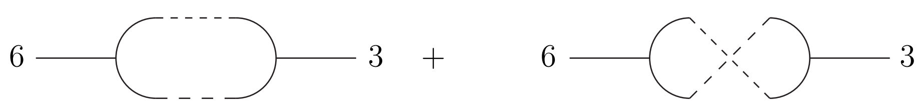
$$\Sigma(p^2) \;=\; i\frac{1}{2}\lambda_R^2\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2-m^2)((k-p)^2-m^2)}. \tag{2.20}$$

λ_R should really be λ_0, but to this order in perturbation theory we can use the renormalized coupling - (finally we want an expansion in terms of the renormalized coupling since this is related directly to a physically measurable quantity). Moreover m should be taken to mean m_R.

The factor of $\frac{1}{2}$ is a "combinatorial" factor and is determined as follows:

- Expanding the exponential of the interacting part of the action we have, at order λ^2

$$\frac{1}{2!}\left(\frac{\lambda}{3!}\right)^2\left(\int d^4x\phi^3(x)\right)^2$$

- 

  There are six ways, to select one of the external lines, three ways to select the other external line (which must be attached to the other vertex) and two ways to join the remaining lines together as internal propagators.

- This gives a total combinatorial factor of

$$6 \times 3 \times 2 \times \frac{1}{2!} \times \left(\frac{1}{3!}\right)^2 \ = \ \frac{1}{2}.$$

It is necessary to determine the combinatorial factor for *each* graph. had we done so for the vertex graphs we would have obtained a combinatorial factor of unity.

Write Eq.(2.20) as

$$\Sigma(p^2) \ = \ i\frac{1}{2}\lambda_R^2 \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m^2)(k^2-2p\cdot k+p^2-m^2)}. \tag{2.21}$$

Note that we cannot set $p^2 = m^2$ here.

Now introduce the trick of Feynman parametrization

$$\Sigma(p^2) \ = \ i\frac{1}{2}\lambda_R^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha d\beta \frac{\delta(1-\alpha-\beta)}{(k^2-2\alpha p\cdot k+p^2\alpha-m^2)^2}., \tag{2.22}$$

(we have used $\alpha+\beta=1$ in the coefficents of k^2 and m^2).

Shift $k^\mu \to k^\mu + \alpha p^\mu$ to get

$$\Sigma(p^2) \ = \ i\frac{1}{2}\lambda_R^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha \frac{1}{(k^2+p^2\alpha(1-\alpha)-m^2)^2}. \tag{2.23}$$

(We have performed the integral over β absorbing the $\delta-$function).

In Euclidean space, after integrating over the angles this is

$$\Sigma(p^2) \ = \ -\frac{1}{16\pi^2}\lambda_R^2 \int_0^1 d\alpha \int k^3 dk \frac{1}{(k^2-p^2\alpha(1-\alpha)+m^2)^2}. \tag{2.24}$$

This integral is *divergent*. The divergence is called "ultraviolet" as it arises from the $l \to \infty$ end of the integral.

The modern view of such divergences is that there is some 'new physics' at some high scale which serves to regulate these divergences. The most popular such theory is string theory in which what we call point particles are really extended objects with a length of order $1/\Lambda$. The point-like field theory that we use is valid up to a scale of order Λ, above which the string-like properties provide a cutoff for these effective integrals which is of order Λ. Several string theories have been identified which have been shown to be ultraviolet finite when treated correctly - these divergences occurring only when one makes the approximation that the strings can be treated as point-like particles.

What we need to ensure is that physically measurable quantities are independent of the cut-off Λ.

For the self-energy in the ϕ^3 theory introducing the cut-off gives

$$\Sigma(p^2) \;=\; -\frac{\lambda_R^2}{32\pi^2}\int_0^1 d\alpha \ln\left(\frac{\Lambda^2}{(m^2 - p^2\alpha(1-\alpha))}\right) \tag{2.25}$$

The divergence means that the bare mass depends on the cut-off

$$m_0^2 = m_R^2 - \frac{\lambda_R^2}{32\pi^2}\int_0^1 d\alpha \ln\left(\frac{\Lambda^2}{m^2(1-\alpha(1-\alpha)}\right) \tag{2.26}$$

As we go to higher orders in perturbation theory the bare mass m_0 is adjusted by terms which depend on the cutoff, in such a way that the renormalized mass m_R is the physical mass that is measured. m_0 is not directly observable and so its cut-off dependence is not important.

In most renormalizable theories, such as QED and QCD, the renormalization constants Z and Z_1 are also cut-off dependent (UV divergent). This, in turn, means that the bare coupling is cut-off dependent in such a that the renormalized coupling is related to a physical measurable in a cut-off independent way.

In the ϕ^3 case Z is cut-off independent and is given by

$$(Z-1) \;=\; \frac{\partial}{\partial p^2}\Sigma(p^2)_{|p^2=m^2} \;=\; -\frac{\lambda_R^2}{32\pi^2}\int_0^1 d\alpha \frac{\alpha(1-\alpha)}{m^2(1-\alpha+\alpha^2)}. \tag{2.27}$$

To calculate the scattering amplitude we also need to consider the box-graphs. These are algebraically very complicated, but in a ϕ^3 theory they do not introduce ultraviolet divergences and are not associated with the renormalization of any of the parameters of the theory.

3 Renormalization

The example considered above tells us that if we calculate an $n-$point Green function defined as

$$(2\pi)^4\delta^4(p_1+\cdots p_n)G^{(n)}(p_1\cdots p_{n-1},\lambda_0,m_0,\Lambda) = \int d^4x_1\cdots d^4x_n e^{i(p_1\cdot x_1\cdots p_n\cdot x_n)}\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle,$$

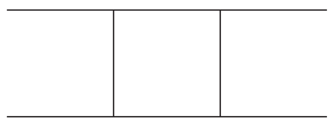
in terms of the bare coupling λ_0 and bare mass m_0, the result will depend explicitly on the cutoff Λ.

This dependence of Λ, however, is such that when expressed in terms of the renormalized quantities λ_R and m_R the "renormalized Green function" defined as

$$G_R^{(n)}(p_1\cdots p_{n-1},\lambda_R,m_R) \;=\; Z^{-(n/2)}G^{(n)}(p_1\cdots p_{n-1},\lambda_0,m_0,\Lambda), \tag{3.1}$$

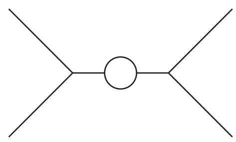
is finite (cut-off independent). It is these renormalized Green functions which are used to construct the S-matrix elements.

It is useful to work in terms of "truncated" or "one-particle irreducible" Green functions. These are Green functions calculated from graphs which cannot be separated into two or more graphs by cutting though one internal line. or a four-point Green function, the box graphs, such as



are one-particle irreducible, since we need to cut through two internal lines to separate them into two graphs.

whereas vertex or self-energy graphs such as



can be cut into two by cutting a single internal line (in several ways), and are therefore not one-particle irreducible.

We use the symbol Γ to refer to one-particle irreducible graphs, and the relation between the renormalized and the bare one-particle irreducible Green functions is

$$\Gamma_R^{(n)}(p_1\cdots p_{n-1},\lambda_R,m_R) \;=\; Z^{n/2}\Gamma^{(n)}(p_1\cdots p_{n-1},\lambda_0,m_0,\Lambda). \tag{3.2}$$

(The self-energy Σ is the same as $\Gamma^{(2)}$.)

3.1 Counterterms

We should think of renormalization as adjusting the masses and coupling constants (by a cut-off dependent amount if necessary), such that the S-matrix elements calculated to higher orders are cut-off independent and expressed in terms of physically measurable masses and couplings. In order to perform these higher order calculations it is convenient to view renormalization as the process of subtracting counterterms, in each order of perturbation theory, for some one-particle-irreducible Green function.

We do this by writing the Lagrangian in terms of bare parameters as a sum of two terms, one being the renormalized Lagrangian in terms of renormalized parameters and renormalized fields and the other being a set of counterterms. Thus for the ϕ^3 theory we have

$$\mathcal{L} \;=\; \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m_0^2\phi^2\right) - \frac{\lambda_0}{3!}\phi^3$$

The fields are the “bare” fields and are related to the renormalized fields by

$$\phi \;=\; \sqrt{Z}\phi_R$$

so we may write the Lagrangian as

$$\mathcal{L} \;=\; \mathcal{L}_R + \mathcal{L}_{CT},$$

where, in terms of renormalized fields, masses and couplings

$$\mathcal{L}_R \;=\; \frac{1}{2}\left(\partial_\mu\phi_R\,\partial^\mu\phi_R - m_R^2\phi_R^2\right) - \frac{\lambda_R}{3!}\phi_R^3$$

and

$$\mathcal{L}_{CT} \;=\; \frac{1}{2}(Z-1)\partial_\mu\phi_R\,\partial^\mu\phi_R - \frac{1}{2}(Z_m-1)m_R^2\phi_R^2 - (Z_1-1)\frac{\lambda_R}{3!}\phi_R^3,$$

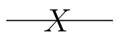
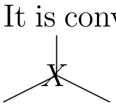
with

$$m_0^2 = \frac{Z_m}{Z}m_R^2$$

and

$$\lambda_0 = \frac{Z_1}{Z^{3/2}}\lambda_R.$$

We now calculate a Green function to any order using $\mathcal{L}_R$, i.e. in terms of renormalized masses and couplings. We will sometimes obtain UV divergences, which are cancelled when we consider a graph of lower order with a counterterm insertion.

It is convenient to view these counterterms as extra Feynman diagrams such as  or , where X represents the counterterm. The subtraction of the counterterm graphs from the unrenormalized Green functions renders them finite up to a factor of $\sqrt{Z}$ for each external line.

For example, for the two-point function (self-energy) we calculate the quantity

$$\Sigma(p^2, \lambda_R, m_R, \Lambda),$$

which has an explicit Λ dependence, but the subtracted quantity

$$\Sigma_R(p^2, \lambda_R, m_R) \;=\; \left(\Sigma(p^2, \lambda_R, m_R, \Lambda) - (Z-1)p^2 - (Z_m - 1)m_R^2\right), \tag{3.3}$$

is *finite* and equal (by eq.(3.2)) to $Z\,\Sigma(p^2, \lambda_0, m_0, \Lambda)$. The contribution from these counterterms is therefore equivalent to replacing the renormalized masses by bare masses and multiplying by Z.

For the three-point function, we have

$$\Gamma^{(3)}(p_1, p_2, \lambda_R, m_R, \Lambda)$$

and the renormalized (finite) quantity is obtained as

$$\Gamma^{(3)}_R(p_1, p_2, \lambda_R, m_R) \;=\; \Gamma^{(3)}(p_1, p_2, \lambda_R, m_R, \Lambda) + (Z_1 - 1)\lambda_R,$$

which, by eq.(3.2) is equal to $Z^{3/2}\,\Gamma^{(3)}(p_1, p_2, \lambda_0, m_0, \Lambda)$

How many counterterms are needed to make all renormalized Green functions finite? The superficial degree, $\omega(G)$, of divergence for some Feynman graph, G, is

$$\omega(G) \;=\; 4L + \sum_{vertices} \delta_v - I_F - 2I_B \;=\; \sum_{vertices} (\delta_V - 4) + 3I_F + 2I_B + 4, \tag{3.4}$$

where L is the number of loops, δ_V is the number of derivatives in the Feynman rule for the vertex (each introduces a power of momentum), and $I_{F(B)}$ is the number of internal fermion (boson) lines. Internal fermion lines carry a power of momentum in the numerator of their propagators.

$$L = I_B + I_F + 1 - V,$$

where V is the number of vertices.

Define

$$\omega_V \;=\; \delta_V + \frac{3}{2} f_V + b_V,$$

where f_V, (b_V) are the number of fermions (bosons) emerging from a vertex. Using the fact that one end of each internal line must end on a vertex we have

$$\sum_{vertices} \omega_V = \sum_{vertices} \delta_V + 3I_F + 2I_B + \frac{3}{2}E_F + E_B,$$

where $E_{F(B)}$ are the number of external fermions (bosons).

Thus we end up with

$$\omega(G) \;=\; 4 - \frac{3}{2}E_F - E_B + \sum_{vertices} (\omega_V - 4) \tag{3.5}$$

If $\omega_V > 4$ then as we go to higher orders more and more graphs will have a non-negative degree of divergence (E_F and E_B increase for a given degree of divergence). These are

non-renormalizable theories, since we need more and more counterterms as we go to higher orders.
Examples of such non-renormalizable theories are theories with interaction terms of the form $\lambda\phi^5$, or $g\bar{\Psi}\gamma^\mu\Psi\partial_\mu\phi$, for which $\omega_V = 5$

If $\omega_V = 4$ we have a renormalizable theory. We require counterterms for all one-particle-irreducible Green functions for which $\frac{3}{2}E_F + E_B \leq 4$, but once these counterterms appear at the one-loop level, no further counterterms are required in higher order.
Examples of such renormalizable theories are those with interaction terms of the form $\lambda\phi^4$, $g\bar{\Psi}\gamma^\mu\Psi A_\mu$, $g\bar{\Psi}\phi\Psi$, $g\phi\partial_\mu\phi A^\mu$. These have $\omega_V = 4$.

There is one exception:
The above analysis of the degree of divergence assumes that propagators for bosons of momentum p always behave as $1/p^2$ as $p \to \infty$. For massive vector particles the propagator is

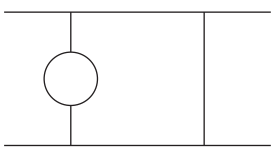
$$-i\left(\frac{g_{\mu\nu} - p_\mu p_\nu/m^2}{p^2 - m^2}\right)$$

and some of the components are constant as $p \to \infty$. Theories involving massive vector particles are in general NOT renormalizable. The exception is the case in which the mass of the vector boson is generated by Spontaneous Symmetry Breaking of a gauge theory. In that case it is possible to choose a gauge in which the propagator of the vector boson does indeed vanish like $1/p^2$ as $p \to \infty$.

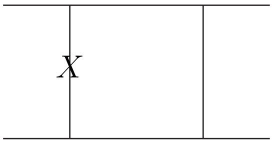
If $\omega_V < 4$ we have a super-renormalizable theory in which the number of counterterms needed to render the Green functions finite decreases as the number of loops increases. The interaction $\lambda\phi^3$ is an example of such a theory. The only cut-off dependent counterterm is the mass renormalization and this is only cut-off dependent at one loop. What this actually means is that beyond one loop the counterterms which we introduce in order to express S-matrix elements in terms of physical quantities are cut-off independent.

For the rest of this section we assume that we are dealing with a renormalizable, rather than a super-renormalizable, theory with generic coupling constant λ so that we assume that vertex correction graphs are ultraviolet divergent and that the divergences do persist in higher orders.

The real degree of divergence of a Feynman graph is the largest superficial divergence of any subgraph

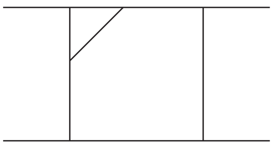
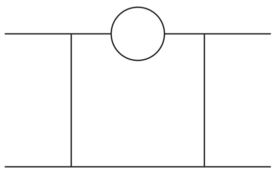


For such graphs, in a renormalizable theory such as QED, the superficial divergence of the entire graph is negative (box-graphs are UV finite [†]), but the real degree of divergence is the divergence of the self-energy insertion on one of the internal lines. This means that in association with the above graph we require a counterterm graph

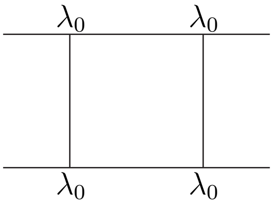


This counterterm renormalizes the mass of the internal line to which it is attached and also contributes to the renormalization of the couplings at either end of that line.

The other graphs which contribute to the renormalization of the couplings are all the remaining vertex corrections at all four vertices and all other self-energy insertions on internal and external lines, e.g.

 ,  etc.

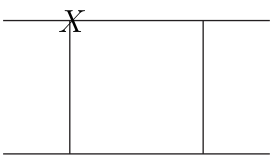
If we look at the one-loop graph with the coupling constants taken to be the bare couplings

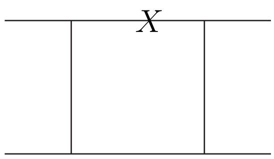


and write

$$\lambda_0 \;=\; \frac{Z_1}{Z^{3/2}}\lambda_R,$$

expand Z and Z_1 to order λ_R^2 we get the above diagram with λ_0 replaced by λ_R everywhere plus all the possible counterterm graphs such as



, 

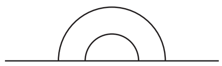
etc.

[†]In a non-abelian gauge theory box diagrams with four external gauge-bosons also require renormalization. This is because there is a four-point coupling between four gauge-bosons at the tree level proportional to g^2 and the renormalization of the coupling constant, g gives rise to a counterterm for the four-point gauge-boson graph.

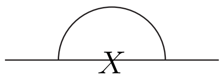
Together with the counterterms associated with mass renormalization, these counterterms render all the divergent subgraphs finite up to a factor of $\sqrt{Z}$ for each external line. The renormalized Green function expressed as a power series in the renormalized coupling and using renormalized masses is therefore cut-off independent, provided the counterterms associated with all the superficially divergent subgraphs have been accounted for.

This technique can also be used for the higher order computation of a Green function which itself has a non-negative degree of superficial divergence.

For example, at two loop level there is a graph contributing to the self-energy which is

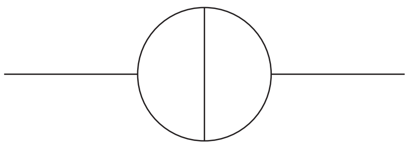


To this we must add a counterterm which renormalizes the mass of the lower propagator, and a counterterm $(Z-1)$ which contributes to the renormalization of the couplings of the one-loop graph.

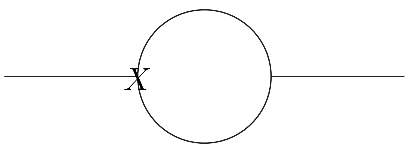


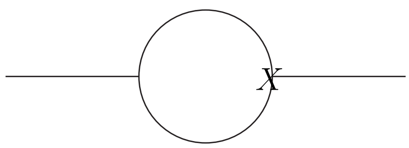
The remaining contribution (which for a general renormalizable theory will still be cut-off dependent), contributes to the λ_R^4 term in the expansions of Z and δm.

Sometimes we will have overlapping divergent sub-diagrams such as

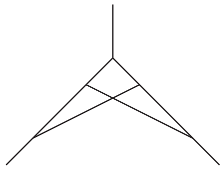


Associated with this we have two counterterm graphs, corresponding to renormalizations of the vertex on the left- and right- of the graph.





There are also some graphs which have no divergent subgraphs, but which are nevertheless divergent and contribute to a counterterm at order λ_R^4

 The subgraphs are all four-point, which we assume to be finite in the theory we are considering, but there is an overall divergence which contributes to Z_1 at order λ_R^4.

The central theorem of renormalization (proved by Bogoliobov, Parasiuk, Hepp and Zimmerman (BPHZ)) states that this procedure can be used to render cut-off independent all renormalized Green functions provided their superficial degree of divergence is negative. In other words, for a renormalizable theory we have a finite number of counterterms, which, in general, have cut-off dependent contributions in all orders in perturbation theory. Provided these counterterms are used in association with all divergent subgraphs, then as we go to higher orders we do *not* have to introduce further counterterms to cancel off infinities that occur in subgraphs.

3.2 Regularization:

A regulator is a process which renders finite a momentum integral which is superficially divergent. Ideally, we would like the regulator to preserve the symmetries of the theory, so that the counterterms calculated using that regulator automatically preserve the symmetries.

The simple cut-off procedure used previously does not in general do this.

Pauli-Villars Regulator
Before the relevance of gauge-theories was recognized, the most popular method of regulating ultraviolet divergent integrals was to replace a propagator

$$\frac{1}{k^2 - m^2}$$

by the regulated propagator

$$\sum_{i=0}^{\infty} a_i \frac{1}{k^2 - m_i^2},$$

where $a_0 = 1$ and $m_0 = m$.

If we expand each term of this sum as a power series in k^2 we get

$$\sum_{i=0}^{\infty}\frac{a_i}{k^2}+\sum_{i=0}^{\infty}\frac{a_i m_i^2}{k^4}+\mathcal{O}\left(\frac{1}{k^6}\right).$$

For a renormalizable theory the maximum superficial power of divergence of any integral is quadratic, so that the $\mathcal{O}(1/k^6)$ terms are ultraviolet finite. The finiteness of the regulated integral is then guaranteed by requiring that

$$\begin{aligned}\sum_{i=0}^{\infty} a_i &= 0,\\ \sum_{i=0}^{\infty} a_i m_i^2 &= 0.\end{aligned}$$

Dimensional Regularization

The above method of regularization is unsuitable for gauge-theories, because gauge invariance requires that the gauge-bosons should be massless, so that the Pauli-Villars regulated propagator, which introduces masses, would break this gauge invariance.

A more useful method is the method of “dimensional regularization”, which relies on the fact that most symmetries (excluding supersymmetry, which will be discussed briefly later) do not depend on the number of dimensions of the space in which we are working.

The integral that we wish to regulate is performed not in four dimensions, but in a number of dimensions, d, for which the integral is finite. An analytic continuation is made in the variable d. This analytic function can be expanded as a Laurent series about $d = 4$ and the fact that the symmetry is preserved in all dimensions means that each term in the series will respect the symmetry. The divergences appear as poles at $d = 4$ and the regularization is effected by removing these poles.

In d dimensions the typical integrals that we obtain after going through the steps of Feynman parametrization, shifting the variable of integration, and rotating to Eucidean space is of the form

$$I_0(\alpha) \equiv i\frac{(-1)^\alpha}{(2\pi)^d}\int\frac{d^dk}{(k^2+A^2)^\alpha} = i\frac{(-1)^\alpha}{(2\pi)^d}\int\frac{d\Omega_d\, k^{d-1}\, dk}{(k^2+A^2)^\alpha}.$$

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

is the area of the surface of a sphere in d dimensions, and

$$\int_0^\infty \frac{x^{d/2-1}\, dx}{(x+1)^\alpha} = \frac{\Gamma(d/2)\,\Gamma(\alpha-d/2)}{\Gamma(\alpha)},$$

so that the integral we are left with is

$$I_0(\alpha) = i\frac{(-1)^\alpha}{(4\pi)^{d/2}}\frac{\Gamma(\alpha-d/2)}{\Gamma(\alpha)}A^{d-2\alpha} \tag{3.6}$$

If $\alpha > (d/2)$, the integral in finite. As an analytic function of d, it has poles if $(\alpha - d/2)$ is zero or a negative integer.

We perform a Laurent expansion about $d = 4$, defining the quantity ϵ † by

$$d \; = \; 4 - 2\epsilon,$$

giving rise to a pole term at $\epsilon = 0$ for any integral which is superficially divergent in four dimensions, plus terms which are finite as $\epsilon \to 0$. Each of these terms preserves the (dimension independent) symmetries of the theory.

For example of $\alpha = 2$ we get

$$i\frac{\Gamma(\epsilon)}{16\pi^2}\left(\frac{4\pi}{A^2}\right)^{\epsilon} \; = \; \frac{i}{16\pi^2}\left(\frac{1}{\epsilon} + \ln\left(\frac{4\pi}{A^2}\right) - \gamma_E \; + \mathcal{O}(\epsilon)\right),$$

where we have used

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon).$$

In the so-called "minimal subtraction (MS)" renormalization scheme the pole part

$$\frac{i}{16\pi^2\,\epsilon}$$

is associated with the counterterm and the regularized integral is the remaining part

$$\frac{i}{16\pi^2}\left(\ln\left(\frac{4\pi}{A^2}\right) - \gamma_E\right).$$

However, the $\ln(4\pi)$ and the Euler constant γ_E. always appear in the finite part of the integral. They have no physical significance and are merely an artifact of the subtraction scheme. A more convenient scheme is the so-called "$\overline{MS}$" scheme, in which the $\ln(4\pi)$ and the γ_E are subtracted off along with the pole part, so that the counterterm is

$$\frac{i}{16\pi^2}\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right)$$

and the regulated integral is

$$-\frac{i}{16\pi^2}\,\ln\left(A^2\right).$$

We note that choosing the counterterms to be the pole part as in the MS scheme or using the $\overline{MS}$ scheme are perfectly valid renormalization prescriptions. However, in these schemes the renormalized coupling constant is not directly related to any physical measurement. Furthermore the renormalized mass is not the physically measured mass. Nevertheless the

†Many authors use a definition of $d = 4 - \epsilon$. We choose this definition because $d/2$ appears very often in the formulae for the integrals.

renormalized couplings and masses obtained in these schemes can be used as parameters, and *all* physical observables - including masses - can be calculated in terms of these renormalized parameters.

A further integral, which will occur often in higher order calculations is of the form

$$i\frac{(-1)^\alpha}{(2\pi)^d}\int\frac{d^dk\,k^\mu k^\nu}{(k^2+A^2)^\alpha}$$

Now by symmetry this will only be non-zero if μ and ν are in the same direction, so we may write

$$i\frac{(-1)^\alpha}{(2\pi)^d}\int\frac{d^dk\,k^\mu k^\nu}{(k^2+A^2)^\alpha} \quad = -I_2(\alpha)g^{\mu\nu}, \tag{3.7}$$

(the minus coming from the fact that we have rotated to Euclidian space). Contracting both sides with $g_{\mu\nu}$ are recalling that $g^{\mu\nu}g_{\mu\nu}=d$, we have

$$I_2(\alpha) = -i\frac{(-1)^\alpha}{d(2\pi)^d}\int\frac{d^dk\,k^2}{(k^2+A^2)^\alpha}$$

Writing $k^2=(k^2+A^2)-A^2$, we have

$$I_2(\alpha) \;=\frac{1}{d}\left(I_0(\alpha-1)-A^2I_0(\alpha)\right)$$

Using Eq.(3.6), this gives

$$I_2(\alpha) \;=\; -i\frac{(-1)^\alpha}{(4\pi)^{d/2}}A^{d+2-2\alpha}\left(\frac{\Gamma(\alpha-1-d/2)}{d\,\Gamma(\alpha-1)}-\frac{\Gamma(\alpha-d/2)}{d\,\Gamma(\alpha)}\right).$$

Manipulating the Γ functions this reduces to

$$I_2(\alpha) \;=\; -i\frac{(-1)^\alpha}{(4\pi)^{d/2}}A^{d+2-2\alpha}\frac{\Gamma(\alpha-1-d/2)}{2\,\Gamma(\alpha)} \tag{3.8}$$

Another useful integral is obtained by contracting both sides of eq.(3.7) and using eq.(3.8)

$$i\frac{(-1)^\alpha}{(2\pi)^d}\int\frac{d^dk\,k^2}{(k^2+A^2)^\alpha} \quad = d\,I_2(\alpha) \;=\; -i\frac{(-1)^\alpha}{(4\pi)^{d/2}}\frac{d}{2}A^{d+2-2\alpha}\frac{\Gamma(\alpha-1-d/2)}{\Gamma(\alpha)} \tag{3.9}$$

Other consequences of dimensional regularization are:

1. The action, which must be dimensionless, is now

$$S \;=\; \int d^{4-2\epsilon}x\mathcal{L}.$$

From the quadratic part of the Lagrangian density, we conclude that a fermion field, which enters as $\bar{\Psi}\gamma\cdot\partial\Psi$ must have dimension $\frac{3}{2}-\epsilon$, whereas a bosonic field, which enters as $(\partial_\mu\phi)^2$ or $F_{\mu\nu}F^{\mu\nu}$, must have dimension $1-\epsilon$.

When we consider the interaction terms, this in turn implies that the couplings acquire a dimension which differs from its dimensionality in four dimensions. For a renormalizable theory, the couplings are dimensionless in four dimensions. However, in $4-2\epsilon$ dimensions this will not be the case. For example, the bare electromagnetic coupling in QED, defined by the interaction term in the action

$$\int d^{4-2\epsilon}x e_0 \bar{\Psi}\gamma^\mu A_\mu \Psi,$$

must be replaced by

$$\int d^{4-2\epsilon}x \tilde{e}_0(\mu)\mu^\epsilon \bar{\Psi}\gamma^\mu A_\mu \Psi,$$

where μ is some mass scale and $\tilde{e}_0(\mu)$ is a dimensionless quantity. In other words the bare coupling e_0 has dimension ϵ.

When this is expanded as a power series in ϵ, we find that there is always a term $\ln(\mu^2)$ accompanying the pole at $\epsilon = 0$. This scale μ serves as the subtraction point in the renormalization procedure.

2. In dimensional regularization, the Dirac algebra must also be carried out in $4-2\epsilon$ dimensions.

$$\{\gamma^\mu \gamma^\nu\} = 2g^{\mu\nu},$$

but

$$\begin{aligned} g_{\mu\nu}g^{\mu\nu} &= 4-2\epsilon \\ \gamma^\mu\gamma_\mu &= (4-2\epsilon)I \end{aligned}$$

Thus, for example

$$\begin{aligned} \gamma^\mu\gamma^\nu\gamma_\mu &= -2(1-\epsilon)\gamma^\nu, \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu &= 4g^{\nu\rho} - 2\epsilon\gamma^\mu\gamma^\rho, \\ \gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu &= -2\gamma^\sigma\gamma^\rho\gamma^\nu + 2\epsilon\gamma^\nu\gamma^\rho\gamma^\sigma. \end{aligned}$$

Strictly, we should also have

$$\mathrm{Tr}\, I = 2^{d/2},$$

but it turns out that one can always use

$$\mathrm{Tr}\, I = 4,$$

and absorb the factor of $2^{d/2-2}$ into the renormalization of the coupling.

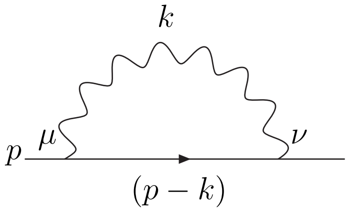
Dimensional Reduction:
This method of regularization is not suitable for dealing with supersymmetry which only holds in a given number of dimensions. For example, in a four dimensional supersymmetric theory a Majorana fermion has two degrees of freedom and is accompanied by two scalar superpartners. A vector field has four degrees of freedom and is accompanied by a Dirac fermion, which also has four degrees of freedom.

The solution to this problem is to introduce extra scalar particles called "$\epsilon-$scalars" which compensate for the "lost" bosonic degrees of freedom as the number of dimensions is reduced below four. Thus, for example, in three dimensions a vector field is replaced by a vector field with three degrees of freedom plus a new scalar field which interacts with the fermions with the same coupling. In this way, the fermion, the vector field and the extra scalar can be combined into a supermultiplet.

The upshot of this scheme is that the Dirac algebra is once again carried out in four dimensions, using the rules of the four dimensional, whereas the integrals over the loop momentum is carried out in $4-2\epsilon$ dimensions.

4 One-Loop Counterterms in QED

4.1 Fermion Self-energy



We work in Feynman gauge. Applying the rules of QED we have (in d dimensions)

$$\Sigma(p^2,m) \;=\; i\mu^{(4-d)}\int\frac{d^dk}{(2\pi)^d}(-ie\gamma^\mu)\frac{i\left(\gamma\cdot(p-k)+m\right)}{(p-k)^2-m^2}(-ie\gamma^\nu)\frac{-ig_{\mu\nu}}{k^2}, \tag{4.1}$$

where we have displayed explicitly the scale dependence of the coupling outside four dimensions.

Introducing Feynman parameters, we get

$$\Sigma(p^2,m) \;=\; -i\,e^2\mu^{(4-d)}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\int_0^1 d\beta\delta(1-\alpha-\beta)\frac{\gamma^\mu\left(\gamma\cdot(p-k)+m\right)\gamma_\mu}{\left(k^2(\alpha+\beta)-2p\cdot k\,\alpha+(p^2-m^2)\,\alpha\right)^2} \tag{4.2}$$

Now shift $k\to k+p\,\alpha$ (and perform the trivial integral over β absorbing the δ-function)

$$\Sigma(p^2,m) \;=\; -i\,e^2\mu^{(4-d)}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\frac{\gamma^\mu\left(\gamma\cdot p(1-\alpha)+m\right)\gamma_\mu}{\left(k^2+p^2(\alpha(1-\alpha)-m^2\alpha)\right)^2} \tag{4.3}$$

We have omitted a term

$$-i\,e^2\mu^{(4-d)}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\frac{\gamma^\mu\gamma\cdot k\gamma_\mu}{\left(k^2+p^2(\alpha(1-\alpha)-m^2\alpha)\right)^2},$$

which vanishes by symmetry since the numerator is odd in k, the denominator is even in k and we must integrate over all directions of the vector k.

Setting $d=4-2\epsilon$, using $\gamma^\mu\gamma\cdot p\,\gamma_\mu=-2(1-\epsilon)\gamma\cdot p$, $\gamma^\mu\,\gamma_\mu=4-2\epsilon$ and the result from eq.(3.6) we have

$$\Sigma(p^2,m) \;=\; -\frac{e^2}{(16\pi^2)}\Gamma(\epsilon)\int_0^1 d\alpha\left[2(1-\epsilon)(1-\alpha)\gamma\cdot p-(4-2\epsilon)m\right]\left(\frac{4\pi\mu^2}{m^2\alpha-p^2\alpha(1-\alpha)}\right)^\epsilon \tag{4.4}$$

Expanding in ϵ and keeping only the terms which do not vanish as $\epsilon \to 0$, we get

$$\begin{aligned}\Sigma(p^2, m) &= -\frac{e^2}{(16\pi^2)}\left[\int_0^1 d\alpha\,(2(1-\alpha)\gamma\cdot p - 4\,m)\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right)\right. \\ &+ \left.\int_0^1 d\alpha\left(2\gamma\cdot p(1-\alpha) - 2\,m + (2(1-\alpha)\gamma\cdot p - 4m)\ln\left(\frac{m^2\alpha - p^2\alpha(1-\alpha)}{\mu^2}\right)\right)\right] \end{aligned} \tag{4.5}$$

Performing the integral over α except in the last term, this reduces to

$$\begin{aligned}\Sigma(p^2, m) &= -\frac{e^2}{(16\pi^2)}\left[(\gamma\cdot p - 4\,m)\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right)\right. \\ &+ \left.\left(\gamma\cdot p - 2\,m + \int_0^1 d\alpha\,(2(1-\alpha)\gamma\cdot p - 4m)\ln\left(\frac{m^2\alpha - p^2\alpha(1-\alpha)}{\mu^2}\right)\right)\right] \end{aligned} \tag{4.6}$$

In order to obtain the (physical) mass subtraction term, δm, and the wavefunction renormalization constant Z_2, we must expand this in a power series in $(\gamma\cdot p - m)$, making use of the relation

$$p^2 - m^2 = (\gamma\cdot p - m)(\gamma\cdot p + m) = 2\,m(\gamma\cdot p - m) + \mathcal{O}((\gamma\cdot p - m)^2).$$

This enables us to expand the logarithm about $p^2 = m^2$. This gives

$$\begin{aligned}\Sigma(p^2, m) &= \frac{e^2}{(16\pi^2)}\left[3m\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right) - m + 2m\int_0^1 d\alpha(1-\alpha)\ln\left(\frac{m^2\alpha^2}{\mu^2}\right)\right] \\ &+ \frac{e^2}{(16\pi^2)}\left[1 - \frac{1}{\epsilon} - \ln(4\pi) + \gamma_E + 2\int_0^1 (1-\alpha)\ln\left(\frac{m^2\alpha^2}{\mu^2}\right) + 4\int_0^1 d\alpha\frac{(1-\alpha^2)}{\alpha}\right](\gamma\cdot p - m) \\ &+ \mathcal{O}((\gamma\cdot p - m)^2) \end{aligned} \tag{4.7}$$

The terms which are $\mathcal{O}((\gamma\cdot p - m)^2$ and higher are finite and independent of the scale μ. They make up the renormalized self-energy $\Sigma_R(p^2, m)$. The last integral over α in eq.(4.7) diverges at $\alpha = 0$. This is a new type of divergence caused by the fact that the photon is massless - it is called an "infrared divergence". For the moment we regularize this infrared divergence by assigning a small mass, λ to the photon wherever necessary (i.e. we only keep terms in λ which are *not* regular as $\lambda \to 0$. When we do this the last integral in eq.(4.7) becomes

$$\int_0^1 d\alpha\frac{\alpha(1-\alpha^2)}{\alpha^2 - (1-\alpha)\lambda^2/m^2} = \frac{1}{2}\left(\ln(\frac{m^2}{\lambda^2} - 1\right) + \mathcal{O}(\lambda^2).$$

Writing (to this order in perturbation theory),

$$\Sigma(p^2, m) = \delta m + (Z_2 - 1)(\gamma\cdot p - m) + \Sigma_R(p^2, m).$$

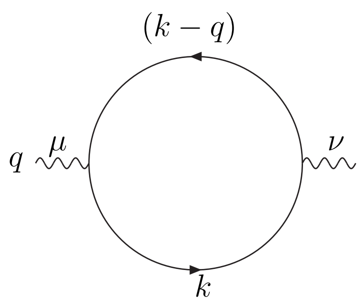
we have for the mass renormalization (introducing the fine-structure constant $\alpha = e^2/(4\pi)$),

$$\begin{aligned}\delta m &= m\frac{\alpha}{4\pi}\left[3\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right) - 1 - 2\int_0^1 d\alpha(1-\alpha)\ln\left(\frac{m^2\alpha^2}{\mu^2}\right)\right] \\ &= 3m\frac{\alpha}{4\pi}\left(\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E + \frac{4}{3} + \ln\left(\frac{m^2}{\mu^2}\right)\right), \end{aligned} \tag{4.8}$$

and for the wavefunction renormalization constant,

$$
\begin{aligned}
Z_2 &= 1+\frac{\alpha}{4\pi}\left[1-\frac{1}{\epsilon}-\ln(4\pi)+\gamma_E+2\int_0^1(1-\alpha)\ln\left(\frac{m^2\alpha^2}{\mu^2}\right)+4\int_0^1 d\alpha\frac{\alpha(1-\alpha^2))}{\alpha^2+(1-\alpha)\lambda^2/m^2}\right] \\
&= 1+\frac{\alpha}{4\pi}\left[-\frac{1}{\epsilon}-\ln\left(\frac{\mu^2}{m^2}\right)+\gamma_E-\ln(4\pi)-4+2\ln\left(\frac{m^2}{\lambda^2}\right)\right]. \qquad (4.9)
\end{aligned}
$$

4.2 Photon Self-energy (Vacuum polarization)



The photon self-energy $\Pi^{\mu\nu}(q^2)$ is, in general, a two-rank tensor, which is formed from the four-momentum of the photon , q^μ and the (invariant) metric tensor. It must therefore have the form

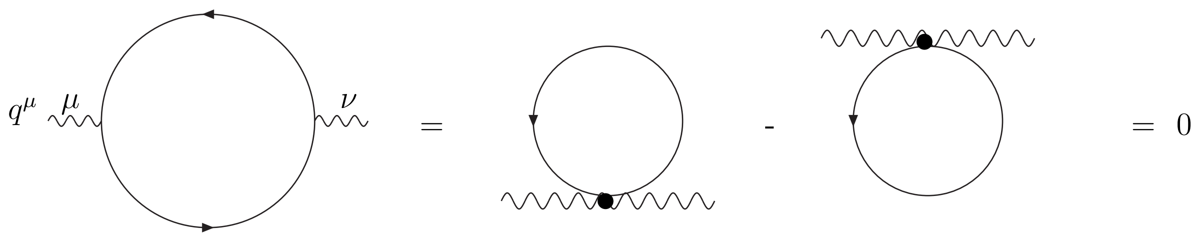
$$\Pi^{\mu\nu}(q^2) \;=\; A(q^2)g^{\mu\nu}+B(q^2)q^\mu q^\nu .$$

On the other hand $\Pi^{\mu\nu}(q^2)$ obeys a Ward identity

$$q_\mu\Pi^{\mu\nu}(q^2) \;=\; 0.$$

This can be seen by writing

$$\frac{1}{(\gamma\cdot k-m)}q^\mu\gamma_\mu\frac{1}{(\gamma\cdot(k-q)-m)}=\frac{1}{(\gamma\cdot(k-q)-m)}-\frac{1}{(\gamma\cdot k-m)}$$



Applying this to the one-loop graph representing the photon self-energy, we get the difference between two graphs in which one of the two internal fermion propagates has been killed. But these two graphs are identical and so the difference is zero.

We may therefore write

$$\Pi^{\mu\nu}(q^2) \;=\; \left(-g^{\mu\nu}q^2 + q^\mu q^\nu\right)\Pi(q^2) \tag{4.10}$$

In other words only the transverse part of the photon propagator acquires a higher order correction.

The photon has no mass and therefore no mass renormalization. There is only a photon wavefunction renormalization constant Z_3.

$$\Pi(q^2) \;=\; \frac{1}{Z_3}\left((Z_3-1)+\Pi_R(q^2)\right), \tag{4.11}$$

where $\Pi_R(q^2)$ is the renormalized (finite) part of the self-energy. At the one-loop level the prefactor $1/Z_3$ in eq.(4.11) may be set to unity.

The fact that only the transverse part of the photon-propagator acquires a higher-order correction means that the gauge parameter, ξ is renormalized. If we write the leading order propagator as

$$-i\frac{\left[\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)-(\xi-1)\frac{q^\mu q^\nu}{q^2}\right]}{q^2}$$

The renormalized propagator is

$$-i\frac{Z_3\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)}{q^2\left(1-\Pi_R(q^2)\right)}+i(\xi-1)\frac{q^\mu q^\nu}{q^4}.$$

The transverse part of the propagator is renormalized but not the longitudinal part. Near $q^2=0$, the renormalized propagator looks like

$$-i\frac{Z_3\left[\left(g^{\mu\nu}-\frac{q^\mu q^\nu}{q^2}\right)-(\xi_R-1)\frac{q^\mu q^\nu}{q^2}\right]}{q^2},$$

where

$$(\xi_R-1)=\frac{(\xi-1)}{Z_3}.$$

Now returning to the one-loop graph and inserting the Feynman rules, we get

$$\Pi^{\mu\nu}(q^2) \;=\; -i\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\mathrm{Tr}\left[(-ie\gamma^\mu)\frac{i(\gamma\cdot(k-q)+m)}{((k-q)^2-m^2)}(-ie\gamma^\nu)\frac{i(\gamma\cdot k+m)}{k^2-m^2)}\right]. \tag{4.12}$$

An overall minus sign has been introduced for a loop of fermions. This arises form the fact that the Wick contraction required to construct the Feynman graph requires an interchange of two fermion fields, thereby introducing a minus sign.

Feynman parametrization gives

$$\Pi^{\mu\nu}(q^2) \;=\; -ie^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha d\beta\delta(1-\alpha-\beta)\frac{\mathrm{Tr}\left[\gamma^\mu(\gamma\cdot(k-q)+m)\gamma^\nu(\gamma\cdot k+m)\right]}{\left(k^2-2k\cdot q\alpha+q^2\alpha-m^2\right)^2}. \tag{4.13}$$

Performing the trace (and integrating over β) gives

$$\Pi^{\mu\nu}(q^2) \;=\; 4\,ie^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\frac{g^{\mu\nu}\left(k\cdot(k-q)-m^2\right)-2k^\mu k^\nu+k^\mu q^\nu+q^\mu k^\nu}{\left(k^2-2k\cdot q\alpha+q^2\alpha-m^2\right)^2}. \tag{4.14}$$

Shifting $k^\mu\to k^\mu+q^\mu\alpha$ we get

$$\Pi^{\mu\nu}(q^2) \;=\; 4\,ie^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha\frac{g^{\mu\nu}\left(k^2-\alpha(1-\alpha)q^2-m^2\right)-2k^\mu k^\nu+2\alpha(1-\alpha)q^\mu q^\nu}{\left(k^2+q^2\alpha(1-\alpha)-m^2\right)^2}, \tag{4.15}$$

where once again we have omitted terms linear in k, which vanish by symmetric integration.

From eq.(4.10) it is sufficient to extract only the terms in the above integral which are proportional to $g^{\mu\nu}$. Using eqs.(3.6), (3.8) and (3.9) we have (setting $d=4-2\epsilon$)

$$\begin{aligned}-q^2\Pi(q^2) \;=\; & -\frac{e^2}{4\pi^2}\int_0^1 d\alpha\left(\frac{4\pi\mu^2}{m^2-q^2\alpha(1-\alpha)}\right)^\epsilon\left[\Gamma(\epsilon-1)\left(\frac{1}{2}(4-2\epsilon)-1\right)\left(q^2\alpha(1-\alpha)-m^2\right)\right. \\ & \left.-\Gamma(\epsilon)\left(q^2\alpha(1-\alpha)+m^2\right)\right]\end{aligned} \tag{4.16}$$

Using

$$\Gamma(\epsilon-1)=-\frac{\Gamma(\epsilon)}{(1-\epsilon)},$$

it can be seen that the RHS of eq.(4.16) becomes proportional to q^2, so we have

$$\Pi(q^2) \;=\; -\frac{e^2}{2\pi^2}\Gamma(\epsilon)\int_0^1 d\alpha\alpha(1-\alpha)\left(\frac{4\pi\mu^2}{m^2-q^2\alpha(1-\alpha)}\right)^\epsilon \tag{4.17}$$

Expanding in ϵ up to terms which vanish as $\epsilon\to 0$, and performing the integral over α where appropriate, this gives

$$\Pi(q^2) \;=\; -\frac{e^2}{12\pi^2}\left[\frac{1}{\epsilon}+\ln(4\pi)-\gamma_E-6\int_0^1 d\alpha\alpha(1-\alpha)\ln\left(\frac{m^2-q^2\alpha(1-\alpha)}{\mu^2}\right)\right] \tag{4.18}$$

We define Z_3 to be $1+\Pi(0)$, so that we have (in terms of the fine-structure constant, α)

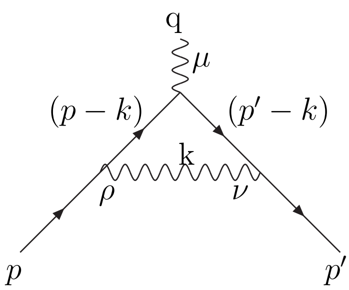
$$Z_3 \;=1-\frac{\alpha}{3\pi}\left[\frac{1}{\epsilon}+\ln(4\pi)-\gamma_E-\ln\left(\frac{m^2}{\mu^2}\right)\right], \tag{4.19}$$

and the renormalized photon self energy

$$\Pi_R(q^2) \;=\; \Pi(q^2)-(Z_3-1),$$

is proportional to q^2 so that it vanishes as the photon goes on mass-shell.

4.3 The Vertex Function



The fermions have momenta p and p' and the photon has momentum $q = p' - p$. In general, we will have processes (such as Compton scattering of photons off electrons) in which one of the fermion legs are internal and therefore off-shell. Here we restrict ourselves to fermion scattering in which both the fermion legs are on-shell, i.e. $p^2 = p'^2 = m^2$.

Using the Feynman rules the vertex correction factor, $\Gamma^\mu(p, p')$ is given (in d-dimensions and in Feynman gauge) by

$$\Gamma^\mu(p,p') \;=\; \mu^{4-d}\int \frac{d^dk}{(2\pi)^d}(-ie\gamma^\nu)\frac{i(\gamma\cdot(p'-k)+m)}{((p'-k)^2-m^2)}(\gamma^\mu)\frac{i(\gamma\cdot(p-k)+m)}{((p-k)^2-m^2)}(-ie\gamma^\rho)\frac{-ig_{\nu\rho}}{(k^2-\lambda^2)}, \tag{4.20}$$

where we have introduced a small photon mass λ in anticipation of the fact that we will also have infrared divergences here.

Introducing Feynman parameters (see eq.(2.4)), this may be written

$$\Gamma^\mu(p,p') \;=\; -i2e^2\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha d\beta d\gamma\frac{\delta(1-\alpha-\beta-\gamma)\mathcal{N}}{(k^2-2k\cdot(p\alpha+p'\beta)-\lambda^2\gamma)^3}, \tag{4.21}$$

where $\mathcal{N}$ is the numerator

$$\mathcal{N} \;=\; \gamma^\nu(\gamma\cdot(p'-k)+m)\gamma^\mu(\gamma\cdot(p-k)+m)\gamma_\nu$$

Shift $k \to k + p\alpha + p'\beta$ (and perform the integral over γ), to get

$$\Gamma^\mu(p,p') \;=\; -i2e^2\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\int_0^1 d\alpha d\beta d\gamma \frac{\theta(1-\alpha-\beta)(\mathcal{N}_2+\mathcal{N}_0)}{\left(k^2-\lambda^2(1-\alpha-\beta)-m^2(\alpha+\beta)^2+q^2\alpha\beta\right)^3}, \tag{4.22}$$

where we have made use the on-shell condition of the fermions and written $p\cdot p' = m^2 - q^2/2$.

$$\mathcal{N}_2 \;=\; k_\rho k_\sigma \gamma^\nu\gamma^\rho\gamma^\mu\gamma^\sigma\gamma_\nu,$$

is the part of the integral which will give an ultraviolet divergence. Using eq.(3.8), the contribution to the vertex correction function from this part is

$$\begin{aligned}\Gamma^{\mu\,div}(p,p') \;&=\; \frac{e^2}{32\pi^2}\Gamma(\epsilon)\int d\alpha d\beta\theta(1-\alpha-\beta)\gamma^\nu\gamma^\sigma\gamma^\mu\gamma^\sigma\gamma_\nu\left(\frac{4\pi\mu^2}{(m^2(\alpha+\beta)^2-q^2\alpha\beta)}\right)^\epsilon \\ &=\; \frac{e^2}{8\pi^2}\Gamma(\epsilon)\int d\alpha d\beta\theta(1-\alpha-\beta)\gamma^\mu(1-\epsilon)^2\left(\frac{4\pi\mu^2}{(m^2(\alpha+\beta)^2-q^2\alpha\beta)}\right)^\epsilon =\end{aligned}$$

$$\frac{e^2}{16\pi^2}\gamma^\mu\left[\frac{1}{\epsilon}+\ln(4\pi)-\gamma_E-2-2\int d\alpha d\beta\theta(1-\alpha-\beta)\ln\left(\frac{(m^2(\alpha+\beta)^2-q^2\alpha\beta)}{\mu^2}\right)\right] \tag{4.23}$$

The $\mathcal{N}_0$ term does not lead to an ultraviolet divergence and may be calculated in four dimensions † . We have

$$\mathcal{N}_0 \;=\; \gamma^\nu(\gamma\cdot p'(1-\beta)-\gamma\cdot p\,\alpha+m)\gamma^\mu(\gamma\cdot p(1-\alpha)-\gamma\cdot p'\,\beta+m)\gamma_\nu,$$

which we may write as

$$\mathcal{N}_0 \;=\; -2(\gamma\cdot p(1-\alpha)-\gamma\cdot p'\,\beta)\gamma^\mu(\gamma\cdot p'(1-\beta)-\gamma\cdot p\,\alpha)-2m^2\gamma^\mu+4m(1-\alpha-\beta)(p+p')^\mu,$$

where we have used the symmetry under $\alpha \leftrightarrow \beta$.

We can consider $\mathcal{N}_0$ to be sandwiched between fermion spinors $\bar{u}(p',m)$ and $u(p,m)$. We have the identity

$$(p+p')^\mu \;=\; \frac{1}{2}\left\{\gamma\cdot(p+p'),\gamma^\mu\right\} \;=\; \gamma\cdot p'\gamma^\mu+\gamma^\mu\gamma\cdot p-\frac{1}{2}q^\nu\left[\gamma^\nu,\gamma^\mu\right] \;=\; 2m\gamma^\mu+iq_\nu\sigma^{\mu\nu}$$

where in the last step we have used the fact that $\gamma\cdot p'$ on the left or $\gamma\cdot p$ on the right generates m since they are adjacent to fermion spinors. The matrices $\sigma^{\mu\nu}$ are the generators of Lorentz transformations in the spinor representation

$$\sigma^{\mu\nu} = -\frac{i}{2}\left[\gamma^\mu,\gamma^\nu\right]$$

Furthermore, we have

$$\gamma\cdot q\gamma^\mu\gamma\cdot q \;=\; 2q^\mu\gamma\cdot q+q^2\gamma^\mu.$$

†Once again, the numerator term $\mathcal{N}_1$, which is linear in k after shifting, is omitted since it vanishes by symmetric integration.

The first term vanishes when sandwiched between fermion spinors leaving only the term $q^2\gamma^\mu$.

Using these relations the numerator $\mathcal{N}_0$ becomes

$$\mathcal{N}_0 = \left(8(1-\alpha-\beta) - 2 - 2(1-\alpha-\beta)^2)m^2 - 2(1-\alpha)(1-\beta)q^2\right)\gamma^\mu$$
$$+4im\left((1-\alpha-\beta) - (1-\alpha)(1-\beta)\right)q_\nu\sigma^{\mu\nu}.$$

The counterterm associate wih the vertex is $(Z_1 - 1)\gamma^\mu$, so we write

$$\Gamma^\mu_R = \Gamma^\mu + (Z_1 - 1)\gamma^\mu,$$

where Γ^μ_R is the finite renormalized vertex correction.

The usual definition of the renormalized electromagnetic coupling is the coupling at zero momentum transfer. In other words we must choose the renormalization constant Z_1 such that $\Gamma^\mu_R(p,p) = 0$, so that we get for

$$Z_1 = 1 + \frac{e^2}{16\pi^2}\left[-\frac{1}{\epsilon} - \ln(4\pi) + \gamma_E - \ln\left(\frac{\mu^2}{m^2}\right) + 2\right.$$
$$\left. +4\int d\alpha d\beta\theta(1-\alpha-\beta)\left\{\ln(\alpha+\beta) + m^2\frac{2(1-\alpha-\beta) - 1 - (1-\alpha-\beta)^2}{m^2(\alpha+\beta)^2 + \lambda^2(1-\alpha-\beta)^2}\right\}\right]. \quad (4.24)$$

The last integral has an infrared divergence as $\lambda \to 0$

The nested integral over α and β is most easily performed by the change of variables

$$\alpha = \rho\omega$$
$$\beta = \rho(1-\omega)$$

The range of ρ and ω are now both from 0 to 1, and there is a factor of ρ from the jacobian. The integrand depends only of ρ so the integral over ω just gives a factor of unity. We now have (in terms of the fine-structure constant, α)

$$Z_1 = 1 + \frac{\alpha}{4\pi}\left[-\frac{1}{\epsilon} - \ln(4\pi) + \gamma_E - \ln\left(\frac{\mu^2}{m^2}\right) + 2 + 4\int_0^1 \rho d\rho\left\{\ln(\rho) + \frac{2(1-\rho) - 1 - (1-\rho)^2}{\rho^2 + (1-\rho)\lambda^2/m^2}\right\}\right]$$
$$= 1 + \frac{\alpha}{4\pi}\left[-\frac{1}{\epsilon} - \ln(4\pi) + \gamma_E - \ln\left(\frac{\mu^2}{m^2}\right) - 4 + 2\ln\left(\frac{m^2}{\lambda^2}\right)\right] \quad (4.25)$$

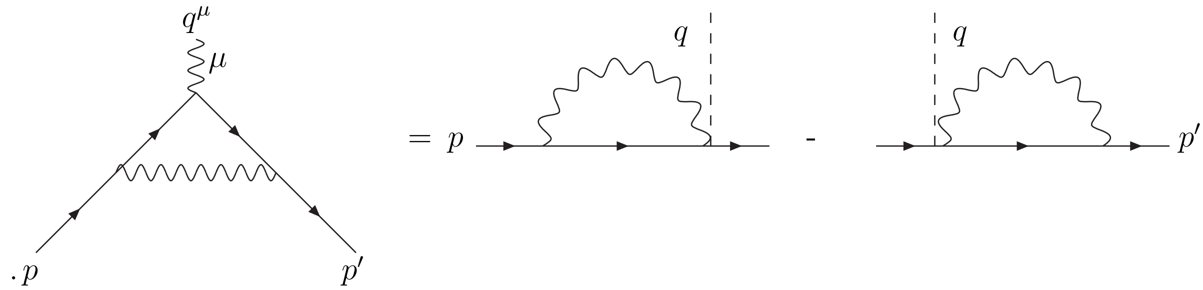
Examination of eq.(4.9) shows that we have

$$Z_1 = Z_2.$$

This is to be expected fropm another Ward identity. As in the case of the photon propagator, we can write

$$\frac{1}{\gamma\cdot(p'-k)-m}\gamma\cdot q\frac{1}{\gamma\cdot(p-k)-m} \;=\; \frac{1}{\gamma\cdot(p-k)-m}-\frac{1}{\gamma\cdot(p'-k)-m}$$

which lead diagrammatically to the "Ward identity"



The dashed line just represents the insertion of the momentum q into the fermion line.

Thus we see a relation between the vertex correction and the self-energy of the fermion. One of the consequences of this is that the divergent parts are related such that

$$q_\mu \Gamma^{\mu\, div} \;=\; \Sigma^{div}(p)-\Sigma^{div}(p')$$

The LHS is $(Z_1-1)q\cdot\gamma$. The mass renormalization cancels out from the two terms on the RHS and we are left with $Z_2(p'\cdot\gamma - p\cdot\gamma)$.

The renormalized vertex function has a term proprtional to γ^μ and a term proprtional to $q_\nu\sigma^{\mu\nu}$, and may be written as

$$\Gamma^\mu_R \;=\gamma^\mu F_1(q^2)\;+\;\frac{i}{2m}q_\nu\sigma^{\mu\nu}F_2(q^2) \tag{4.26}$$

The functions F_1 and F_2 which depend on q^2 for the case of on-shell fermion legs are known as the "electric" and "magnetic" form factors respectively.

The exact expressions for F_1 and F_2 are very complicated, but simplify in the limits $q^2 \gg m^2$ and $q^2 \ll m^2$. For $q^2 \gg m^2$ we have:

$$F_1(q^2)\;\to\;1-\frac{\alpha}{2\pi}\left[\left(\ln\left(\frac{-q^2}{m^2}\right)-1\right)\ln\left(\frac{m^2}{\lambda^2}\right)+\ln\left(\frac{-q^2}{m^2}\right)-2\right]$$

$$F_2\;\sim\;\frac{q^2}{m^2}$$

For $q^2 \ll m^2$ we have:

$$F_1(q^2) \to 1 + \frac{\alpha}{3\pi}\frac{q^2}{m^2}\left[\frac{1}{2}\ln\left(\frac{m^2}{\lambda^2}\right) - \frac{3}{8}\right]$$

$$F_2(q^2) \;\to \frac{\alpha}{2\pi}$$

The magnetic form-factor in the limit $q^2 \to 0$ acts as a correction to the magnetic moment of the electron, μ. In leading order

$$\mu = g_s \frac{e}{2m},$$

with $g_s = 2$. But in higher order

$$g_s - 2 \;=\; \frac{\alpha}{2\pi} + \cdots .$$

This has now been measured for the muon up to one part in 10^9 and calculated in QED up to five loops. The calculation up to three loops in QED agrees with experiment. There has recently been reported a two standard deviation discrepancy between the experimental observation and the theoretically calculated value. This discrepancy is assumed to be evidence for physics beyond the Standard Model rather than a breakdown of the validity of QED.

4.4 Ward Identities

The fact that $q_\mu \Pi^{\mu\nu}(q^2) = 0$ and $Z_1 = Z_2$ are examples of Ward identities derived from the fact that the interaction Hamiltonian density may be written $e\, A_\mu(x) j^\mu(x)$, where $j^\mu(x)$ is the conserved electromagnetic current, i.e. $\partial_\mu j^\mu = 0$.

The photon propagator, $G^{\mu\nu}(q^2)$, can always be written as the tree-level contribution, $G_0^{\mu\nu}(q^2)$ plus a correction which may be expressed as the tree-level propagator multiplying the vacuum expectation value of the time-ordered product of the electromagnetic current and the photon field. This is because the first interaction of the free photon is always with the electromagnetic current, i.e.

$$G^{\mu\nu}(q^2) \;=\; G_0^{\mu\rho}(q^2)\left[g^{\rho\nu} - i\int d^4x e^{-iq\cdot x}\langle 0|T j_\rho(x) A_\nu(0)|0\rangle\right], \tag{4.27}$$

where, in a general gauge, the tree-level propagator is

$$G_0^{\mu\nu}(q^2) \;=\; -i\frac{\left(g^{\mu\nu} - \xi\frac{q^\mu q^\nu}{q^2}\right)}{q^2}$$

$$q_\mu G_0^{\mu\rho}(q^2) \;=\; -i\xi\frac{q^\rho}{q^2}$$

$$\begin{aligned} -iq^\rho \int d^4x e^{-iq\cdot x}\langle 0|T j_\rho(x) A_\nu(0)|0\rangle \;&=\; -\int d^4x e^{-iq\cdot x}\frac{\partial}{\partial x_\rho}\langle 0|T j_\rho(x) A_\nu(0)|0\rangle \\ &=\; -\int d^4x e^{-iq\cdot x}\langle 0|\left[j_0(x), A_\nu(0)\right]|0\rangle\delta(x_0) \;=\; 0 \end{aligned}$$

where the last term arises because the derivative w.r.t. x_ρ has to act on the time ordering operator T giving rise to $\delta(x_0)$, as well as acting on the current, giving a term which vanishes by current conservation. The result is zero because the electromagnetic current and the photon field commute.

This then implies that

$$q_\mu G^{\mu\nu}(q^2) \;=\; q_\mu G_0^{\mu\nu}(q^2),$$

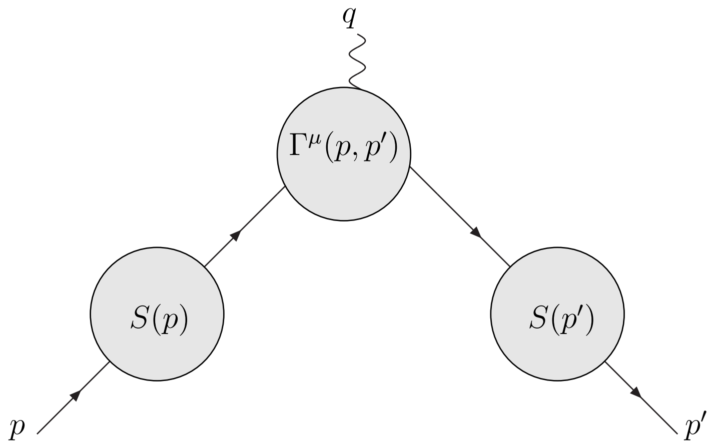
i.e. the longitudinal part of the photon propagator does not acquire higher order corrections to any order in perturbation theory.

We now apply the same technique to the quantity

$$\int d^4x d^4y d^4z e^{i(p'\cdot z - p\cdot y - q\cdot x)}\langle 0|T j_\mu(x)\Psi(y)\overline{\Psi}(z)|0\rangle \;=\; S(p')\Gamma^\mu(p,p')S(p)(2\pi)^4\delta^4(p+q-p'),$$

where $S(p)$ is the full electron propagator

$$S^{-1}(p) \;=\; -i\left(\gamma\cdot p - m - \Sigma(p)\right)$$



Here the quantity denoted as Γ^μ is the one-particle-irreducible vertex calculated to all orders and includes the tree level term $e_R\gamma^\mu$.

Contracting with q_μ,

$$q_\mu S(p')\Gamma^\mu(p,p')S(p)(2\pi)^4\delta^4(p+q-p') \;=\; i\int d^4x d^4y d^4z e^{i(p'\cdot z - p\cdot y - q\cdot x)}\frac{\partial}{\partial x_\mu}\langle 0|Tj^\mu(x)\Psi(y)\overline{\Psi}(z)|0\rangle$$

Since $\partial_\mu j^\mu = 0$, we only pick up the contributions from the derivative acting on the time-ordering operator so this gives

$$\int d^4x d^4y d^4z e^{i(p'\cdot z - p\cdot y - q\cdot x)}\left\{\langle 0|T\left[j_0(x),\Psi(y)\right]\overline{\Psi}(z)|0\rangle\delta(x_0-y_0)\right.$$
$$\left. -\langle 0|T\left[j_0(x),\overline{\Psi}(z)\right]\Psi(y)|0\rangle\delta(x_0-z_0)\right\},$$

where the relative minus sign arises from commuting two fermion fields. Using

$$j_0(x) \;=\; \Psi^\dagger(x)\Psi(x)$$

and

$$\left\{\Psi(x),\Psi^\dagger(y)\right\}\delta(x_0-y_0) \;=\; \delta^4(x-y)$$

the commutation relations give

$$\left[j_0(x),\Psi(y)\right]\delta(x_0-y_0) \;=\; -\Psi(y)\delta^4(x-y)$$
$$\left[j_0(x),\overline{\Psi}(z)\right]\delta(x_0-z_0) \;=\; -\overline{\Psi}(z)\delta^4(x-y).$$

Integrating over x to absorb the $\delta-$functions, we get

$$\begin{aligned} q_\mu S(p')\Gamma^\mu(p,p')S(p)(2\pi)^4\delta^4(p+q-p') \;&=\; \\ -i\int d^4y d^4z e^{-i\frac{1}{2}(q-p'+p)\cdot(z+y)}&\left[e^{ip'\cdot(z-y)}\langle 0|T\Psi(y)\overline{\Psi}(z)|0\rangle - e^{ip\cdot(y-z)}\langle 0|T\Psi(y)\overline{\Psi}(z)|0\rangle\right] \\ &=\; i\left(S(p)-S(p')\right)(2\pi)^4\delta^4(p+q-p') \end{aligned} \tag{4.28}$$

(here Γ^μ includes the tree diagram γ^μ as well as all higher order corrections).

Dividing both sides the the external fermion propagators, this gives

$$q_\mu \Gamma^\mu(p,p') \;=\; i\left(S^{-1}(p') - S^{-1}(p)\right) \;=\; \left(\Sigma(p) - \Sigma(p') + \gamma\cdot p - \gamma\cdot p'\right). \tag{4.29}$$

This identity is clearly obeyed in leading order, where it becomes

$$\gamma\cdot q \;=\; (\gamma\cdot p' - m) - (\gamma\cdot p - m)\,.$$

We have shown that this works explicitly at the one-loop level. The above derivation establishes the result to all orders in perturbation theory.

For very small momentum transfer $q_\mu \to 0$ the identity reduces to

$$\Gamma^\mu(p,p) \;=\; -\frac{\partial}{\partial p_\mu}\left(\Sigma(p) - \gamma\cdot p\right).$$

Before renormalization, $\Gamma^\mu(p,p)$ (recall that this includes the tree-level contribution) is $Z_1\,\gamma^\mu$ so we have:

$$\frac{\gamma^\mu}{Z_1} \;=\; \gamma^\mu\left(1 - \frac{(Z_2-1}{Z_2}\right),$$

where we have written

$$\Sigma(p) = \frac{1}{Z_2}\left(Z_2\delta m + (Z_2-1)(\gamma\cdot p - m)\right) + \mathcal{O}((\gamma\cdot p - m)^2).$$

We have thus established the relation

$$Z_1 \;=\; Z_2, \tag{4.30}$$

to all orders in perturbation theory.

Had we calculated Z_1 and Z_2 in a different gauge we would have obtained different values. Z_1 and Z_2 do not themselves correspond to physically measurable quantities and may therefore be gauge dependent, but we always have the relation $Z_1 = Z_2$. Z_3 is gauge invariant. Piecing this together we therefore have the fact that the bare coupling, which is related to the renormalized coupling simply by $e_0 = \sqrt{Z_3}e_R$, is gauge invariant.

4.5 Finite Renormalization

We have defined the renormalized electromagnetic coupling constant to be the value of the coupling of an electron to a zero momentum photon. This is a sensible definition but it is *not* unique. We could have chosen a different experiment, e.g. $e^+ e^-$ scattering at the threshold $s = 4m^2$, to determine the coupling.

Alternatively, we could have chosen a definition which did not directly correspond to a real experiment at all. For example, we could have made $\Gamma^\mu(p,p')$ finite by subtracting the contribution from the Feynman graph at an unphysical point where all three external legs had square momentum $p^2 = -\mu^2$. Furthermore, we could have subtracted the infinities in the electron propagator by defining

$$\begin{aligned} \delta m &= \Sigma(p,m)_{|\gamma\cdot p = i\mu} \\ (Z_2 - 1) &= \frac{\partial}{\partial(\gamma\cdot p)}\Sigma(p,m)_{|\gamma\cdot p = i\mu} \end{aligned}$$

and for the photon propagator

$$(Z_3 - 1) = \Pi(-\mu^2).$$

Such definitions would be sufficient to subtract all the infinities rendering the renormalized Green-functions finite.

Such a renormalization scheme has the following consequences:

- The renormalized coupling constant does not correspond to a physical quantity. All such quantities (including zero momentum transfer potential scattering) would have to be calculated in terms of the renormalized coupling e_R defined in this scheme.
- The renormalized fermion self-energy would not be proportional to $(\gamma\cdot p - m)^2$, but would have the form

  $$-\Delta m + \Delta Z_2(\gamma\cdot p - m) + \mathcal{O}((\gamma\cdot p - m)^2),$$

  where Δm and ΔZ_2 are finite. The physical mass (position of the pole of the propagator) would not be at $m_R = m + \delta m$ but at $m_R + \Delta m$.
- Z_2 would not be the same as the Z_2 which appears in the LSZ reduction formula for the S-matrix elements, but would differ from it by a finite amount.

Nevertheless, such unphysical definitions of the counterterms are often useful.

- QED provides a natural definition of the physical coupling. But in other field theories, such as those with massless gauge particles, no such physical definition arises naturally.
- It may not always be possible to perform "physical" renormalizations for all masses and couplings in a given theory without introducing counterterms that violate the internal symmetries of the theory. Spontaneously broken gauge theories in which the gauge-bosons acquire different masses is an example of this. One cannot perform on-shell subtractions for each the gauge bosons, because gauge invariance only allows one wavefunction renormalization constant for *all* of the gauge-bosons.

- A simpler way of defining counterterms can help higher order calculations.
- General renormalizations introduce a subtraction scale μ. The renormalized Green functions depend explicitly on μ, but so do the renormalized parameters e_R and m_R in such a way that the physical S-matrix elements are μ independent. This can be used to obtain information about the behaviour of renormalized Green functions as the momenta are scaled up or down.

Dimensional regularization introduces a scale μ associated with the dimension of the coupling constant outside four dimensions. A simple and practical renormalization prescription is to define the couterterms to be the pole parts of any given graph. This is the "MS" scheme - we can also use the "$\overline{MS}$" scheme in which the countertems consist of the pole part along with the $\ln(4\pi)-\gamma_E$ that always accompanies it. Such a renormalization automatically generates counterterms which generate the (dimensionality independent) symmetries of the theory.

In the $\overline{MS}$ scheme we have, for the fermion propagator

$$\begin{aligned}\delta m &= \frac{3\alpha}{4\pi} m \left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right] \\ (Z_2 - 1) &= \frac{\alpha}{4\pi}\left[-\frac{1}{\epsilon} - \ln(4\pi) + \gamma_E\right]\end{aligned}$$

and the renormalized propagator is

$$\Sigma_R(p,m) = -\frac{\alpha}{4\pi}\left[(\gamma\cdot p - 2m) + \int_0^1 d\alpha\,(2(1-\alpha)\gamma\cdot p - 4m)\ln\left(\frac{m^2\alpha - p^2\alpha(1-\alpha)}{\mu^2}\right)\right]$$

We again have $Z_1 = Z_2$. This is obeyed exactly because the $\overline{MS}$ renormalization scheme preserves the gauge invariance. However, for a general "unphysical" renormalization scheme only the infinite (pole) parts would necessarily obey this relation.

For the photon propagator in the $\overline{MS}$ scheme we have

$$(Z_3 - 1) = -\frac{\alpha}{3\pi}\left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma_E\right]$$

with the renormalized propagator

$$\Pi_R(q^2) = \frac{2\alpha}{\pi}\left[\int_0^1 d\alpha\alpha(1-\alpha)\ln\left(\frac{m^2 - q^2\alpha(1-\alpha)}{\mu^2}\right)\right]$$

($\Pi_R(q^2)$ does *not* vanish as $q^2 \to 0$ in this scheme and nor does $\Sigma_R(p,m)$ vanish as $\gamma\cdot p \to m$).

In dimensional regularization the dimensionless renormalized coupling, α_R, ($\equiv g_R^2/(4\pi)$) is related to the bare coupling α_0 by the relation

$$\alpha_B = \mu^{2\epsilon}\alpha_R\left(1 + \beta_0\frac{\alpha_R}{4\pi}\frac{1}{\epsilon} + \cdots\right), \tag{4.31}$$

so that α_R is an implicit function of the scale μ - despite the fact that it is dimensionless.

5 The Renormalization Group

5.1 The β-function

The value of renormalized coupling constants are renormalization prescription dependent. In particular, they depend on the value of the momentum μ, at which the infinite Green functions are subtracted - in dimensional regularization this is the mass scale, μ, which controls the dimensionful bare coupling outside four dimensions.

In $d = 4-2\epsilon$ dimensions in the MS scheme, the renormalized dimensionless coupling, $g_R(\mu)$, is related to the dimensionful bare coupling g_0 by

$$g_R(\mu) \;=\; \mu^{-\epsilon} Z_g(g_R) g_0, \tag{5.1}$$

with

$$Z_g \;=\; \frac{Z_2 Z_3^{1/2}}{Z_1}$$

We define the beta function in d dimensions, $\tilde{\beta}(g_R)$ to be a function of the renormalized coupling, such that

$$\tilde{\beta}(g_R) \;=\; \frac{\partial}{\partial(\ln(\mu))} g_R(\mu)). \tag{5.2}$$

From eq.(5.1) this gives

$$\tilde{\beta}(g_R) \;=\; -\epsilon g_R + \frac{g_R}{Z_g(g_R)} \frac{\partial Z_g(g_R)}{\partial g_R} \tilde{\beta}(g_R) \tag{5.3}$$

$\tilde{\beta}$ is finite as $\epsilon \to 0$, so by comparing powers of ϵ (the coefficient of ϵ) we must have

$$\tilde{\beta} \;=\; -\epsilon g_R + \beta, \tag{5.4}$$

where β is the value of $\tilde{\beta}$ in four dimensions.

Using the fact that in the MS scheme, Z_g contains only poles at $\epsilon = 0$, (at n^{th} order it will contain poles up to order n) so it can be written as

$$Z_g \;=\; 1 + \sum_n \sum_{k=1}^{n} a_k^n \frac{g_R^{2n}}{\epsilon^k}$$

and substituting eq.(5.4) into the RHS of eq.(5.3)

$$Z_g(g_R)\beta \;=\; -\epsilon g_R^2 \frac{\partial Z_g(g_R)}{\partial g_R} + \beta \frac{\partial Z_g(g_R)}{\partial g_R} g_R$$

Comparing coefficients of ϵ^0 we get

$$\beta \;=\; -2\, g_R^2 \sum_n n a_1^n g_R^{2n-1}, \tag{5.5}$$

i.e. in any order of perturbation theory the β-function may be obtained from (minus $2\times$) the simple pole part of the coupling constant renormalization factor, Z_g.

β has a perturbative expansion

$$\beta(g_R) \;=\; g_R \left[\beta_0 \frac{g_R^2}{16\pi^2} + \beta_1 \left(\frac{g_R^2}{16\pi^2} \right)^2 \;+\; \cdots \right]$$

Suppose that in a different renormalization scheme

$$g_R' \;=\; g_R + A \frac{g_R^2}{16\pi^2}$$

$$\begin{aligned} \beta'(g_R') \;&=\; g_R' \left[\beta_0' \frac{g_R'^2}{16\pi^2} + \beta_1' \left(\frac{g_R'^2}{16\pi^2} \right)^2 \;+\; \cdots \right] \\ &=\; g_R \left[\beta_0 \frac{g_R^2}{16\pi^2} + (\beta_1 + 3A\beta_0) \left(\frac{g_R^2}{16\pi^2} \right)^2 \;+\; \cdots \right] \\ &=\; g_R' \left[\beta_0 \frac{g_R'^2}{16\pi^2} + \beta_1 \left(\frac{g_R'^2}{16\pi^2} \right)^2 \;+\; \cdots \right] \end{aligned} \tag{5.6}$$

We see here that the first two terms in the β-function are renormalization prescription independent - so we may use the MS scheme for convenience (this statement is *not* true beyond the first two terms in the expansion.)

In the case of QED ($g \equiv e$), we have $Z_1 = Z_2$ so that

$$Z_g \;=\; \sqrt{Z_3}.$$

From eq.(4.19) we see that the simple pole part of Z_3 is given at the one loop level by

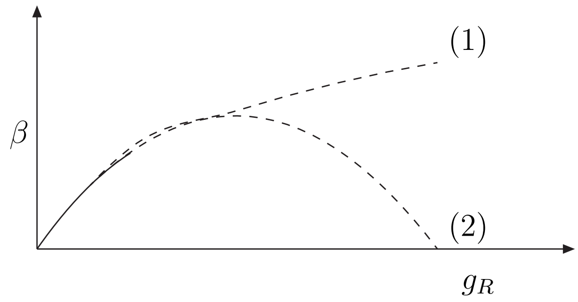
$$-\frac{4}{3} \frac{e^2}{16\pi^2}$$

So we see that in this case β_0 is given by

$$\beta_0^{QED} \;=\; \sum_i \frac{4}{3} Q_i^2, \tag{5.7}$$

the sum and factor Q_i^2 arising from the fact that for the QED calculation we considered the electron as the only charged particle. In reality all charged particles, i, contribute to the higher order corrections to the photon propagator with a coupling proportional to the square of their electric charge, Q_i.

The fact that β_0 is positive means that at least for small values of the coupling, β itself is positive and this means that as the renormalization scale increases, the renormalized coupling constant also rises.



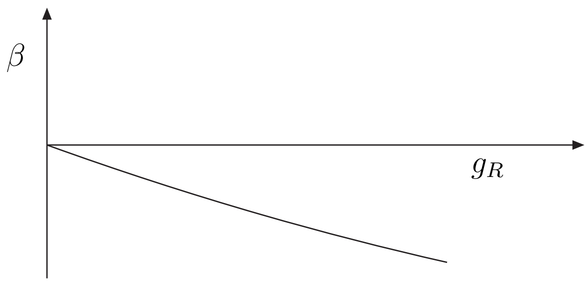
Whether the renormalized coupling continues to increase indefinitely or stops at some "ultraviolet fixed point" depends on whether the higher order contributions to β are such that it remains positive for all values of the coupling (curve(1)) , or whether it acquires negative contributions in higher orders such that it decreases again and crosses the axis at a value of g_R equal to that fixed point (curve(2)). In the second case, once the fixed point has been reached, $\beta = 0$, so that the renormalized coupling ceases to change as μ is further increased.

For non-Abelian gauge theories, the calculation of β_0 is more involved. Here we do *not* have $Z_1 = Z_2$. Furthermore there are extra Feynman graphs for the vertex correction and the gauge boson propagator correction which account for the self-interactions of the gauge bosons (and where appropriate the Faddeev-Popov ghosts). The couplings carry colour factors, so the contributions from each graph will be proportional to Casimir operators of the gauge group in various different representations.

For an $SU(N)$ gauge theory with n_f copies of fermions transforming as the defining representation of $SU(N)$, β_0 takes the value

$$\beta_0 \;=\; -\frac{11N - 2n_f}{3} \tag{5.8}$$

Provided the number of fermions is not too large this is negative. Theories with negative values of β_0 are called "asymptotically free" theories.



As $\mu \to \infty$, the renormalized coupling goes to zero. Non-Abelian gauge theories are the only known examples of such asymptotically free gauge theories.

For such theories we may solve the differential equation

$$\beta(g_R) \;=\; \frac{\partial\, g_R(\mu)}{\partial\, \ln(\mu)} \;=\; \beta_0 \frac{g_R^3}{16\pi^2} \;+\; \beta_1 \frac{g_R^5}{(16\pi^2)^2} \;+\; \cdots$$

It is usually more convenient to work in terms of

$$\alpha_R \;\equiv\; \frac{g_R^2}{4\pi},$$

for which the differential equation becomes

$$\frac{\partial\, \alpha_R(\mu^2)}{\partial\, \ln(\mu^2)} \;=\; \beta_0 \frac{\alpha_R^2}{4\pi} \;+\; \beta_1 \frac{\alpha_R^3}{(4\pi)^2}.$$

We have truncated the series at the two-loop level. This is a first order differential equation with a constant of integration, which is usually expressed in terms of the value of the renormalized coupling, α_0, at some reference scale μ_0 - the renormalization prescription must be specified. † The solution of the differential equation, accurate to the order of the truncation is

$$\alpha_R(\mu^2) \;=\; \frac{\alpha_0}{\left[1 + \left(\frac{\alpha_0}{4\pi}|\beta_0| - \beta_1 \frac{\alpha_0^2}{(4\pi)^2}\right) \ln\left(\frac{\mu^2}{\mu_0^2}\right)\right]} \tag{5.9}$$

This expression is valid provided μ_0 is sufficiently large that $\alpha_0 \ll 1$ and $\mu \geq \mu_0$.

An older way of writing the solution is to introduce a scale Λ_{QCD} (which is again renormalization prescription invariant) and expressing the renormalized coupling as

$$\alpha_R(\mu^2) \;=\; \frac{4\pi}{\left[|\beta_0| \ln\left(\frac{\mu^2}{\Lambda_{QCD}^2}\right) + \frac{\beta_1}{\beta_0} \ln\left(\ln\left(\frac{\mu^2}{\Lambda_{QCD}^2}\right)\right)\right]}$$

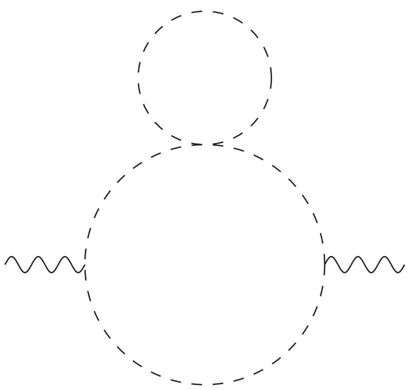
These two expressions are equivalent (up to corrections of order α_0^3) provided we identify

$$\alpha_0 \;=\; \frac{4\pi}{\left[|\beta_0| \ln\left(\frac{\mu_0^2}{\Lambda_{QCD}^2}\right) + \frac{\beta_1}{\beta_0} \ln\left(\ln\left(\frac{\mu_0^2}{\Lambda_{QCD}^2}\right)\right)\right]}$$

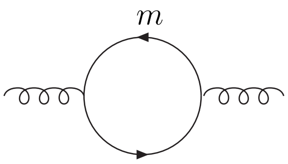
†Nowadays the most popular reference scale is the mass of the $Z-$boson and the $\overline{MS}$ prescription is used - i.e. the renormalized coupling is expressed in terms of $\alpha_{\overline{MS}}(M_Z^2)$.

If we consider a Green function whose invariant square momenta are of order q^2 and we subtract the ultraviolet infinities at μ^2, we are always left with corrections which are proportional to $\ln(q^2/\mu^2)$ in the renormalized Green functions. If $q^2 \ll \mu^2$ or $q^2 \gg \mu^2$ these logarithms can become large and give large coefficients of the renormalized coupling, $\alpha_R(\mu^2)$ so that $|\alpha_R(\mu^2)\ln(q^2/\mu^2)| \sim 1$ and this may spoil the convergence of the perturbation expansion. It is therefore convenient to choose $\mu^2 \approx q^2$, i.e. we choose a subtraction scale to be of the order of the typical energy of the process under consideration. This way we obtain a perturbative expansion in the "effective coupling", $\alpha_R(q^2)$, with no large logarithms. For asymptotically free theories the effective coupling decrease as the energy scale increases. Strong interactions are believed to be described by such a theory (QCD), so that the interactions actually become sufficiently small at large energies for a perturbation expansion to be valid.

In general, spontaneously broken gauge theories are *not* asymptotically free. This is not because the β-function for the gauge coupling is positive at small couplings (the β-function acquires a small positive contribution from the interaction of the gauge-bosons with the Higgs scalar fields, but unless there are a very large number of fermions or scalar fields β_0 remains negative), but rather because the β-function for the self-interaction of the Higgs scalar field is positive, so the ϕ^4 coupling grows as μ increases and this increasing scalar coupling feeds into the other couplings at higher orders.



Thresholds
When calculating the effective coupling, $\alpha_R(q^2)$, we include in the number of fermion multiplets, n_f, only the fermions whose masses are less than $\sqrt{q^2}$. This is because the derivative w.r.t. $\ln(q^2)$ of the contribution to the gauge-boson propagator from a graph with a massive fermion loop

$\frac{\partial}{\partial \ln(q^2)}$ 

$\sim q^2/m^2, \quad (q^2 \ll m^2)$

On the other hand the contribution it clearly reaches its full value, if $q^2 \gg m^2$. Thus a fermion only contributes to the q^2 development of the effective coupling for $q^2 \geq m^2$. The exact threshold can be calculated from the finite part of the Feynman graph, but a good approximation for the threshold is $q^2 = 4m^2$, which is the energy threshold at which a gauge-boson can produce a quark-antiquark pair of mass m.

5.2 Callan-Symanzik equation

Consider an n-point 1-particle-irreducible Green function, which depends on external momenta p_i and masses m_j, the coupling g, and, in general, the gauge parameter ξ, and possibly an ultraviolet cut-off Λ. The renormalized green function is independent of the ultraviolet cutoff Λ, but depends explicitly on the subtraction scale μ.

$$\Gamma_0(p_1 \cdots p_{n-1}, g_0, m_0, \xi_0, \Lambda) \;=\; \prod_{i=1}^{n} Z_i^{-1/2}(\mu)\Gamma_R(p_1 \cdots p_{n-1}, g_R(\mu), m_R(\mu), \xi_R(\mu), \mu) \quad (5.10)$$

The renormalized quantities, g_R, m_R, ξ_R depend on the subtraction scale μ as does the wavefunction renormalization constant Z_i, for each external particle.

However, the LHS of eq.(5.10) is independent of μ, and this leads to a first-order (partial) differential equation for the renormalized Green function, Γ_R (the "Callan-Symanzik equation"),

$$\frac{\partial}{\partial \ln(\mu)}\Gamma_R + \beta\frac{\partial \Gamma_R}{\partial g_R} + m_{Rj}\gamma_{m_j}\frac{\partial \Gamma_R}{\partial m_{Rj}} + \xi_R\gamma_\xi\frac{\partial \Gamma_R}{\partial \xi_R} - \sum_i \gamma_i\Gamma_R \;=\; 0, \quad (5.11)$$

where

$$\gamma_{m_j} \;=\; \frac{1}{m_{Rj}}\frac{\partial m_{Rj}}{\partial \ln(\mu)}$$

$$\gamma_\xi \;=\; \frac{1}{\xi_R}\frac{\partial \xi_R}{\partial \ln(\mu)}$$

and

$$\gamma_i \;=\; \frac{1}{2\,Z_i}\frac{\partial Z_i}{\partial \ln(\mu)}$$

γ_i are called the "anomalous dimensions" of the field corresponding to particle i.

If we are in a high-energy region of momentum-space in which the masses may be neglected we may neglect the terms involving
$$\frac{\partial\,\Gamma_R}{\partial\, m_{Rj}}.$$
Such a region is the "deep-Euclidean region" in which all the external square momenta and scalar products between momenta are negative (space-like) and large compared with all the masses, ($p_i^2 < 0$ and $|p_i^2|$, $|p_i \cdot p_k \gg m_j^2$).

In this region we may write $p_i = p\, y_i$, where y_i are dimensionless vectors and p sets the scale for the momenta. We can write the renormalized Green function as
$$\Gamma_R(p_1 \cdots p_{n-1}, g_R, \xi_R, \mu) \;=\; p^d \tilde{\Gamma}_R\left(y_1 \cdots y_{n-1}, g_R, \xi_R, \frac{\mu^2}{p^2}\right),$$
where d is the naive ("engineering") dimension of the Green function (obtained from simple power counting). $\tilde{G}_R$ depends explicitly on μ^2, through the dimensionless ratio μ^2/p^2. This gives us the Callan-Symanzik equation for the p dependence of the renormalized Green-function
$$\frac{\partial}{\partial\,\ln p}\Gamma_R \;=\; \left(d - \frac{\partial}{\partial\,\ln(\mu)}\right)\Gamma_R. \tag{5.12}$$
We restrict ourselves to gauge invariant quantities, so that we may now discard the terms involving the derivative w.r.t. ξ. G_R now obeys the equation
$$\left(\frac{\partial}{\partial\,\ln(\mu)} + \beta(g_R(\mu))\frac{\partial}{\partial\, g_R(\mu)} + \sum_i \gamma_i(g_R(\mu))\right)\Gamma_R \;=\; 0. \tag{5.13}$$
β and γ_i depend on μ through their dependence on $g_R(\mu)$. Using the relation between the explicit μ dependence and the p dependence (eq.(5.12)), we get
$$\left(\frac{\partial}{\partial\,\ln p} - \beta(g_R(p))\frac{\partial}{\partial\, g_R(p)} - d + \sum_i \gamma_i(g_R(p))\right)\Gamma_R \;= 0, \tag{5.14}$$
where $g_R(p)$ is the value for the renormalized coupling at $p^2 = \mu^2$ (the effective coupling).

The solution to eq.(5.14) is
$$\Gamma_R(p_1 \cdots p_{n-1}, g_R(\mu), \mu) \;=\; p^d \tilde{\Gamma}_R(y_1 \cdots y_{n-1}, g_R(p), 1)\exp\left\{\int_{g_R(\mu)}^{g_R(p)} \frac{-\sum_i \gamma_i(g')}{\beta(g')} dg'\right\}. \tag{5.15}$$
$\tilde{\Gamma}_R(y_1 \cdots y_{n-1}, g_R(p), 1)$ is $\tilde{\Gamma}_R$ at $p^2 = \mu^2$, calculated in an ordinary perturbation expansion in $g_R(p)$. For asymptotically free theories $g_R(p) \to 0$ as $p \to \infty$. This then tells us something about the high energy behaviour of Euclidean Green functions.

Using perturbative expansions
$$\beta(g') \;=\; \frac{{g'}^3}{16\pi^2}\beta_0 + \frac{{g'}^5}{(16\pi^2)^2}\beta_1 + \cdots$$

and

$$\gamma_i(g') \;=\; \frac{{g'}^2}{16\pi^2}\gamma_{i\,0} + \frac{{g'}^4}{(16\pi^2)^2}\gamma_{i\,1} + \cdots$$

The power series for $\tilde{\Gamma}_R(y_1 \cdots y_{n-1}, g_R(\mu), 1)$ may be written as

$$\tilde{\Gamma}^0_R(y_1, \cdots y_{n-1}) \left[1 + c(y_1, \cdots y_{n-1}) \frac{g_R^2}{16\pi^2} + \cdots \right].$$

We can use this to determine the behaviour of the renormalized Green function under a change of the momentum scale from p to p'. To leading order (in terms of $\alpha_R \equiv g_R^2/(4\pi)$) we have

$$\begin{aligned} \Gamma_R(p'_1, \cdots p'_{n-1}, g_R(\mu), \mu) \;&=\; \Gamma_R(p_1, \cdots p_{n-1}, g_R(\mu), \mu) \left(\frac{p'}{p}\right)^d \left(\frac{\alpha_R({p'}^2)}{\alpha_R(p^2)}\right)^{-\sum_i \gamma_{i\,0}/(2\beta_0)} \\ &\times \;\; \left[1 + \mathcal{O}\left(\alpha_R(p^2) - \alpha_R({p'}^2)\right)\right] \end{aligned} \tag{5.16}$$

Calculation of the Anomalous Dimensions

In the MS scheme

$$Z_i \;=\; 1 + \sum_n \sum_{k=1}^{n} a_k^n \frac{g_R^{2n}}{\epsilon^k}$$

In $4 - 2\epsilon$ dimensions

$$\frac{\partial\, g_R}{\partial\, \ln(\mu)} \;=\; \beta(g_R) - \epsilon g_R,$$

so that (differentiating w.r.t $\ln(\mu)$

$$Z_i\,\gamma_i \;=\; \frac{1}{2} \sum_n \sum_{k=1}^{n} (2n) a_k^n \frac{g_R^{2n-1}}{\epsilon_k} \left(\beta(g_R) - \epsilon g_R\right)$$

Comparing coefficients of ϵ^0, we have

$$\gamma_i \;=\; -\sum_n n a_1^n g_R^{2n}$$

We see that to any order in perturbation theory, the anomalous dimension is simply related to the coefficient of the simple pole.

For a fermion interacting with a non-Abelian gauge-boson, we have to one loop order (in Feynman gauge)

$$Z_2 \;=\; 1 - \frac{g_R^2}{16\pi^2}\frac{C_F}{\epsilon} + \cdots$$

($C_F = (N^2-1)/(2N)$ for an $SU(N)$ theory). This gives

$$\gamma_{i\,0} \;=\; +\frac{\alpha_R}{4\pi}C_F.$$

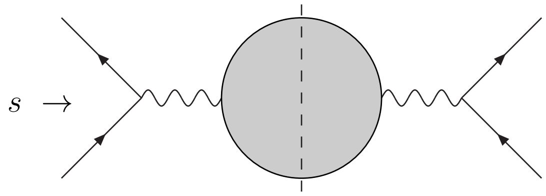
Likewise the next coefficient $\gamma_{i\,1}$ may be determined from the two-loop calculation of the wavefunction renormalization Z_2. Only the leading order term $\gamma_{i\,0}$ is renormalization scheme independent - the higher order terms are different in different renormaliztion schemes.

Deep Euclidean region Green functions are not physical, but there are a number of techniques available for using the scale dependence of such Green functions to calculate the energy dependence of physical processes.

The simplest example of this is the total cross-section in electron-positron annihilation. This is usually expressed in terms of a ratio of the cross-section into hadrons to the pure QED process in which the electron-positron pair annihilates into a muon pair

$$R \equiv \frac{\sigma(e^+\,e^- \;\rightarrow\; \text{hadrons})}{\sigma(e^+\,e^- \;\rightarrow\; \mu^+\,\mu^-)}.$$

Unitarity and the Optical Theorem relate the total cross-section to the imaginary part of the off-shell photon propagator

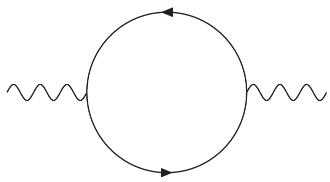


Here the shaded blob refers to the sum of all graphs with a quark loop plus gluon corrections. A graph with a quark loop with quarks of mass m_i has an imaginary part provided $s > 4m_i^2$. We can make an analytic continuation into negative (space-like) s and for such values of s we are indeed in the Deep Euclidean region where the Callan-Symanzik equation is valid. However, in this case the external line is the off-shell photon with square momentum s. Since this is not a strongly interacting particle there is no QCD contribution to the anomalous dimension. The solution to the Callan-Symanzik equation then greatly simplifies and we have

$$\Pi(s, \alpha_S(\mu^2), \mu) \;=\; \Pi(\alpha_S(s), s=\mu^2),$$

(α_S means the renormalized strong coupling)

To leading order, Π is just given by the one-loop graph

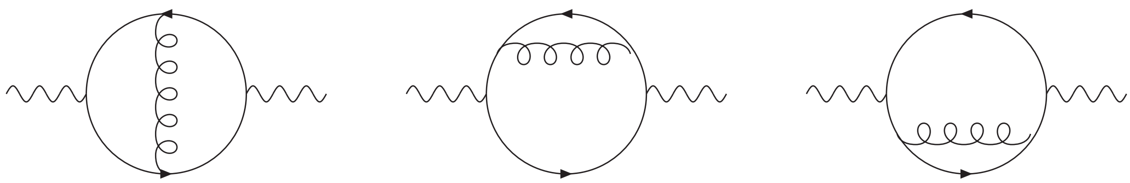


The contribution to the ratio R is just the sum of the squares of the electric charges Q_i^2, of all the quarks with masses below s, multiplied by a phase-space factor which accounts for the non-zero quark masses.

$$R \;=\; \sum_{i:\ 4m_i^2<s} Q_i^2 \times \left[\sqrt{1-4\frac{m_i^2}{s}}\left(1+2\frac{m_i^2}{s}\right)\right],$$

where the factor inside the square brackets is the phase-space factor.

In next order in perturbation ($\mathcal{O}(\alpha_S)$ corrections) we have the graphs



These are calculated at $s=\mu^2$ with α_S renormalized at $\mu^2=s$. This gives

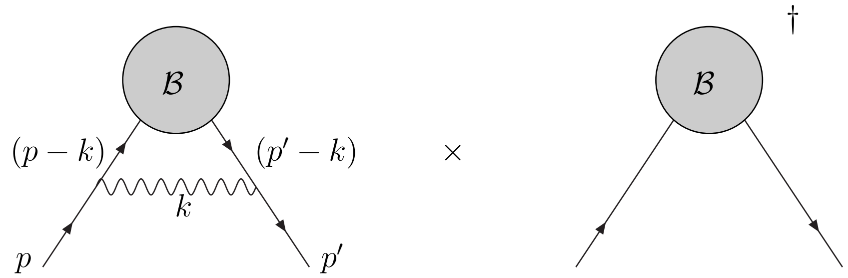
$$R(s) \;=\; \sum_{i:\ 4m_i^2<s} Q_i^2 \left[1+3C_F\frac{\alpha_S(s)}{4\pi}+\cdots\right] \times \left(\sqrt{1-4\frac{m_i^2}{s}}\left(1+2\frac{m_i^2}{s}\right)\right),$$

Deep inelastic electron proton scattering is another example of a process whose energy dependence can be determined using the Callan-Symanzik equation.

6 Infrared Divergences

We have already seen that some QED graphs have a divergence associated with the masslessness of the photon. The divergence occurs at small values of the photon momentum k. In a general graph there are infrared divergences when *both* ends of a photon are attached to an external charged line.

The contribution to a transition probability or cross-section from such a correction is the interference between the correction graph and the graph without the photon attached to the external lines



The shaded blob stands for any other part of the graph, which could be simply a tree-level or process or it may contain any number of loops of internal photons and fermions.

For small k we neglect any powers of k in the numerator (this is the "eikonal approximation") and similarly in the denominator of the fermion propagator we neglect k^2 and write

$$\frac{i}{((p-k)^2 - m^2)} \;\to\; \frac{-i}{2p\cdot k}.$$

In Feynman gauge the numerator may be written (in the eikonal approximation)

$$-\left(-ie\gamma^\mu\right)\left(\gamma\cdot p' + m\right)\mathcal{B}\left(\gamma\cdot p + m\right)\left(-ie\gamma_\mu\right)$$

This is sandwiched between on-shell spinors, so that we can anti-commute $\gamma\cdot p'$ or $\gamma\cdot p$ through γ_μ and use the Dirac equation to reduce this to

$$4e^2 p\cdot p'\mathcal{B},$$

where $\mathcal{B}$ represents the contribution from the shaded blob.

The infrared divergent part of this interference may therefore be written

$$-ie^2|\mathcal{B}|^2 8p\cdot p' \int \frac{d^4k}{(2\pi)^4}\frac{1}{(k^2+i\epsilon)(2p\cdot k - i\epsilon)(2p'\cdot k - i\epsilon)} \tag{6.1}$$

We have reinstated the $i\epsilon$ in the propagators from the time ordering operator. We do this because we choose to perform the above integral by integrating first over the time component k_0 of the loop momentum.

We therefore rewrite eq.(6.1) as

$$-ie^2 2|\mathcal{B}|^2 p \cdot p' \int \frac{dk_0}{(2\pi)} \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{1}{(k_0^2 - \mathbf{k}^2 + i\epsilon)((p_0 k_0 - \mathbf{p}\cdot\mathbf{k} - i\epsilon)(p_0' k_0 - \mathbf{p}'\cdot\mathbf{k} - i\epsilon)} \tag{6.2}$$

and integrate over k_0 by closing the contour in such a way as to pick up the pole at $k_0 = |\mathbf{k}|$. This gives

$$-e^2 2|\mathcal{B}|^2 \int \frac{|\mathbf{k}|^2 d\mathbf{k} d\Omega}{(2\pi)^3 2|\mathbf{k}|} \frac{1}{((p_0|\mathbf{k}| - \mathbf{p}\cdot\mathbf{k})(p_0'|\mathbf{k}| - \mathbf{p}'\cdot\mathbf{k})}$$

This integral diverges at $|\mathbf{k}| \to 0$, so we cut off this lower limit at $|\mathbf{k}| = \lambda$. We also impose an upper limit E above which the infrared approximation is no longer valid (this upper limit is rather arbitrary but we are only interested here in the infrared divergences. We therefore get for the infrared divergent part

$$-\frac{\alpha}{2\pi} p \cdot p' |\mathcal{B}|^2 \ln\left(\frac{E}{\lambda}\right) \int \frac{d\cos\theta \, d\phi}{2\pi} \frac{1}{(E - p\cos\theta)(E' - p'\cos\theta')}, \tag{6.3}$$

E, p and E', p' are the energies and magnitudes of 3-momenta of the external fermion lines and

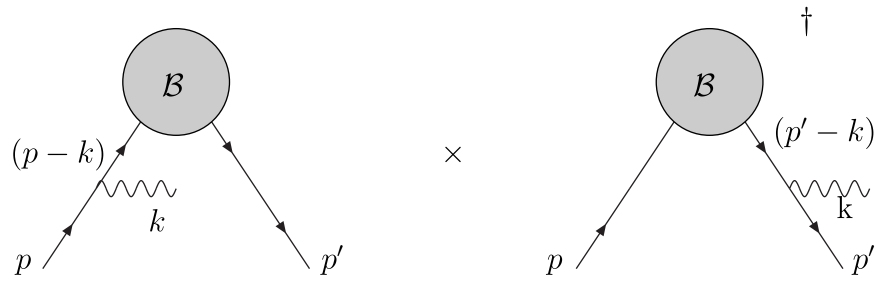
$$\cos\theta' \;=\; \cos\alpha\cos\theta - \sin\alpha\sin\theta\cos\phi,$$

where α is the angle between $\mathbf{p}$ and $\mathbf{p}'$ (θ is the angle between $\mathbf{k}$ and $\mathbf{p}$, whereas θ' is the angle between $\mathbf{k}$ and $\mathbf{p}'$).

These infrared divergences do not cancel within the process considered above. Whenever a process occurs with electromagnetic corrections there is an experimental limit to the accuracy with which the initial and final state energies can be measured. This means that there will *always* be some energy loss in emitted photons (Bremsstrahlung). What is actually observed is the *sum* of the elastic process (no emitted photons) and the process in which a small quantity of energy up to the energy resolution, ΔE, is lost in photon emission. Keeping track of orders of the electromagnetic coupling, we see that a one-loop correction to a tree-level process with no emitted photons is of the same order as the tree-level process involving a single emitted photon. This generalizes to the statement that the α^n correction to a tree-level process consists of the sum of all the process with $n - r$ virtual loops and r emitted photons.

The emission of a (real) photon from an external charged line also introduces an infrared divergences as the energy of the emitted photon goes to zero. It is *this* infrared divergence that cancels the infrared divergence associated with the virtual correction.

Consider the interference between the graphs for the above process in which the photon is emitted from different charge lines



Again, using the eikonal approximation for the numerator (in this case we have exactly $k^2 = 0$ since the emitted photon is on-shell - this means that the denominators of the internal fermion lines are $2p \cdot k$ and $2p' \cdot k$ respectively), the infrared part of this process is

$$e^2 |\mathcal{B}|^2 4p \cdot p' \int \frac{d^3\mathbf{k}}{(2\pi)^3 2|\mathbf{k}|} \frac{1}{(2p \cdot k)(2p' \cdot k)}, \tag{6.4}$$

We have performed a summation over the polarization of the emitted fermion (which gives a factor $-g_{\mu\nu}$, being the counterpart of the Feynman gauge propagator in the virtual correction). The integral over $\mathbf{k}$ is the integral over the phase-space of the emitted photon. We note that this integral also has a divergence as $|\mathbf{k}| \to 0$ with the opposite sign from that of the virtual correction. Again we cut this lower limit off at $|\mathbf{k}| = \lambda$. We take the upper limit of the integration over $\mathbf{k}$ to be the energy resolution, ΔE. This interference then contributes an infrared divergent part

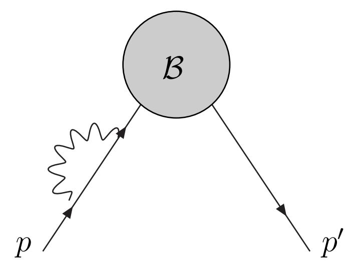
$$\frac{\alpha}{2\pi} p \cdot p' \ln\left(\frac{\Delta E}{\lambda}\right) \int \frac{d\cos\theta \, d\phi}{2\pi} \frac{1}{(E - p\cos\theta)(E' - p'\cos\theta')}. \tag{6.5}$$

If we sum the contributions from eqs.(6.3) and (6.5) we see that the dependence on the infrared cut-off λ cancels and we are left with

$$-\frac{\alpha}{2\pi} p \cdot p' \ln\left(\frac{E}{\Delta E}\right) \int \frac{d\cos\theta \, d\phi}{2\pi} \frac{1}{(E - p\cos\theta)(E' - p'\cos\theta')}, \tag{6.6}$$

to which we must add the contributions from the hard (virtual and real) photons that we have neglected in the eikonal approximation.

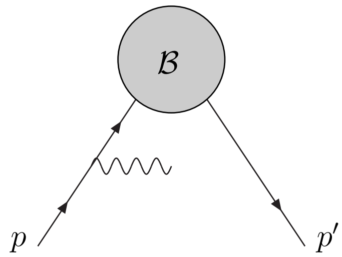
There are other infrared divergent graphs, such as the fermion self-energy insertions, which give rise to an infrared divergence when on-shell renormalization is performed (Z_2 is infrared divergent).



This gives an infrared divergent contribution

$$\frac{\alpha}{\pi}\ln\left(\frac{E}{\lambda}\right)$$

which cancels the infrared divergent part of the square of the graph



This graph squared gives a contribution

$$e^2|\mathcal{B}|^2m^2\int\frac{d^3\mathbf{k}}{2\pi^3 2|\mathbf{k}|}\frac{1}{(2k\cdot p)^2}.$$

This also has an infrared divergence as $|\mathbf{k}|\to 0$.

The complete soft photon (i.e. small $\mathbf{k}$) contribution to the sum of the two processes is

$$-|\mathcal{B}|^2\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{E}{\Delta E}\right), \tag{6.7}$$

where

$$\mathcal{K} \;=\; \int\frac{d\Omega}{4\pi}\frac{p\cdot p'}{(E-p\cos\theta)(E'-p'\cos\theta')}-1 \tag{6.8}$$

It can be shown that this cancellation of infrared divergences between the elastic (only virtual photon corrections) part and the inelastic part (one or more real photon emissions with total energy less than ΔE) persists to all orders in perturbation theory. In fact, the infrared divergences can be shown to exponentiate so that for the elastic cross section the sum to all orders of the infrared divergent part is

$$|\mathcal{B}|^2\exp\left\{-\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{E}{\lambda}\right)\right\}$$

and for the elastic amplitude plus any number of real photons with total energy up to ΔE we have

$$|\mathcal{B}|^2\exp\left\{-\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{E}{\lambda}\right)\right\}\exp\left\{+\frac{\alpha}{\pi}\mathcal{K}\ln\left(\frac{\Delta E}{\lambda}\right)\right\}$$

So that the soft photon parts of the complete inelastic cross-section is proportional to

$$\left(\frac{\Delta E}{E}\right)^{\alpha\mathcal{K}/\pi}$$

Since $\mathcal{K}$ is positive, we see that this vanishes as $\Delta E\to 0$, meaning that the probability of a purely elastic process with no energy loss into emitted photons is zero.

6.1 Dimensional Regularization of Infrared Divergences

The method of dimensional regularization can also be used to regularize infrared divergences. After the integration over the energy component k_0 in the case of virtual corrections we have an integral over the $d-1$ space-like components of the photon momentum $d^{d-1}\mathbf{k}$. Likewise the integral over the phase space of the emitted photon for the bremsstrahlung process is carried out in $d-1 = 3-2\epsilon$ dimensions.

$$\int \frac{d^{3-2\epsilon}\mathbf{k}}{(2\pi)^{3-2\epsilon}} = \frac{1}{8\pi^2}\Gamma(1-2\epsilon)(4\pi)^{\epsilon}4^{\epsilon}\int (\sin\theta)^{1-2\epsilon}d\theta|\mathbf{k}|^{2-2\epsilon}d|\mathbf{k}|, \tag{6.9}$$

where we have integrated over all but one of the polar angles.

For real photon emission with photon energy up to ΔE we have the phase-space integral

$$-e^2|\mathcal{B}|^2\int_0^{\Delta E}\frac{d^{3-2\epsilon}\mathbf{k}}{(2\pi)^{3-2\epsilon}2|\mathbf{k}|}\left(\frac{m^2}{(k\cdot p)^2}+\frac{m^2}{(k\cdot p')^2}-2\frac{p\cdot p'}{(k\cdot p)(k\cdot p')}\right) \tag{6.10}$$

The last term is handled using the Feynman parametrization trick, so we integrate over the Feynman parameter α and define the momentum

$$p_\alpha^\mu = p^\mu\alpha + p'^\mu(1-\alpha).$$

The expression (6.10) becomes

$$-e^2|\mathcal{B}|^2\int_0^{\Delta E}\frac{d^{3-2\epsilon}\mathbf{k}}{(2\pi)^{3-2\epsilon}2|\mathbf{k}|}\left(\frac{m^2}{(k\cdot p)^2}+\frac{m^2}{(k\cdot p')^2}-2\int_0^1 d\alpha\frac{p\cdot p'}{(k\cdot p_\alpha)^2}\right) \tag{6.11}$$

Using eq.(6.9) this is

$$-\frac{\alpha}{2\pi}(4\pi)^\epsilon\Gamma(1-2\epsilon)|\mathcal{B}|^2 4^\epsilon\int_0^{\Delta E}|\mathbf{k}|^{-1-2\epsilon}(\sin\theta)^{1-2\epsilon}d|\mathbf{k}|d\theta\left(\frac{m^2}{(k\cdot p)^2}+\frac{m^2}{(k\cdot p')^2}-2\int_0^1 d\alpha\frac{p\cdot p'}{(k\cdot p_\alpha)^2}\right) \tag{6.12}$$

The term

$$4^\epsilon\int(\sin\theta)^{1-2\epsilon}\frac{1}{(E-p\cos\theta)^2}$$

is a hypergeometric function whose expansion about $\epsilon=0$ is

$$\frac{2}{(E^2-p^2)}\left[1-\epsilon\frac{E}{p}\ln\left(\frac{E-p}{E+p}\right)+\mathcal{O}(\epsilon^2)\right].$$

The integral over $|\mathbf{k}|$ gives a pole at $\epsilon=0$. This pole signals the infrared divergence. When dimensional regularization is used to regularize infrared divergences we must think of this as performing the integral initially in *more than* four dimensions (negative ϵ) for which there is no infrared divergence and then performing an analytic continuation to four dimensions.

The expression (6.12) gives a pole term

$$\frac{\alpha}{2\pi}\Gamma(1-2\epsilon)|\mathcal{B}|^2(4\pi)^\epsilon\frac{(\Delta E)^{-\epsilon}}{2\epsilon}\left[4-4p\cdot p'\int_0^1\frac{d\alpha}{(E_\alpha^2-p_\alpha^2)}\right] \tag{6.13}$$

and a finite term

$$-\frac{\alpha}{2\pi}|\mathcal{B}|^2\left[\frac{E}{p}\ln\left(\frac{E+p}{E-p}\right)+\frac{E'}{p'}\ln\left(\frac{E'+p'}{E'-p'}\right)-2p\cdot p'\int_0^1 d\alpha\frac{E_\alpha}{p_\alpha}\frac{1}{(E_\alpha^2-p_\alpha^2)}\ln\left(\frac{E_\alpha+p_\alpha}{E_\alpha-p_\alpha}\right)\right] \tag{6.14}$$

Expanding the term

$$\frac{(\Delta E)^{-\epsilon}}{2\epsilon}$$

in the pole part gives the $\ln(\Delta E)$ dependence found previously.

Now compare this with the virtual correction term. The soft photon contribution is

$$-ie^2|\mathcal{B}|^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2}\left[\frac{4m^2}{(k^2-2p\cdot k)^2}+\frac{4m^2}{(k^2-2p'\cdot k)^2}-\frac{8p\cdot p'}{(k^2-2p\cdot k)(k^2-2p'\cdot k)}\right] \tag{6.15}$$

The first two terms in square parenthesis coming from the self-energy insertions after performing on-shell wavefunction renormalization and the third term from the correction term in which the virtual photon connects the two external fermions. We have used the eikonal approximation in the numerator, but kept the denominators exact.

Using Feynman parametrization this becomes

$$\begin{aligned}-ie^2|\mathcal{B}|^2\int\frac{d^dk}{(2\pi)^d}&\left[\int_0^1 2\alpha\,d\alpha\frac{4m^2}{(k^2-2p\cdot k\alpha)^3}+\int_0^1 2\alpha\,d\alpha\frac{4m^2}{(k^2-2p'\cdot k\alpha)^3}\right.\\ &\left.-\int d\alpha d\beta\theta(1-\alpha-\beta)\frac{16p\cdot p'}{(k^2-2k\cdot(p\alpha+p'\beta))^3}\right]\end{aligned} \tag{6.16}$$

Shifting the momentum k as appropriate this gives

$$-ie^2|\mathcal{B}|^2\int\frac{d^dk}{(2\pi)^d}\left[\int_0^1 16\alpha\,d\alpha\frac{m^2}{(k^2-m^2\alpha^2)^3}-\int d\alpha d\beta\theta(1-\alpha-\beta)\frac{16p\cdot p'}{(k^2-(p\alpha+p'\beta)^2)^3}\right] \tag{6.17}$$

Performing the integration over k gives

$$\frac{\alpha}{4\pi}\Gamma(1+\epsilon)(4\pi)^\epsilon|\mathcal{B}|^2\left[(m^2)^{-\epsilon}\int_0^1 8\alpha^{-1-2\epsilon}d\alpha-8p\cdot p'\int_0^1 d\alpha d\beta\frac{\theta(1-\alpha-\beta)}{((p_\alpha+p'\beta)^2)^{1+\epsilon}}\right]. \tag{6.18}$$

The double nested integral over α and β is performed by making the change of variables

$$\alpha=\rho\omega$$

$$\beta=\rho(1-\omega)$$

In the first term we change variable $\alpha \to \rho$, to get

$$4\frac{\alpha}{4\pi}\Gamma(1+\epsilon)(4\pi)^{\epsilon}|\mathcal{B}|^2 \int \rho^{-1-2\epsilon} d\rho \left[(m^2)^{-\epsilon} - p\cdot p' \int d\omega \frac{1}{\left((p\omega + p'(1-\omega))^2 \right)^{1+\epsilon}} \right] \tag{6.19}$$

Integrating over ρ we are left with

$$4\frac{\alpha}{4\pi}\Gamma(1+\epsilon)(4\pi)^{\epsilon}|\mathcal{B}|^2 \frac{1}{2\epsilon} \left[(m^2)^{-\epsilon} - p\cdot p' \int d\omega \frac{1}{\left((p\omega + p'(1-\omega))^2 \right)^{1+\epsilon}} \right] \tag{6.20}$$

We see that the pole term in this expression cancels against the pole term for the real emission. The infrared finite term is obtained by expanding up to order ϵ^0

6.2 Collinear Divergences

The expressions (6.13) and (6.20) for the pole parts of the real emission and virtual corrections respectively contain a factor

$$\int_0^1 d\omega \frac{1}{(p\omega + p'(1-\omega))^2}$$

For $p\cdot p' \gg m^2$ this integral is approximately

$$\frac{1}{p\cdot p'} \ln\left(\frac{2p\cdot p'}{m^2}\right),$$

and diverges as $m \to 0$.

A study of such divergences gives information about the behaviour of processes as the momentum scale increases (the high energy limit, $p\cdot p' \gg m^2$) of QED. For non-Abelian gauge theories such as QCD we have to deal with interacting particles that are strictly massless. In such cases there is a further "collinear" divergence which occurs even if the emitted photon (or gluon) does not carry small momentum, but when it is emitted parallel to the parent particle.

In the case of real photon emission, the double divergence we get when the electron mass is neglected arises from the term in the phase-space integral (in 3+1 dimensions)

$$\int \frac{d|\mathbf{k}|d\Omega}{(2\pi)^3 2|\mathbf{k}|} \frac{1}{(E - p\cos\theta)(E' - p'\cos\theta')}$$

We see that not only is there a divergence as $|\mathbf{k}| \to 0$ but for massless particles for which $E = p$ and $E' = p'$ there is a divergence at angles $\theta = 0$ and $\theta' = 0$. These are the collinear divergences.

These collinear divergences can also conveniently be treated using dimensional regularization. For, example the term under consideration from the expression (6.18) for the virtual correction is

$$\frac{\alpha}{2\pi}|\mathcal{B}|^2\frac{(4\pi)^\epsilon\Gamma(1+\epsilon)}{2\epsilon}2p\cdot p'\int_0^1\frac{d\omega}{\left(p\omega+p'(1-\omega)\right)^{1+\epsilon}}$$

If $p^2 = p'^2 = 0$ this is

$$\frac{\alpha}{2\pi}|\mathcal{B}|^2\left(\frac{(4\pi)}{2p\cdot p'}\right)^\epsilon\frac{\Gamma(1+\epsilon)}{2\epsilon}\int_0^1\frac{d\omega}{\left(\omega(1-\omega)\right)^{1+\epsilon}}$$

The integral over ω may now be performed

$$\int_0^1 d\omega w^{-1-\epsilon}(1-\omega)^{-1-\epsilon} \;=\; \frac{\Gamma^2(-\epsilon)}{\Gamma(-2\epsilon)} \;=\; -\frac{2}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

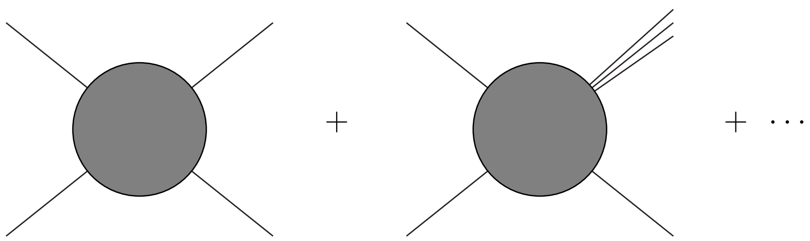
We therefore get

$$-\frac{\alpha}{2\pi}|\mathcal{B}|^2\frac{1}{\epsilon^2}\left(\frac{4\pi}{p\cdot p'}\right)^\epsilon\frac{\Gamma(1+\epsilon)\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}$$

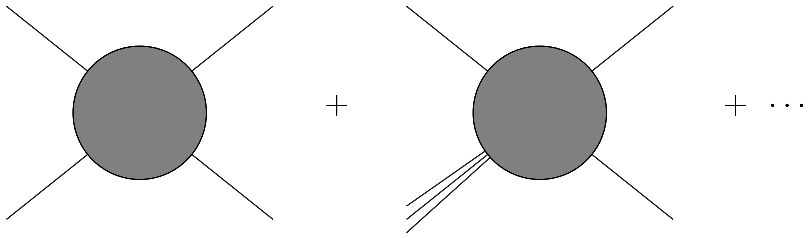
The double pole indicates that we have both a soft photon and a collinear photon divergence.

There is a similar double pole term from the real photon emission, such that the double pole cancels. However, the single pole will *not* cancel in the case of a massless electron. The cancellation between real emission and virtual corrections refers to the soft photon divergences but *not* in general to the collinear divergences. Indeed, for the collinear divergences the eikonal approximation which has been used to extract the numerators of the various graphs is not valid, so we expect more collinear divergent terms than those we have considered here.

The cancellation of both infrared and collinear divergences in massless QED or non-Abelian gauge theories with massless self-interacting particles is far more restrictive. In the case of (massive) QED the cancellation of infrared divergence occurs provided we sum over all processes involving final states that give rise to infrared divergences. For massless interacting particles this means not only summing over processes in which soft massless particles are emitted, but also over states in which hard massless particles are emitted (nearly) parallel to their parent particles. For example we need to sum over processes in which a massless outgoing particle is replaced by a jet of nearly parallel outgoing massless particles.



Unfortunately, this is not sufficient. Kinoshita, Lee & Nauenberg showed that in order to guarantee the cancellation of both soft and collinear divergences we must sum over processes involving all possible *initial* states which can give rise to soft or collinear divergences. For example we need to sum over processes in which an incoming massless particle is replaced by an incoming jet of nearly parallel massless particles.

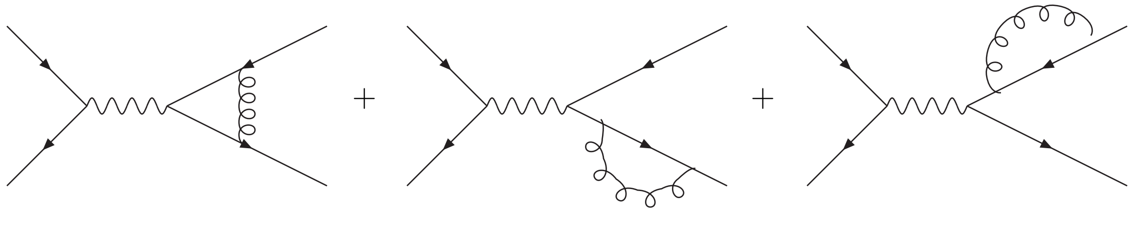


For the summation over final states, this is not really a problem, since one cannot distinguish experimentally between a single particle and a sufficiently narrow jet of particles any more than one can detect soft photon (or gluon) radiation which takes off energy less than the energy resolution of the experiment.

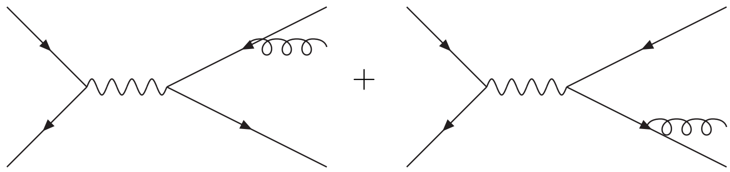
The requirement that one sums over incoming jets in order to cancel the collinear divergences is more problematic. What this means is that if we calculate in perturbation theory the QCD process of quark-quark (or quark-gluon, or gluon-gluon) scattering, we will not get a finite result even when summing over all possible final states. On the other hand, it is important to note that in practice one cannot prepare an initial state which consists of free quarks and/or gluons. The initial states are hadrons which contain quarks and gluons. The remaining divergence arising from the calculation of a process with initial quarks and/or gluons is absorbed into the (momentum scale dependence) of the parton "distribution function", i.e. the probability that a parent hadron contains a parton with a given flavour and momentum fraction.

One case in which we do not need to worry about summing over initial states is the case of electron-positron annihilation. Here the initial state consists of particles which do not have strong interactions and so there are no other initial states that are connected by strong interactions to the initial electron-positron state.

For example, in perturbative QCD to order α_S, the Kinoshita-Lee-Nauenberg theorem tells us that the soft and collinear divergences which arise in the one gluon exchange virtual correction to the cross-section for a quark-antiqark pair



cancels against corresponding divergences in the tree-level process for the production of a quark-antiquark pair *plus* a single gluon



provided we integrate over all phase space for the final state gluon.

At order α_S^2 the cancellation is between the two-loop correction to the quark-antiquark production process, the one-loop correction to the quark-antiquark-gluon production process, and the tree-level quark-antiquark-gluon-gluon production process.

It is not necessary to integrate over the whole of the phase space of the final state particles. Some differential cross-sections are also infrared finite. In such cases we would be able to calculate the differential decay rate into a state in which the final state particles had a particular variable t set equal to a value T. This variable t would be a function of the momenta of the final state particles which would depend on how many particles there were in the final state. For n final state particles we would require

$$t_n(p_1 \cdots p_n) \ = T$$

If $d\sigma^{(n)}(p_1 \cdots p_n)$ is the differential cross-section for an electron-positron pair to decay into n particles with momenta $(p_1 \cdots p_n)$ (which will in general contain soft and collinear divergences from the virtual corrections), then the total cross-section with respect to the variable T is obtained by inserting a $\delta-$function inside the phase space integral for each of the processes.

$$\frac{d\sigma}{dT} \ = \ \sum_n d\sigma^{(n)}(p_1 \cdots p_n)\delta(t(p_1 \cdots p_n) - T)d\{P.S.\}^n, \tag{6.21}$$

where $d\{P.S.\}^n$ means n-particle phase space integration. Each term in the sum of eq.(6.21) contains infrared divergences, but the sum will be finite provided t is what is known as an "infrared-safe" quantity. At order α_S the sum over n with be the two and three particles final states, whereas at order α_S^2 we would also need the four-particle final state.

For t to be an infrared safe quantity we require that that n-particle function becomes equal to the corresponding $n-1$-particle expression if any two final state particles become parallel or if any final stae particle becomes soft, i.e. for any pair of particles i, j we must have

$$t_n(p_1 \cdots p_i, p_j \cdots p_n) \ \stackrel{(p_i+p_j)^2 \to 0}{\to} \ t_{n-1}(p_1 \cdots (p_i+p_j) \cdots p_n).$$

7 Unitarity, Causality and Analyticity

The propagator for a scalar particle can be written in terms of a "dispersion relation" sometimes called the "Källén-Lehmann representation"

$$-i\Delta_F(q^2) \;\equiv\; \int \frac{d^4x}{(2\pi)^4} e^{iq\cdot x}\langle 0|T\phi(0)\phi(x)|0\rangle \;=\; i\int \frac{\rho(\sigma^2)d\sigma^2}{q^2-\sigma^2+i\epsilon}. \tag{7.1}$$

Taking the imaginary part we have

$$\Im m\left\{\Delta_F(q^2)\right\} \;=\; \pi\int \rho(\sigma^2)\delta(q^2-\sigma^2)d\sigma^2 \;=\; \rho(q^2).$$

The interpretation of the "spectral function" $\rho(q^2)$ is that it is the probability for a one particle state with square momentum q^2 to decay into all possible (energetically allowed) final states

$$\rho(\sigma^2) \;=\; (2\pi)^3\sum_n \delta^4(p_n-\sigma)|\langle 0|\phi(0)|n\rangle|^2,$$

where p_n is the total momentum of the particles in the state $|n\rangle$ (this is seen by inserting a complete set of states $\sum_n |n\rangle\langle n|$ between the fields in (7.1)).

The vacuum expectation value of the commutator of two fields may also be related to this spectral function

$$\begin{aligned}
\Delta(x) \;&\equiv\; \langle 0|\left[\phi(0),\phi(x)\right]|0\rangle \;=\; \sum_n |\langle 0|\phi(0)|n\rangle|^2\left(e^{-ip_n\cdot x}-e^{+ip_n\cdot x}\right)\\
&=\; \frac{1}{(2\pi)^3}\int d^4q(2\pi)^3\sum_n \delta^4(p_n-q)|\langle 0|\phi(0)|n\rangle|^2\left(e^{-ip_n\cdot x}-e^{+ip_n\cdot x}\right)\\
&=\; \frac{1}{(2\pi)^3}\int d^4q d\sigma^2\rho(\sigma^2)\delta(\sigma^2-q^2)\left(e^{-iq\cdot x}-e^{+iq\cdot x}\right)
\end{aligned} \tag{7.2}$$

Performing the integration over the energy component q_0 this becomes

$$-i\int d\sigma^2\rho(\sigma^2)\int \frac{d^3\mathbf{q}}{(2\pi)^3}e^{i\mathbf{q}\cdot\mathbf{x}}\frac{\sin(\sqrt{\mathbf{q}^2+\sigma^2})t)}{\sqrt{\mathbf{q}^2+\sigma^2}}.$$

The integral over $\mathbf{q}$ can be performed and a result given in terms of Bessel functions, which can be shown to vanish if $|\mathbf{x}| > t$. Actually we can see by inspection that for $t = 0$ this integral vanishes for any non-zero $|\mathbf{x}|$ and so by Lorentz invariance it must always vanish if the four-vector x is space-like. This result is expected from causality - it tells us that the commutator of two fields vanishes if the arguments of the fields are separated by a space-like quantity.

The above argument can be inverted to show that causality implies that the propagator is analytic in the upper half of the plane in q^2 (this explains the sign of the $i\epsilon$ term in the denominator of eq.(7.1)).

The argument can be extended to show that causality implies that all scattering amplitudes are analytic in the upper half complex plane for all dynamical variables.

Unitarity further implies that scattering amplitudes are analytic in a plane which is cut along the real axis.

Define the $\mathcal{T}$-matrix from the $\mathcal{S}$-matrix by

$$\mathcal{S}_{ab} \;\equiv\; \langle a_{out}|b_{in}\rangle \;=\; \delta_{ab} + i\mathcal{T}_{ab}(2\pi)^4\delta^4(p_a - p_b)$$

then the unitarity of the $\mathcal{S}$-matrix, $\mathcal{S}\,\mathcal{S}^\dagger = 1$ gives

$$\mathcal{T}_{ab} - \mathcal{T}^*_{ba} \;=\; i\sum_n (2\pi)^4\delta^4(p_a - p_n)\mathcal{T}_{an}\mathcal{T}^*_{bn} \tag{7.3}$$

The sum over n means that for each possible final state c, consisting of a certain set of final state particles, we must integrate over the whole of the available phase-space. Putting $a = b$ we have an expression for the imaginary part of the forward amplitude, known as the "optical theorem"

$$\Im m\,\{\mathcal{T}_{aa}\} \;=\; \frac{1}{2}\int\sum_c |\mathcal{T}_{ac}|^2 d\{P.S.\} \tag{7.4}$$

The RHS is proportional to the total probability for the state $|a\rangle$ to propagate into some other state $|c\rangle$. If $|a\rangle$ is a two-body state with masses m_1 and m_2 then

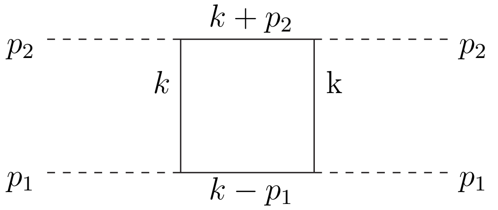
$$\Im m\,\{\mathcal{T}_{aa}\} = \;\lambda^{1/2}(s, m_1^2, m_2^2)\,\sigma^{(a)}_{TOT}(s) \tag{7.5}$$

where

$$\lambda(x, y, z) \;=\; x^2 + y^2 + z^2 - 2xy - 2xz - 2yx.$$

As an example we consider the forward scattering of two massless scalar particles (ϕ) theory interacting via a cubic interaction, $\frac{1}{2}g\phi\chi^2$ with a massive scalar field (χ).

The forward scattering amplitude is calculated from the graph



(the solid line represents the χ particles which have mass m and the dashed lines the massless external particles).

The contribution from this graph (to the $\mathcal{T}$-matrix) is

$$-ig^4\int\frac{d^4k}{(2\pi)^4}\frac{1}{(k^2 - m^2 + i\epsilon)^2((p_1 - k)^2 - m^2 + i\epsilon)((p_2 + k)^2 - m^2 + i\epsilon)}$$

Feynman parametrizing gives

$$-i6g^4 \int \frac{d^4k}{(2\pi)^4} \int d\alpha d\beta d\gamma d\delta \frac{\delta(1-\alpha-\beta-\gamma-\delta)}{(k^2-m^2-2k\cdot(p_1\alpha-p_2\beta)+i\epsilon)^4}$$

Shift $k \to k + p_1\alpha - p_2\beta$ and make use of the relations $p_1^2 = p_2^2 = 0, \;\; p_1 \cdot p_2 = s/2$

$$-i6g^4 \int \frac{d^4k}{(2\pi)^4} \int_0^1 d\alpha d\beta d\gamma d\delta \frac{\delta(1-\alpha-\beta-\gamma-\delta)}{(k^2-m^2+s\alpha\beta+i\epsilon)^4}$$

Now integrate over k to give

$$\frac{g^4}{16\pi^2} \int d\alpha d\beta d\gamma d\delta \frac{\delta(1-\alpha-\beta-\gamma-\delta)}{(m^2-s\alpha\beta-i\epsilon)^2}$$

Integrating over δ and then over β this gives

$$\frac{g^4}{16\pi^2} \int_0^1 d\alpha d\gamma \frac{\theta(1-\alpha-\gamma)}{s\alpha} \left[\frac{1}{m^2-s\alpha(1-\alpha-\gamma)-i\epsilon} - \frac{1}{m^2-i\epsilon} \right] \tag{7.6}$$

The second term in square parenthesis has no imaginary part. The imaginary part of the first term is

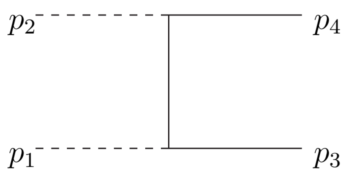
$$\begin{aligned} & \frac{g^4}{(16\pi)} \int d\alpha d\gamma \frac{1}{s\alpha} \delta(m^2 - s\alpha(1-\alpha-\gamma)\theta(1-\alpha-\gamma) \\ = \;& \frac{g^4}{16\pi} \int_0^1 \frac{d\rho d\omega}{s\omega} \delta(m^2 - s\omega\rho(1-\rho)) \\ = \;& \frac{g^4}{16\pi\, s\, m^2} \sqrt{1-\frac{4m^2}{s}} \end{aligned} \tag{7.7}$$

This imaginary part only exists if $s > 4m^2$, which is the physical threshold for the production of two $\chi-$ particles in the intermediate state. Note that the maximum value of $\omega\rho(1-\rho)$ is $\frac{1}{4}$, and that $\sqrt{1-4m^2/s}$ is the range in ρ over which we can pick up a zero of the $\delta-$function when integrating over ω.

Now we compare this with the cross-section for the process:

$$\phi + \phi \to \chi + \chi$$

The tree-level amplitude for this process is obtained from the Feynman graph



The amplitude from this graph is

$$\frac{g^2}{(t-m^2)} \qquad (t = (p_1 - p_3)^2)$$

From this we get the total cross-section to be the phase-space integral

$$\sigma = \frac{1}{2\lambda^{1/2}(s,0,0)} \int \frac{d^3p_3}{(2\pi)^3\, 2E_3} \frac{d^4p_4}{(2\pi)^4}(2\pi)\delta(p_4^2 - m^2) \left(\frac{g^2}{(t-m^2)}\right)^2 (2\pi)^4\delta^4(p_1 + p_2 - p_3 - p_4)$$

We carry out the phase space integral in the C.M. frame of p_1 and p_2, for which

$$\begin{aligned}
p_1 &= \left(\frac{\sqrt{s}}{2}, 0, 0, \frac{\sqrt{s}}{2}\right) \\
p_2 &= \left(\frac{\sqrt{s}}{2}, 0, 0, -\frac{\sqrt{s}}{2}\right) \\
p_3 &= \left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s-4m^2}\sin\theta\cos\phi}{2}, \frac{\sqrt{s-4m^2}\sin\theta\sin\phi}{2}, \frac{\sqrt{s-4m^2}\cos\theta}{2}\right) \\
p_4 &= \left(\frac{\sqrt{s}}{2}, -\frac{\sqrt{s-4m^2}\sin\theta\cos\phi}{2}, -\frac{\sqrt{s-4m^2}\sin\theta\sin\phi}{2}, -\frac{\sqrt{s-4m^2}\cos\theta}{2}\right) \\
t &= \frac{\left(2m^2 - s + \sqrt{s}\sqrt{s-4m^2}\cos\theta\right)}{2} \\
d^3p &= (2\pi)\frac{s}{4}dE_3 d\cos\theta
\end{aligned}$$

The integral over E_3 is used to absorb the δ-function $\delta(p_4^2 - m^2)$ and we have finally

$$\begin{aligned}
\sigma &= \frac{1}{\lambda^{1/2}(s,0,0)} \frac{g^4}{8\pi} 2 \int_{-1}^{1} d\cos\theta \frac{\sqrt{s}\sqrt{s-4m^2}}{\left(s - \sqrt{s}\sqrt{s-4m^2}\cos\theta\right)^2} \\
&= \frac{1}{\lambda^{1/2}(s,0,0)} \frac{g^4}{16\pi s m^2} \sqrt{1 - \frac{4m^2}{s}}. \qquad (7.8)
\end{aligned}$$

Comparing this expression with (7.7) we see that we get agreement with the unitarity condition, (7.5).

7.1 Analytic Structure of scattering amplitudes

In general, we expect a scattering amplitude to be a real analytic function of its dynamical variables (e.g. s and t) except for cuts along the real axis corresponding to a physical region. A real analytic function $f(z)$ of a complex variable z, obeys the relation

$$f(z) = f^*(z^*),$$

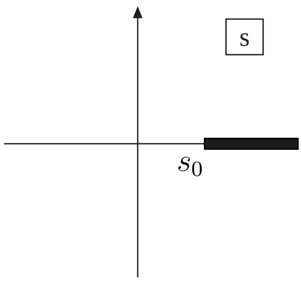
which implies

$$f(z) - f^*(z) = f(z) - f(z^*).$$

Thus the imaginary part of the forward scattering amplitude is one half the discontinuity across the cut in the complex s-plane, i.e.

$$2i\Im m\left\{F(s,t=0)\right\} \;=\; F(s+i\epsilon,0) - F(s-i\epsilon,0),$$

and by the optical theorem this can be deduced from the total cross-section.



If s is below the threshold for the production of intermediate state particles, s_0, the imaginary part vanishes which implies that the discontinuity vanishes. This means that the cut along the real axis starts at the physical threshold, s_0, which then becomes a branch point. Further cuts open at higher values of s as more and more physical states become energetically allowed. At each such threshold there will be a branch-point singularity.

For a general Feynman graph (for any number of loops) the amplitude, after integrating out the loop momenta, is a function of the momentum invariants, the masses, and a set of Feynman parameters.

$$\mathcal{A} \;\sim\; \int d\alpha_1 \cdots d\alpha_n \delta(1-\sum \alpha_i) \frac{1}{(J(\{\alpha_i\},\{p_j\cdot p_k\},\{m_l^2\})+i\epsilon}$$

At some points in the space of Feynman parameters, $\alpha_i = \alpha_i^0$, the function J will vanish. We can usually use the $i\epsilon$ prescription to integrate through such singularities in the integrand. The exceptions are if the J is also at a turning point at the point where it vanishes, *or* any of the the Feynman parameters are at the end-points of the range of integration. At such points the contribution to the amplitude from the Feynman graph has a (branch-point) singularity.

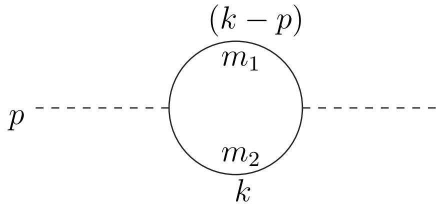
The conditions for a branch point are therefore

$$J \;=\; 0, \quad \alpha_i = \alpha_i^0$$

$$\text{either } \frac{\partial J}{\partial \alpha_i} = 0, \quad \text{or } \alpha_i^0 = 0, \; \textit{or} \;\; \text{or } \alpha_i^0 = 1 - \sum_{j \neq i} \alpha_j^0.$$

Examples:

1. **Scalar propagator in cubic interaction theory:**
   We will allow the internal particles to have arbitrary masses m_1 and m_2

   The finite part of the self-energy is

   $$-\frac{g^2}{(16\pi^2)} \int_0^1 d\alpha \ln\left(m_1^2\alpha + m_2^2(1-\alpha) - p^2\alpha(1-\alpha)\right).$$

   This gives an imaginary part if the argument of the logarithm becomes negative. The minimum value of p^2 for which this can happen is when

   $$J \;\equiv m_1^2\alpha + m_2^2(1-\alpha) - p^2\alpha(1-\alpha) \;=\; 0$$

   and

   $$\frac{\partial J}{\partial \alpha} \;=\; m_1^2 - m_2^2 - p^2(1-2\alpha) = 0 \;\; \text{or } \alpha = 0 \;\; \text{or } \alpha = 1$$

   The solution to this is

   $$p^2 = (m_1+m_2)^2, \quad \left(\alpha = \frac{m_2}{m_1+m_2}\right)$$

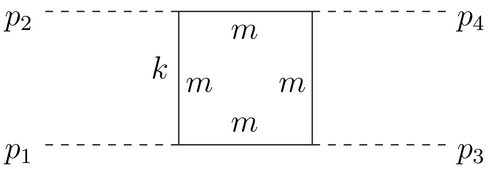
   This is the threshold for the production of two particles with masses m_1 and m_2 in the intermediate state.

   There is also a solution

   $$p^2 = (m_1-m_2)^2, \quad \left(\alpha = -\frac{m_2}{m_1+m_2}\right),$$

   but this ("pseudo-threshold") is outside the range of integration of α and so we discard it.

2. **The one-loop amplitude for the scattering of four massless scalar particles (ϕ) which interact via a cubic interaction term $\frac{1}{2}g\phi\chi^2$**
We have previously looked at the forward amplitude - we now consider the amplitude for a general square momentum transfer $t = (p_1 - p_3)^2$.



After Feynman parametrization, shifting the loop momentum and integrating over k the amplitude form this graph is

$$\frac{g^4}{(16\pi^2)}\int_0^1 d\alpha d\beta d\gamma \frac{\theta(1-\alpha-\beta-\gamma)}{(s\alpha(1-\alpha-\beta-\gamma)+t\beta\gamma - m^2 + i\epsilon)^2}$$

(we have performed the integral over the Feynman parameter δ absorbing the δ-function). The threshold is at the values of α, β, γ that obey the relations

$$J \equiv s\alpha(1-\alpha-\beta-\gamma)+t\beta\gamma - m^2 = 0$$

and

$$\frac{\partial J}{\partial \alpha} = s(1-\beta-\gamma-2\alpha) = 0, \quad \text{or } \alpha = 0, \quad \text{or } \alpha = 1-\beta-\gamma$$

and

$$\frac{\partial J}{\partial \beta} = -s\alpha + t\gamma = 0, \quad \text{or } \beta = 0, \quad \text{or } \beta = 1-\alpha-\gamma$$

and

$$\frac{\partial J}{\partial \gamma} = -s\alpha + t\beta = 0, \quad \text{or } \gamma = 0, \quad \text{or } \gamma = 1-\beta-\alpha$$

This has a solution within the range of integration at

$$\alpha = \frac{1}{2}, \quad \beta = \gamma = 0, \quad s = 4m^2, \quad t \le 0$$

or

$$\alpha = 0, \quad \beta = \gamma = \frac{1}{2}, \quad t = 4m^2, \quad s \le 0$$

The second solution is the physical threshold for the crossed (t-channel) process.

7.2 Cutkosky Rules

The discontinuity across a cut in the variable s of any Feynman graph is written as

$$\mathcal{A}(s+i\epsilon) - \mathcal{A}(s-i\epsilon).$$

Since by causality amplitudes are analytic in the upper-half plane we can define

$$\mathcal{S}^+_{ab} \;=\; \delta_{ab} + i(2\pi)^4\delta^4(p_a - p_b)\mathcal{T}^+_{ab} \;=\; \lim_{\epsilon\to 0} \langle a_{out}|b_{in}\rangle_{|s+i\epsilon}.$$

and its Hermitian conjugate

$$\mathcal{S}^-_{ab} \;=\; \delta_{ab} - i(2\pi)^4\delta^4(p_a - p_b)\mathcal{T}^-_{ab} \;=\; \lim_{\epsilon\to 0} \langle a_{in}|b_{out}\rangle_{|s-i\epsilon}.$$

$$\left(\mathcal{S}^+_{ab}\right)^* \;=\; \left(\mathcal{S}^-_{ba}\right).$$

The quantity $\lim_{\epsilon\to 0}\langle a_{in}|b_{out}\rangle$ would be calculated (following the steps of the LSZ reduction formula) using the anti-time ordered product (T^*), rather than the time-ordered product in the Green functions. For, example, for the two-point Green function of two scalar fields we have

$$\langle 0|T^*\phi(x)\phi(0)|0\rangle \;=\; \int \frac{d^4q}{(2\pi)^4} e^{iq\cdot x} \frac{-i}{q^2 - m^2 - i\epsilon}$$

This differs form the expression for the time-ordered product by an overall and the sign of the $i\epsilon$ prescription. The perturbative expansion for the anti-time ordered product also introduces a minus sign for every interaction vertex. Collecting all the signs we find that $\mathcal{T}^-_{ab}$ is obtained from $\mathcal{T}^+_{ab}$ by replacing $i\epsilon$ everywhere by $-i\epsilon$. In other words

$$\mathcal{T}^-_{ab} \;=\; \left(\mathcal{T}^+_{ab}\right)^*$$

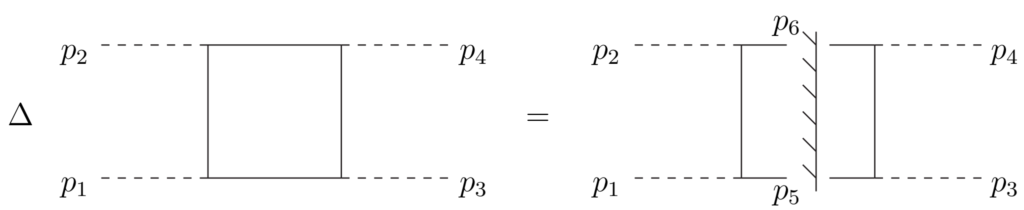
The unitarity of the $\mathcal{S}$-matrix then gives us

$$\Delta\mathcal{T}_{ab} \;\equiv\; \mathcal{T}^+_{ab} - \mathcal{T}^-_{ab} \;=\; i\sum_n (2\pi)^4\delta^4(p_a - p_n)\mathcal{T}^+_{an}\mathcal{T}^-_{nb}, \tag{7.9}$$

where Δ indicates the discontinuity across the cut . This is the generalization of the optical theorem and it is valid away from the forward direction - for example it refers to the discontinuity in the variable s for a fixed value of t away from zero.

Diagramatically the RHS of eq.(7.9) is interpreted as the sum of all cuts in the channel whose discontinuity is being considered. The part of the diagram on the right of the cut is calculated with the $i\epsilon$ replaced by $-i\epsilon$, the cut lines are placed on mass-shell and the phase-space integral for the cut lines is performed (this is implied in the sum σ_n).

For example, the s-channel discontinuity of the one-loop correction to the scattering of two massless scalar particles which interact with massive scalar particles



The amplitude on the left of the cut is

$$\mathcal{T}^{+}_{an} = \frac{g^2}{((p_1 - p_5)^2 - m^2 + i\epsilon}$$

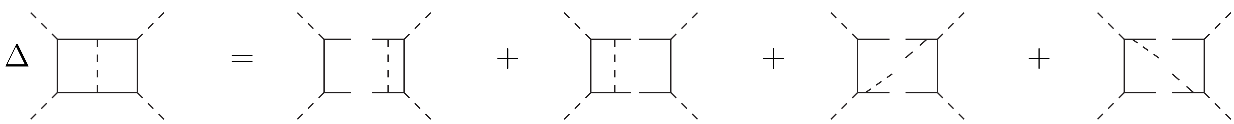
The amplitude on the right of the cut is

$$\mathcal{T}^{-}_{nb} = \frac{g^2}{((p_3 - p_5)^2 - m^2 - i\epsilon}$$

Multiplying these together and integrating over the phase-space for the intermediate particles with momenta p_5, p_6 we get for the discontinuity across the cut

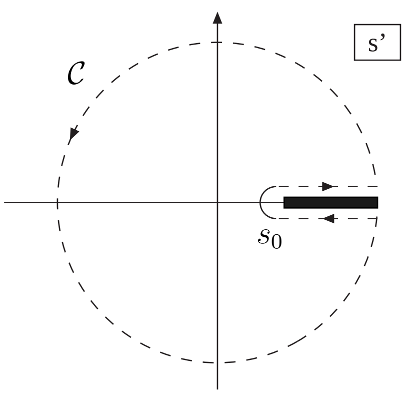
$$\int \frac{d^4 p_5}{(2\pi)^3}\delta(p_5^2 - m^2)\frac{d^4 p_6}{(2\pi)^3}\delta(p_6^2 - m^2)\frac{(2\pi)^4\delta^4(p_5 + p_6 - p_1 - p_2)\, g^4}{((p_1 - p_5)^2 - m^2 + i\epsilon)((p_3 - p_5)^2 - m^2 - i\epsilon)}$$

In higher order there are more cut graphs



The first two graphs on the RHS are integrated over two-body phase-space and the last two over three-body phase-space.

7.3 Dispersion Relations



If we consider the integral

$$\oint_{\mathcal{C}} \frac{F(s')}{(s - s' + i\epsilon)},$$

around the contour shown above, where $F(s)$ is some scattering amplitude (it can also be a function of t and other variables if there are more than two final-state particles), then given that $F(s)$ is analytic inside the contour, the integral is by Cauchy's theorem

$$2\,\pi\,i\,F(s+i\epsilon).$$

If, furthermore, $F(s)$ goes to zero as $|s| \to \infty$ then the contour integral is just the integral over the discontinuity across the cut and is therefore equal to

$$2\,i\int_{s_0}^{\infty} ds' \frac{\Im m\,\{F(s')\}}{(s-s'+i\epsilon)}$$

repeating this with ϵ replaced by $-\epsilon$ and taking th average, we get an expression for the real part of the scattering amplitude i terms of an integral over the imaginary part.

$$\Re e\,\{F(s)\} \;=\; \frac{1}{\pi}\int_{s_0}^{\infty} ds' \frac{\Im m\,\{F(s')\}}{(s-s')}_{(PV)}, \tag{7.10}$$

where "PV" indicates that the singulaity at $s = s'$ is handled using the Principle Value prescription. This is called a "dispersion relation".

If the above integral over s' does not converge, it is necessary to introduce a subtraction and we have a subtracted dispersion relation, which gives the real part in terms of the real part at some subtraction point s_B,

$$\frac{1}{(s_B-s)}\left(\Re e\,\{F(s)\} - \Re e\,\{F(s_B)\}\right) \;=\; \frac{1}{\pi}\int_{s_B}^{\infty} ds' \frac{\Im m\,\{F(s')\}}{(s-s')(s_B-s')} \tag{7.11}$$

A simple example of this is the scalar propagator with equal internal masses m, in the limit $s \gg 4m^2$.

Calculating the one-loop graph we obtain the integral over the Feynman parameter, α as

$$-\frac{g^2}{16\pi^2}\int_0^1 d\alpha \ln\left(m^2 - s\alpha(1-\alpha)\right) \tag{7.12}$$

The imaginary part is $-\pi$ times the range of α over which the argument of the logarithm is negative, which gives

$$\frac{g^2}{16\pi}\sqrt{1-\frac{4m^2}{s}}\theta(s-4m^2).$$

The real part is therefore given by

$$\Re e\left\{\Sigma(s,m^2)\right\} \;=\; \frac{g^2}{16\pi^2}\int_{4m^2}^{\infty} ds' \frac{\sqrt{1-4m^2/s'}}{(s-s')}$$

This integral diverges (it is the standard ultraviolet divergence) and so we need to subtract the dispersion relation For convenience we choose the subtraction point to be the branch-point $s = 4m^2$, to obtain

$$\begin{aligned}\Re e\left\{\Sigma(s,m^2)\right\} - \Re e\left\{\Sigma(4m^2,m^2)\right\} &= \frac{g^2}{(16\pi^2)}(s-4m^2)\int_{4m^2}^{\infty} ds' \frac{\sqrt{(1-4m^2/s')}}{(s-s')(4m^2-s')} \\ &= -\frac{g^2}{(16\pi^2)}\sqrt{1-\frac{4m^2}{s}}\ln\left(\frac{1+\sqrt{(1-4m^2/s)}}{1-\sqrt{(1-4m^2/s)}}\right)\end{aligned}$$

This result could also have been obtained by performing the integral over α in (7.12).

In the case of the forward scattering amplitude for the interacting scalars with equal internal masses m (eq.(7.7)), the real part of the amplitude is given by the integral

$$\begin{aligned}&\frac{g^4}{16\pi^2\, m^2}\int_{4m^2}^{\infty} ds' \frac{1}{(s'-s)s'}\sqrt{1-\frac{4m^2}{s'}} \\ =\;& \frac{g^4}{16\pi^2\, s\, m^2}\left[\sqrt{1-\frac{4m^2}{s}}\ln\left(\frac{1+\sqrt{1-4m^2/s}}{1-\sqrt{1-4m^2/s}}\right)-2\right]\end{aligned}$$

. This could also have been obtained by performing the integral over α and γ in (7.6).

8 Anomalies

Consider a theory with N massless fermions coupled to some gauge fields. The Lagrangian density for the fermions is

$$\mathcal{L} \;=\; i\overline{\Psi}_a\gamma^\mu \left(D_\mu\right)^a_b \Psi^b, \quad (a, b = 1\cdots n)$$

This is invariant (in the absence of fermion masses) under the global group

$$SU(N)_V \otimes SU(N)_A \otimes U(1)_V \otimes U(1)_A$$

where under:

$$\begin{array}{rl} SU(N)_V: & \Psi \to e^{i\boldsymbol{\omega}\cdot\mathbf{T}}\Psi, \quad \overline{\Psi} \to \overline{\Psi}e^{-i\boldsymbol{\omega}\cdot\mathbf{T}} \\ SU(N)_A: & \Psi \to e^{+i\boldsymbol{\omega}\cdot\mathbf{T}\gamma^5}\Psi, \quad \overline{\Psi} \to \overline{\Psi}e^{i\boldsymbol{\omega}\cdot\mathbf{T}\gamma^5} \\ U(1)_V: & \Psi \to e^{i\omega}\Psi, \quad \overline{\Psi} \to \overline{\Psi}e^{-i\omega} \\ U(1)_A: & \Psi \to e^{i\omega\gamma^5}\Psi, \quad \overline{\Psi} \to \overline{\Psi}e^{+i\omega\gamma^5}, \end{array} \tag{8.1}$$

where $\mathbf{T}$ are the generators of $SU(N)$.

[Note that under an infinitesimal axial transformation $\Psi \to \Psi + i\omega\gamma^5\Psi$, which implies that $\Psi^\dagger \to \Psi^\dagger - i\omega\Psi^\dagger\gamma^5$, which in turn implies $\overline{\Psi} \to \overline{\Psi} + i\omega\overline{\Psi}\gamma^5$, using the anti-commutation relation $\{\gamma^0, \gamma^5\} = 0$.]

By Noether's theorem, each of these symmetries has associated with it a conserved current. For example, the $U(1)$ axial current

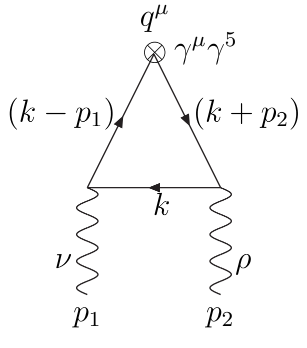
$$j^\mu_A \;\equiv\; \overline{\Psi}_a\gamma^\mu\gamma^5\Psi_a,$$

obeys the relation

$$\partial_\mu j^\mu_A \;=\; 0. \tag{8.2}$$

However, if the current can couple to gauge fields, then this conservation law can be violated at higher order.

Consider, for the case of QED, the triangle graph



which calculates the matrix element

$$\langle 0|\partial_\mu j_A^\mu|p_1,\nu;\, p_2,\rho\rangle$$

where the state $|p_1,\nu;\, p_2,\rho\rangle$ means two photons with momenta p_1 and p_2 respectively and Lorentz indices ν, ρ (to be contracted with the helicity polarization vectors of the photons). Using

$$q^\mu \;=\; p_2^\mu + p_1^\mu.$$

(We must add to this the graph in which the arrows on the fermion triangle flow in the opposite direction).

We can write

$$q_\nu\gamma^\mu\,\gamma^5 \;=\; -\gamma\cdot(k-p_1)\,\gamma^5 - \gamma^5\,\gamma\cdot(k+p_2), \tag{8.3}$$

where in the second term the γ^μ and γ^5 have been anti-commuted. Each of these terms gives an expression in which one of the two fermion propagators in the triangle have been "killed" so that we get two terms of the form

$$\int d^dk \frac{\mathrm{Tr}\left(\gamma^5\,\gamma^\nu\,\gamma\cdot k\,\gamma^\rho\,\gamma\cdot(k+p_2)\right)}{k^2(k+p_2)^2}$$

and

$$\int d^dk \frac{\mathrm{Tr}\left(\,\gamma^5\,\gamma\cdot(k-p_1)\,\gamma^\nu\,\gamma\cdot k\,\gamma^\rho\right)}{k^2(k-p_1)^2}$$

For the first of these these terms the trace gives a term proportional to

$$\epsilon_{\nu\rho\sigma\tau}\,k_\sigma p_{2\,\tau},$$

However the integral over k gives a contribution proportional to $p_{2\,\sigma}$ so this term vanishes. This applies also to the second term, so it looks as though the contribution vanishes as expected.

Unfortunately, this treatment is too glib. Since the integrals over k are ultraviolet divergent, we need to consider carefully the question of regularization of terms involving γ^5. in dimensional regularization this requires care since the matrix γ^5 anti-commutes with the other *four* γ-matrices in four dimensions. Much work has been done on this problem. One consistent procedure is to consider the integrals in *more than* four dimensions (ϵ negative) and to let γ^5 anti-commute with the first four $\gamma-$matrices but commute with the remaining -2ϵ γ-matrices. Once the numerator has been calculated using this prescription we may analytically continue to positive ϵ before performing the integral over k.

If we do this we note that (8.3) is not valid for the components of k outside the 4-dimensions because γ^5 does *not* anticommute with the components of γ^μ outside 4 dimensions and so we get an extra term from the triangle graph

$$4ie^2\int\frac{d^dk}{(2\pi)^d}\frac{\mathrm{Tr}\left((\gamma\cdot(k+p_2)\,\gamma^5\,\gamma\cdot l\,\gamma\cdot(k-p_1)\,\gamma^\nu\,\gamma\cdot k\,\gamma^\rho\right)}{k^2(k-p_1)^2(k+p_2)^2},$$

where l^μ represents the components of k^μ in the extra dimensions. This means that $l \cdot p_1$, $l \cdot p_2$, $l^\nu\, l^\rho$ all vanish and the trace gives

$$4l \cdot k\, \epsilon_{\sigma\nu\tau\rho} p_1^\sigma p_2^\tau$$

Introducing Feynman parameters the contribution from the triangle is

$$-16ie^2 \epsilon_{\sigma\nu\tau\rho} \int \frac{d^d k}{(2\pi)^d} \int_0^1 2d\alpha d\beta \theta(1-\alpha-\beta) \frac{p_1^\sigma p_2^\tau\, l \cdot k}{(k^2 - 2k \cdot (p_1\alpha - p_2\beta))^3}$$

Shifting $k \to k + p_1\alpha - p_2\beta$ gives

$$-16ie^2 \epsilon_{\sigma\nu\tau\rho} \int \frac{d^d k}{(2\pi)^d} \int_0^1 2d\alpha d\beta \theta(1-\alpha-\beta) \frac{p_1^\sigma p_2^\tau\, l \cdot k}{\left(k^2 - (p_1\alpha - p_2\beta)^2\right)^3}$$

Now $l \cdot k = l^2$ and

$$\int \frac{d^d k}{(2\pi)^d} \frac{l^2}{(k^2 + A^2)^3} = \frac{2\epsilon}{d} \int \frac{d^d k}{(2\pi)^d} \frac{k^2}{(k^2 + A^2)^3} = \frac{i}{32\pi^2} + \mathcal{O}(\epsilon).$$

Thus the final result is (in terms of the fine-structure constant)

$$-2\frac{\alpha}{\pi} \epsilon_{\sigma\nu\tau\rho} p_1^\sigma p_2^\tau.$$

This is the coupling between the divergence of the $U(1)$ axial current and two photons. We may express this by stating that the divergence of the $U(1)$ axial current is *not* zero as given by eq.(8.2), but rather has an "anomaly"

$$\partial_\mu j_A^\mu = \frac{\alpha}{2\pi} \epsilon_{\sigma\nu\tau\rho} F^{\sigma\nu} F^{\tau\rho}. \tag{8.4}$$

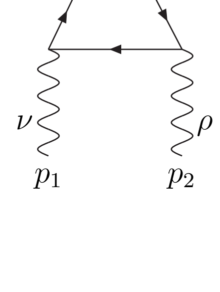
Noting that

$$F^{\sigma\nu} = \partial^\sigma A^\nu - \partial^\nu A^\sigma,$$

and expanding the photon fields in terms of creation and annihilation operators, we see that to leading order we have The RHS of eq.(8.4) has a matrix element

$$\langle 0 | \epsilon_{\sigma\nu\tau\rho} F^{\sigma\nu} F^{\tau\rho} | p_1, \nu;\, p_2, \rho \rangle = 4\epsilon_{\sigma\nu\tau\rho}\, p_1^\sigma p_2^\tau.$$

Adler and Bardeen demonstrated that there are no higher order corrections to this anomaly, i.e. there is no correction to eq.(8.4) from graphs such as

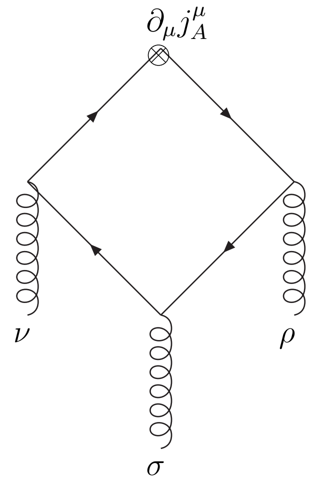


It is not only the $U(1)$ axial current that has an anomalous divergence involving photons. There is a non-zero divergence for all the diagonal elements of $SU(N)_A$, so that more generally we have

$$\partial_\mu j_A^{a\,\mu} \;=\; \frac{\alpha}{2\pi}\left(T^a\right)_{ii} Q_i^2 \epsilon_{\sigma\nu\tau\rho} F^{\sigma\nu} F^{\tau\rho}, \tag{8.5}$$

where (T^a) are the diagonal generators (Cartan sub-algebra) and Q_i is the electron charge of the fermion of flavour i.

In the case of the coupling of the divergence of the axial current to non-Abelian gauge fields we also get an anomalous contribution from the graph



so that the anomalous divergence eq.(8.4) generalizes to the case of a non-abelian gauge theories to

$$\partial_\mu j_A^\mu \;=\; \frac{\alpha_s}{2\pi} \mathrm{T_R} \epsilon^{\mu\nu\rho\sigma} \mathbf{F}_{\mu\nu} \cdot \mathbf{F}_{\rho\sigma}, \tag{8.6}$$

where $\mathrm{T_R}$ implies a sum over all the flavours of fermions to which the gauge bosons couple.

We can view this by assigning to the axial current an anomalous component

$$j_A^{\mu\,(\mathrm{anom})} \;=\; \frac{\alpha_s}{2\pi} \mathrm{T_R} \epsilon^{\mu\nu\rho\sigma} B_{\nu\rho_\sigma},$$

where $B_{\nu\rho_\sigma}$ us an anti-symmetric three-rank tensor known as a "Chern-Simons three-form",

$$B_{\nu\rho_\sigma} \;=\; \mathbf{A}_{[\nu}\partial_\rho \cdot \mathbf{A}_{\sigma]} - g\frac{2}{3}\mathbf{A}_{[\nu} \cdot \mathbf{A}_\rho \wedge \mathbf{A}_{\sigma]},$$

($\mathbf{X} \cdot \mathbf{Y} \wedge \mathbf{Z}$ means $f_{abc}X^aY^bZ^c$).

This "Chern-Simons" form is not gauge invariant, but the gauge dependence is a total derivative

$$\delta_{\boldsymbol{\omega}}\left(B_{\nu\rho_\sigma}\right) = \;\partial_{[\nu}\left(\boldsymbol{\omega} \cdot \mathbf{F}_{\rho\sigma]}\right)$$

This we see that we have an "anomaly", i.e. a conservation law obeyed by the Lagrangian is violated by the quantized theory. The quantized theory is expressed in terms of a partition function (path integral)

$$\mathcal{Z}(\eta, \overline{\eta}, j_\mu) = \int \mathcal{D}[A_\mu]\mathcal{D}[\Psi]\mathcal{D}[\overline{\Psi}] \exp\left\{ i \int d^4x \left(\mathcal{L} + j_\mu A^\mu + \overline{\eta}\Psi + \overline{\Psi}\eta \right) \right\}.$$

Since $\mathcal{L}$ is invariant under axial transformations, it must be the measure $\mathcal{D}[\Psi]\mathcal{D}[\overline{\Psi}]$ which violates the invariance.

Let us parametrize $\mathcal{D}[\Psi]$ in terms of the eigenstates, ψ_n, of $i\gamma^\mu \mathbf{D}_\mu$, for some fixed gauge field, $\mathbf{A}_\mu$, i.e

$$i\gamma \cdot \mathbf{D}\psi_n = \lambda_n \psi_n$$

such that

$$\Psi = \sum_n a_n \psi_n, \quad a_n = \int d^4x \overline{\psi}_n \Psi$$

and similarly

$$\overline{\Psi} = \sum_n a_n^* \overline{\psi_n}, \quad a_n^* = \int d^4x \overline{\Psi} \psi_n$$

The measure is now

$$\mathcal{D}[\Psi] = \Pi_n da_n$$
$$\mathcal{D}[\overline{\Psi}] = \Pi_n da_n^*$$

Under an infinitesimal axial $U(1)$ transformation

$$\Psi \rightarrow \Psi + i\omega\gamma^5\Psi$$

$$a_n \rightarrow a_n + i\omega \int d^4x \overline{\psi}_n \gamma^5 \Psi = a_n + i\omega \sum_m \int d^4x \overline{\psi}_n \gamma^5 \psi_m a_m$$

Likewise

$$a_n^* \rightarrow a_n^* + i\omega \sum_m \int d^4x \overline{\psi}_m \gamma^5 \psi_n a_m^*$$

This means that under the transformation $\mathcal{D}[\Psi]\mathcal{D}[\overline{\Psi}]$ acquires a factor

$$\det(1 + 2\omega\gamma^5) \approx 1 + 2\omega \, \mathrm{tr}\gamma^5.$$

Superficially $\mathrm{tr}\gamma^5 = 0$, but we have a sum over an infinite number of states and so what we really mean by the trace is

$$\int d^4x \sum_n \overline{\psi}_n \gamma^5 \psi_n,$$

where we need to take into account the fact that

$$\int d^4x \sum_n \overline{\psi}_n \psi_n,$$

is divergent.

We regulate this divergence by writing the trace as

$$\lim_{\delta \to 0} \sum_n \overline{\psi}_n e^{-\delta \lambda_n^2} \gamma^5 \psi_n = \lim_{\delta \to 0} \sum_n \overline{\psi}_n e^{\delta \, \gamma \cdot \mathbf{D} \, \gamma \cdot \mathbf{D}} \gamma^5 \psi_n$$

Taking care of the commutation between $\mathbf{D}_\mu$ and $\mathbf{D}_\nu$ we have

$$\gamma \cdot \mathbf{D}\,\gamma \cdot \mathbf{D} \;=\; \mathbf{D}^2 + \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]\left[\mathbf{D}_\mu, \mathbf{D}_\nu\right] = \; \mathbf{D}^2 + ig\frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right]\mathbf{F}_{\mu\nu}$$

So the trace becomes

$$\lim_{\delta\to 0}\sum_n \overline{\psi}_n e^{\delta \mathbf{D}^2}\,\mathrm{tr}\left\{e^{-ig\delta\gamma^\mu\gamma^\nu\mathbf{F}_{\mu\nu}}\,\gamma^5\right\}\psi_n$$

Expanding $\exp\left\{-ig\delta\gamma^\mu\gamma^\nu\mathbf{F}_{\mu\nu}\right\}\gamma^5$, the only term which has a non-zero trace is

$$-\frac{\delta^2}{2}g^2\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\mathbf{F}_{\mu\nu}\cdot\mathbf{F}_{\rho\sigma}$$

and the trace (including the trace over flavours - or fermion multiplets) gives

$$-2\delta^2\epsilon_{\mu\nu\rho\sigma}\,g^2\,\mathrm{T_R}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}_{\rho\sigma}$$

For a free Dirac theory,

$$\sum_n \overline{\psi}_n e^{\delta\mathbf{D}^2}\psi_n$$

means

$$\int\frac{d^4k}{(2\pi)^4}e^{-\delta k^2} \;=\; \frac{1}{16\pi^2\delta^2}$$

and this results also holds true for a Dirac field in the presence of a gauge field.

Piecing together, this means that the change in the partition function under an infinitesimal axial $U(1)$ transformation is

$$\delta_{axial\,U(1)}\mathcal{Z}(0) \;=\; \frac{g^2}{8\pi^2}\int\mathcal{D}[A_\mu]\mathcal{D}[\Psi]\mathcal{D}[\overline{\Psi}]\mathrm{T_R}\epsilon^{\mu\nu\rho\sigma}\mathbf{F}_{\mu\nu}\cdot\mathbf{F}_{\rho\sigma}\exp\left\{i\int d^4x\mathcal{L}\right\}, \qquad (8.7)$$

in agreement with eq.(8.6).

Although these anomalies mean that some of the original symmetries of the theory are violated by the quantization procedure, they can have some constructive use.

An example is the explanation of the decay of a π^0 into two photons. The π^0 couples to the third component of the divergence of the $SU(2)$ axial current

$$\partial_\mu j_A^{3\,\mu} \;=\; f_\pi m_\pi^2\phi_{\pi_0},$$

where ϕ_{π_0} is the pion field, and f_π is the pion decay constant, and

$$j_A^{3\,\mu} \;=\; \frac{1}{2}\sum_{i=1}^{3}\left(\overline{u}^i\gamma^\mu\gamma^5 u_i - \overline{d}^i\gamma^\mu\gamma^5 d_i\right),$$

(the sum being over three colours). $\partial_\mu j_A^{3\,\mu}$ couples to two photons via the anomaly so we get (see eq.(8.5))

$$\partial_\mu j_A^{3\,\mu} \;=\; \frac{1}{2}\,3\times\left(\left(\frac{2}{3}\right)^2-\left(\frac{1}{3}\right)^2\right)\frac{\alpha}{2\pi}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$$

This means that for a stationary pion decaying into two photons with momenta p_1 and p_2, and polarization vectors ϵ_1 and ϵ_2 the decay matrix element is

$$\mathcal{M} \;=\frac{\alpha}{2\pi}\frac{2}{f_\pi}\epsilon_{\mu\nu\rho\sigma}p_1^\mu\epsilon_1^\nu p_2^\rho\epsilon_2^\sigma$$

Squaring this and performing the integral over the two-photon phase-space we get the decay rate to be

$$\Gamma \;=\; \frac{\alpha^2 m_\pi^3}{64\pi^3 f_\pi^2} \;=\; 7.63\,eV$$

This compares favourably with the experimental value of 7.7 ± 0.6 eV.

On the other hand, these anomalies can be disastrous if the gauge theory itself is axial or contains an axial component - such as the GWS model of weak and electromagnetic interactions.

The problem here is that the gauge invariance itself is broken by the anomalies, and in cases such as spontaneous symmetry breaking, gauge invariance is used to demonstrate renormalizability. Thus, in general, the renormalizability of a chiral (or axial-vector) gauge theory is spoilt unless the anomalies are arranged to cancel.

For an axial gauge theory, all the axial currents $j_A^{a\,\mu}$ acquire a non-zero divergence through the anomalies. This divergence is given by

$$\partial_\mu j_{A\,a}^\mu \;=\; \frac{\alpha_s}{4\pi}\sum_i d_{abc}^{(i)}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^bF_{\rho\sigma}^c, , \tag{8.8}$$

where for the fermion multiplet i

$$d_{abc}^{(i)} \;=\; \mathrm{Tr}\left(T^aT^bT^c+T^cT^bT^a\right)$$

For a theory to be anomaly free, we need to ensure that

$$\sum_i d_{abc}^{(i)} \;= 0.$$

For the GWS model, $d_{abc}^{(i)}\neq 0$ only if two of a,b,c are $SU(2)$ labels and are equal and the third is a $U(1)$. In this case the anomaly is proportional to the $U(1)$ charge of the fermions in the triangle. Thus the (gauged part of the) GWS model is anomaly free because the sum of the $U(1)$ charges of all the fermions in the model is zero. Care has to be taken when extending this model, particularly if one wishes to extend the non-abelian sector beyond $SU(2)$ in which case the non-abelian sector by itself can produce an anomalous contribution since th e d-matrices are in general non-zero.