MATH1007

Mathematical Methods for Physical Sciences

Skeletal Notes

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1 Elements of Linear Algebra

1.1 Matrices and simultaneous equations

1.1.1 General idea

Let A be a given square matrix of dimension $n \times n$, and let $\boldsymbol{c}$ be a given column vector of length n. Then

is a *matrix representation* of a set of n simultaneous equations for the n unknown components of the column vector $\boldsymbol{x}$. For example,

represents the set of two simultaneous equations

In general, there is a simple systematic way to solve the set (1): Suppose that A has an inverse, given by A^{-1} (i.e., $A^{-1}A = I$, where I is the Identity matrix). Then, by multiplying both sides on the left by A^{-1}, we obtain

Not all matrices have inverses. A square matrix which has an inverse is said to be **nonsingular**. A matrix that does not possess an inverse is **singular**. Recall that a matrix only has an inverse if its determinant is nonzero, i.e.,

Zero determinant $\Rightarrow$ no inverse, singular matrix.

We first consider examples of nonsingular systems. In the next section we shall look at singular systems.

Example 1:

Solve the following system:

$$\begin{aligned} x + 2y &= 4, \\ 2x + 3y &= 5. \end{aligned}$$

We may write this as the matrix equation $A\boldsymbol{x} = \boldsymbol{c}$, where

Then

Example 2:

Solve the following system:

$$\begin{aligned} x + 2z = 1, \\ 3y + z = 2, \\ x + y = 3. \end{aligned}$$

We may write this as the matrix equation $A\boldsymbol{x} = \boldsymbol{c}$, where

Then

1.1.2 Simultaneous equations and singular matrices

We need to distinguish between **homogeneous** and **inhomogeneous** systems:

(a) Homogeneous case: $A\boldsymbol{x} = \mathbf{0}$

Suppose we have a system with n equations and n unknowns, so that A is an $n \times n$ matrix. If $\det(A) = 0$ this implies that one or more rows of A can be expressed as a linear combination of the others (recall properties of determinants). This row represents the coefficients of an equation in the unknowns, so, in other words, one equation is just a linear combination of the others, and so gives no further information.

Example :

$$\begin{aligned} x + y + z &= 0, \\ x + 2y + z &= 0, \\ 2x + 3y + 2z &= 0. \end{aligned}$$

This is equivalent to $A\boldsymbol{x} = \mathbf{0}$ where

Since $\det A = 0$, we know that we can write one of the equations in terms of the other two. In this example the 3rd equation is just the sum of the first two equation, so this tells us nothing new. Hence, the original system of equations is equivalent to

$$\begin{aligned} x + y + z &= 0, \\ x + 2y + z &= 0, \end{aligned}$$

and we have just two equations for three unknowns.

The important thing is the number of linearly independent equations m:

If $\det A \neq 0$ then $m = n$ and the only solution to $A\boldsymbol{x} = \mathbf{0}$ is the "trivial" null solution, $\boldsymbol{x} = 0$.

If $\det A = 0$ then $m < n$ and there are an infinite number of solutions that depend on $n - m$ parameters.

Example :

$$\begin{aligned} x + y + z &= 0, \\ x + 2y + z &= 0, \\ 2x + 3y + 2z &= 0. \end{aligned}$$

As we have seen above, here $\det A = 0$ so that the equations are not independent. In fact, we have just two independent equations (so $n = 3$ and $m = 2$), so the solution depends on $3 - 2 = 1$ parameters. To solve the system we throw away the third equation (which tells us nothing new) and rewrite the first two equations (taking z as a parameter) as

Alternatively, we could have taken x as the free parameter, in which case the solution to these two equations would have been written in the form $z = -x$, $y = 0$.

Geometrically, the three equations describe three planes, and the solution (in whatever form) represents the line along which these planes intersect.

NB: we could not have picked y as a free parameter as its value is fixed. The fact that there is an infinite number of solutions does not imply that all unknowns are undetermined: once the chosen parameter values have been fed into the solutions, the corresponding values of the unknowns can be found.

Inhomogeneous case: $A\boldsymbol{x} = \mathbf{c} \neq 0$

The inhomogeneous case is slightly more complicated. Consider two examples:

Example 1:

$$\begin{aligned} x + y + z &= 1, \\ x + 2y + z &= 1, \\ 2x + 3y + 2z &= 2. \end{aligned}$$

Example 2:

$$\begin{aligned} x + y + z &= 1 \\ x + 2y + z &= 1 \\ 2x + 3y + 2z &= 3 \end{aligned}$$

1.1.3 Summary of cases

	$\det \neq 0$, nonsingular	$\det = 0$, singular
homogeneous	only trivial null solution	infinite number of solutions
inhomogeneous	unique, nontrivial solution	no solution, or infinite number of solutions

1.2 Eigenvalues and eigenvectors

In many areas in physical sciences, properties of objects are often described by a matrix A. One then often has to solve equations of the form

namely: Find a number λ **and** a vector $\boldsymbol{v}$ such that this equation is satisfied. The number λ is called an **eigenvalue**, the vector $\boldsymbol{v}$ is called an **eigenvector**. ("eigen" is the German word of "own", implying that an eigenvector is somehow "special" to the matrix.)

Example :

The energetic structure of an atom is described in quantum mechanics by a matrix, with its eigenvalues describing the energy levels of the atom, and its eigenvectors describing the electron states corresponding to these level.

Given a matrix, how do we evaluate the eigenvalues and eigenvectors?

Rewrite:

This is now in the form of a homogeneous system, $B\boldsymbol{x} = 0$, where $B = A - \lambda I$ and $\boldsymbol{x} = \boldsymbol{v}$. Thus, as we have seen before, if $\det(A - \lambda I) \neq 0$ it will have only the trivial solution $\boldsymbol{v} = \mathbf{0}$. A more useful solution is only possible if

The expression $\det(A - \lambda I) = 0$ is actually a polynomial in λ and is called the **characteristic polynomial**. The eigenvalues are the roots of the characteristic polynomial. Once the eigenvalues are known, the eigenvectors are any solutions of the system

Example :

Find the eigenvalues and eigenvectors of

First calculate the characteristic polynomial, :

The eigenvalues are the roots of the characteristic polynomial:

Alternatively, use the formula

The eigenvector of the eigenvalue $\lambda = 1$ is the solution of

The eigenvector of the eigenvalue $\lambda = 3$ is the solution of

Geometrically, eigenvectors and eigenvalues can be interpreted in the following way. Suppose matrix A is a transformation of position vectors in a plane, e.g, combinations of rotations, reflections, translations, magnifications. The eigenvectors are those special vectors which remain unchanged in direction after the transformation, but which are scaled by a factor λ. This has important consequences in solid mechanics, where the eigenvectors give the principle directions in which strains caused by external forces take place. For more information on matrix transformations of the plane, see the course website at http://www.maths.soton.ac.uk/teaching/units/math1007/matrices.pdf. In this course you will only be expected to know how to find eigenvalues and eigenvectors of 2×2 matrices.

2 Complex Numbers

2.1 introduction

When we try to solve quadratic equations, such as

$$x^2 + 1 = 0,$$

or

$$x^2 - 2x + 2 = 0,$$

we find that they do not have solutions which are real numbers. In order to solve such equations, we introduce a quantity i which has the property that

$$i^2 = -1.$$

We then define complex numbers by:

$$z = x + iy, \qquad x, y \in \mathbb{R}.$$

Thus $z = 2 + i6$ or $2 + 6i$ is an example of a complex number.

For $z = x + iy$ we call x the **real part** of z and write

$$\mathrm{Re}(z) = \quad ,$$

and we call y the **imaginary part** of z and write

$$\mathrm{Im}(z) =$$

Notice that both the real part and the imaginary part of z are real.

It is now possible to solve quadratic equations such as the ones above.

Example 1

$$x^2 - 2x + 2 = 0$$

2.2 Powers of i

$i^2 = -1$ so, multiplying both sides by i gives

$i^3 =$

$i^4 =$

$i^5 =$

$i^6 =$

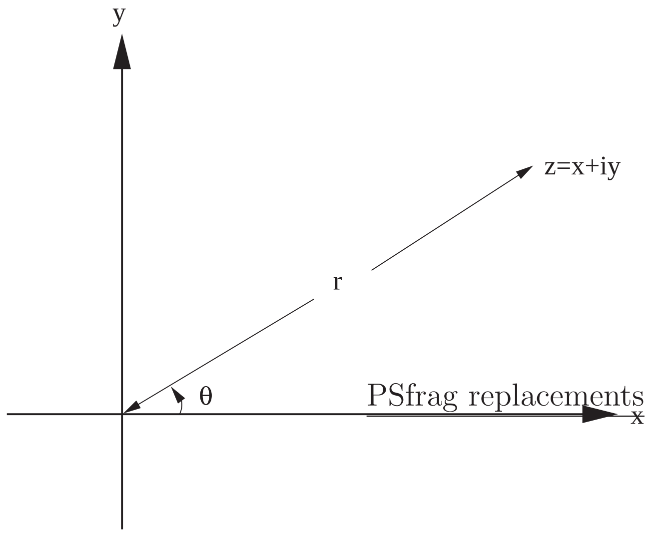
$i^{-1} =$

$i^{-2} =$

$i^{-3} =$

2.3 Argand Diagrams

We can associate the complex number $z = x + iy$ with the point in the Euclidean plane with Cartesian coordinates (x, y). Conversely, any point in the x, y-plane corresponds to a complex number $z = x + iy$.



So we can think of the x-axis as the real axis, and the y-axis as the imaginary axis. A diagram of the complex plane thought of in this way is called an Argand diagram.

2.4 Modulus and Argument of a Complex Number

Instead of using x and y as coordinates, we can use polar coordinates r and θ, where:

$x =$ $\qquad$ $y =$

$r =$ $\qquad$ $\tan\theta =$

Note: The angle θ must be such that it is in the correct quadrant as given by the signs of x and y. This means we may have to add or subtract π to the value given by $\arctan\theta = \tan^{-1}\theta$.

In polar coordinates we may write

$$z = x + iy$$

$$=$$

$$= \qquad .$$

The number r is called the **modulus** of z and we write the modulus of z as $|z|$:

The number θ is called the **argument** of z, which we write as $\arg z$. Note that the argument is not unique but it is only defined up to multiples of 2π.

Example

$$1 + i =$$

$$=$$

$$=$$

In general,

$$1 + i =$$

Although the argument of a complex number is not unique, we define the *principal value* of the argument to be the value of the argument in the range $\quad < \theta \leqslant \quad$.

Thus for $z = x + iy$ the principal value of the argument is given by that value of θ in the range $-\pi < \theta \leqslant \pi$ such that:

$$\tan\theta = \quad ,$$

$$\sin\theta = \quad ,$$

$$\cos\theta = \quad .$$

It should be noted that this value does not always coincide with $\arctan\frac{y}{x} = \tan^{-1}\frac{y}{x}$.

For example, if $z = 1 + i\sqrt{3}$ then $x = 1$ and $y = \sqrt{3}$ so that

$$r = \qquad \text{and} \qquad \tan\theta =$$

and we would conclude corrrectly that

$$\theta =$$

To check this note that

$$z = r\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) =$$

as required.

On the other hand, if $z = -1 + i\sqrt{3}$ then $x = -1$ and $y = \sqrt{3}$ so that

$$r = \qquad \text{and} \qquad \tan\theta =$$

and we would conclude **incorrrectly** that

$$\theta =$$

To check this note that

$$z \neq r\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right) =$$

In fact, in this case we have

$$\theta = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}$$

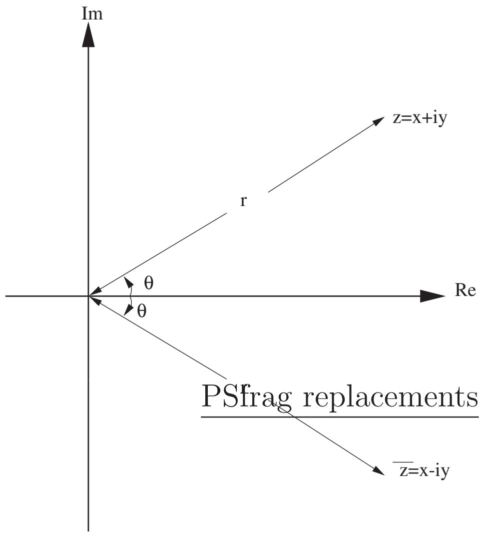
2.5 Complex Conjugates

If $z = x + iy$, then we define the **complex conjugate** $\overline{z}$ by

$$\overline{z} =$$

Note: the complex conjugate of z is often written as z^*.

In the Argand diagram, taking the complex conjugate is just a reflection in the real axis:



Note from this diagram that

$$|\overline{z}| = \qquad\qquad \text{and} \quad \arg \overline{z} =$$

If

$$z = \qquad\qquad\qquad \text{then} \quad \overline{z} =$$

However this is **not** in polar form because of the negative sign.

To put it in polar form we write

$$\overline{z} =$$

so that $|\overline{z}| = r$ and $\arg \overline{z} = -\theta$, as required.

2.6 Addition and subtraction of complex numbers

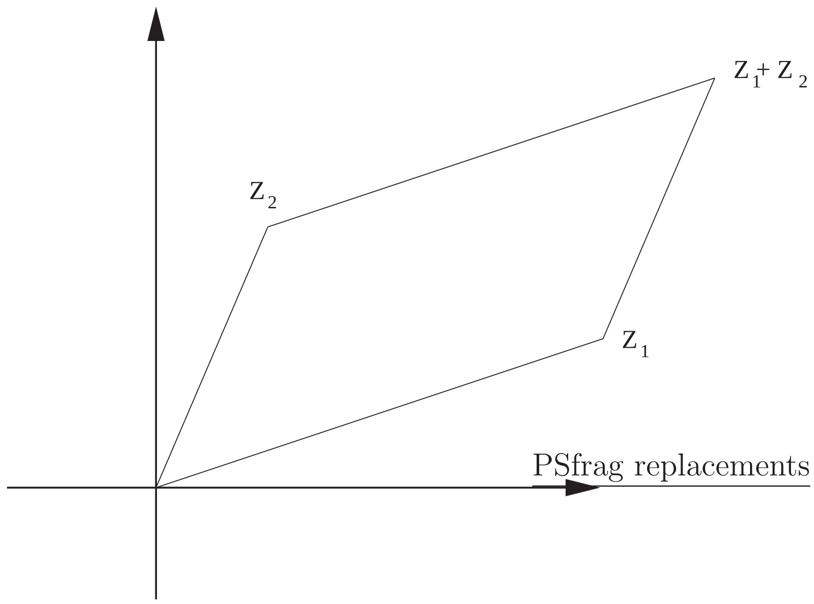
Addition and subtraction Let $z_1 = x_1 + y_1$ and $z_2 = x_2 + y_2$. We then define

$$z_1 + z_2 =$$

and

$$z_1 - z_2 = \qquad\qquad\qquad\qquad .$$

Addition in the Argand Diagram



Note that for $z = x + iy$ and $\overline{z} = x - iy$ we have

$$z + \overline{z} =$$

and

$$z - \overline{z} =$$

Conversely, we have

$$\mathrm{Re}(z) =$$

and

$$\mathrm{Im}(z) = \qquad .$$

2.7 Multiplication of Complex Numbers

To find the rule for the multiplication of complex numbers we assume that complex numbers obey the ordinary rules of arithmetic. Hence the product of $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ is given by

$$\begin{aligned} z_1 z_2 &= \\ &= \\ &= \qquad . \\ \text{Thus} \quad z_1 z_2 &= \qquad . \end{aligned}$$

We in fact take the final line as our **definition** of the product of complex numbers (and the previous lines as motivation).

Example: Let $z_1 = -3 + 5i$ and $z_2 = 2 - 3i$. Then

$$\begin{aligned} z_1 z_2 &= \\ &= \\ &= \qquad , \end{aligned}$$

and

$$\begin{aligned} z_2 z_1 &= \\ &= \\ &= \quad . \end{aligned}$$

Notice that $z_1 z_2 = z_2 z_1$.

More generally, with our definition of multiplication we can verify that complex numbers satisfy all the usual rules of arithmetic, so that:

$$\begin{aligned} z_1 z_2 &= \\ z_1(z_2 z_3) &= \\ (z_1 + z_2)z_3 &= \end{aligned}$$

It is often useful to use the fact that

$$z\overline{z} =$$

Hence

For example, if $z = -3 + 4i$ we have

$$z\overline{z} =$$

The rule for multiplication of complex numbers is simpler if we write the complex number in polar form.

$$\text{Let} \quad z_1 = r_1(\cos\theta_1 + i\sin\theta_1), \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

Then:

$$z_1 z_2 =$$
$$=$$
$$= \qquad .$$

Hence

$$|z_1 z_2| = \qquad ,$$
$$\arg(z_1 z_2) = \qquad .$$

2.8 Division of Complex Numbers

To perform division, the trick is to multiply the numerator and denominator of the quotient by the complex conjugate of the denominator:

$$\frac{z_1}{z_2} =$$
$$= \qquad .$$

In Cartesian form:

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} =$$
$$= \qquad .$$

Example: Let $z_1 = -3 + 5i$ and $z_2 = 2 - 3i$. Then

$$\frac{z_1}{z_2} =$$

$$=$$

$$=$$

$$= \qquad .$$

In polar form:

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$=$$

$$=$$

$$= \qquad .$$

Hence,

$$\left|\frac{z_1}{z_2}\right| =$$

$$\arg\left(\frac{z_1}{z_2}\right) =$$

2.9 Complex Exponentials and Trigonometric Functions

Let z be a complex number. We define

$$e^z =$$

Now let $z = e^{i\theta}$, where $\theta \in \mathbb{R}$, then:

$$\begin{aligned} e^{i\theta} &= \\ &= \underbrace{\qquad\qquad\qquad}_{(1)} \quad \underbrace{\qquad\qquad\qquad}_{(2)} \end{aligned}$$

(1) is the Taylor series for $\cos\theta$ and (2) is the Taylor series for $\sin\theta$. Hence:

$$\boxed{\qquad\qquad\qquad\qquad\qquad (A) \qquad}$$

Since we can write any complex number in polar form,

$$z =$$

we see that any complex number can be written as

$$\boxed{\qquad\qquad\qquad\qquad\qquad\qquad\qquad}$$

Or, since the argument is not unique,

$$z = \qquad\qquad\qquad\qquad\qquad .$$

From (A), replacing θ by $-\theta$ we get

$$e^{-i\theta} =$$

$$\Rightarrow \quad e^{-i\theta} = \qquad\qquad . \qquad\qquad (B)$$

Adding (A) and (B) and dividing by 2 gives

Subtracting (B) from (A) and dividing by $2i$ gives

2.10 Further examples of complex numbers in polar form

Example 1.

Let $z_1 = 3 + 3i$, and $z_2 = 2e^{i\frac{\pi}{3}}$. Calculate $z_1 z_2$ in both Cartesian and polar form.

$$z_2 =$$

$$=$$

$$=$$

so that $z_1 z_2 =$

$$= \qquad\qquad .$$

On the other hand,

$$z_1 = 3 + 3i =$$

(1) is the modulus and to put (2) in the form $\cos\theta + i\sin\theta$ we must have $\theta = \frac{\pi}{4}$, so

$$z_1 =$$

$$\text{and} \qquad z_1 z_2 =$$

$$=$$

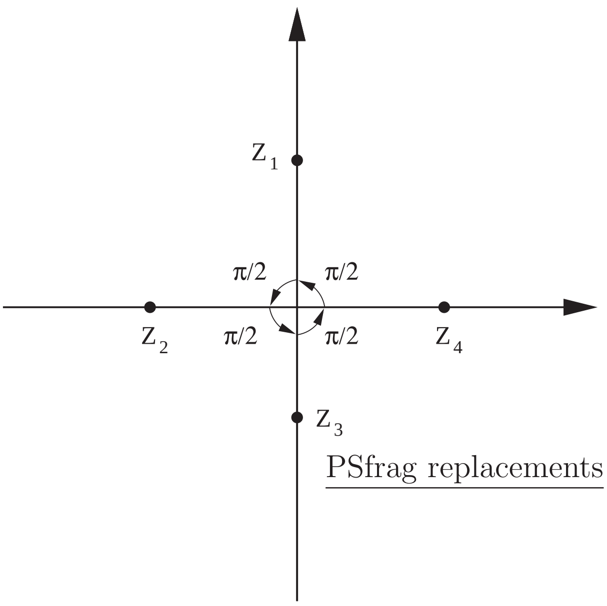
Example 2: Find the possible values of $e^{\frac{i\pi k}{2}}$, where $k \in \mathbb{Z}$.

$k = 1$:

$k = 2$:

$k = 3$:

$k = 4$:



2.11 Multiplying Complex Numbers in Exponential Form

If $z_1 = r_1 e^{i\theta_1}$ and $z_2 = r_2 e^{i\theta_2}$ then,

$$z_1 z_2 =$$

since we multiply the moduli and add the arguments.

These results can be used to prove trigonometric identities. For example,

$$e^{i\theta} e^{i\phi} =$$

$$\Rightarrow$$

$$\Rightarrow$$

Equating real and imaginary parts, we get

$$\cos(\theta + \phi) =$$

$$\sin(\theta + \phi) =$$

2.12 Dividing complex numbers in exponential form

$$\frac{z_1}{z_2} =$$

$$=$$

$$=$$

Thus we divide the moduli and subtract the arguments.

Taking $z_1 = 1$ and $z_2 = z$ gives

$$\frac{1}{z} =$$

$$=$$

so that

$$\left|\frac{1}{z}\right| = \frac{1}{|z|},$$

$$\arg\left(\frac{1}{z}\right) = -\arg z.$$

2.13 Calculating Powers

If we write a complex number in exponential form, $z = re^{i\theta}$, and take the n-th power we get

$$z^n =$$

Note: this result remains true for non-integer values of n, although some care is needed (see details later in section on solving complex equations). For example,

$$\sqrt{z} = \pm\sqrt{r}e^{i\frac{\theta}{2}}.$$

2.14 de Moivre's Theorem

If we take the special case $z = e^{i\theta}$, then $z^n = e^{in\theta}$, and

$$\left(e^{i\theta}\right)^n =$$

so that

This result is known as de Moivre's theorem.

2.15 Applications of de Moivre's Theorem

Example 1:

Express $\cos 3\theta$ and $\sin 3\theta$ in powers of $\sin\theta$ and $\cos\theta$.

$$\cos 3\theta + i\sin 3\theta =$$

$$=$$

Equating real and imaginary parts, we find

$$\cos 3\theta =$$

$$\sin 3\theta =$$

We can use these results to find $\tan 3\theta$:

$$\tan 3\theta =$$

$$=$$

Now divide top and bottom by $\cos^3\theta$

$$\tan 3\theta =$$

Similar calculations can be carried out for any other power.

Example 2:

$$\cos 6\theta + i \sin 6\theta =$$

$$=$$

$$+$$

Equating real and imaginary parts, one obtains

$$\cos 6\theta =$$

$$\sin 6\theta =$$

To go in the other direction and write $\sin^n\theta$ and $\cos^n\theta$ in terms of sines and cosines of multiple angles, we write

$$z = \quad , \qquad z^{-1} = \quad , \qquad z^n = \quad , \qquad z^{-n} = \quad ,$$

so that

$$\cos\theta =$$

$$\cos n\theta =$$

$$\sin\theta =$$

$$\sin n\theta =$$

Example 1:

$$\cos^3\theta =$$

$$=$$

$$=$$

$$=$$

$$\cos^3\theta =$$

Example 2:

$$\cos^6\theta =$$

$$=$$

$$=$$

$$=$$

2.16 Solving Equations

Example 1: Determine the nth roots of unity; that is, solve the equation

$$z^n =$$

We write the RHS as a complex number in polar form using the **general** value of the argument:

$$z^n =$$

$$\Rightarrow \quad z =$$

$$\Rightarrow \quad z =$$

We write the possible values of z as z_k for $k = 1,\ 2,\ \ldots.$

$k = 0:$

$k = 1:$

$k = 2:$

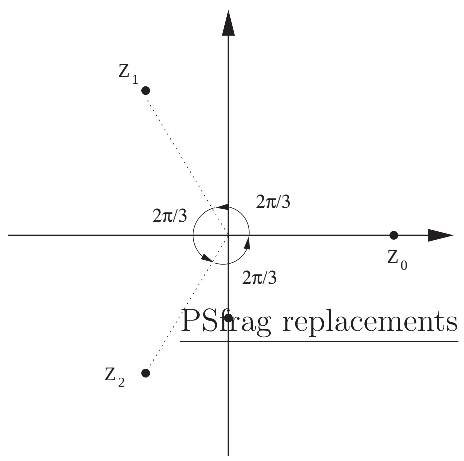
$\vdots$

$k = n - 1:$

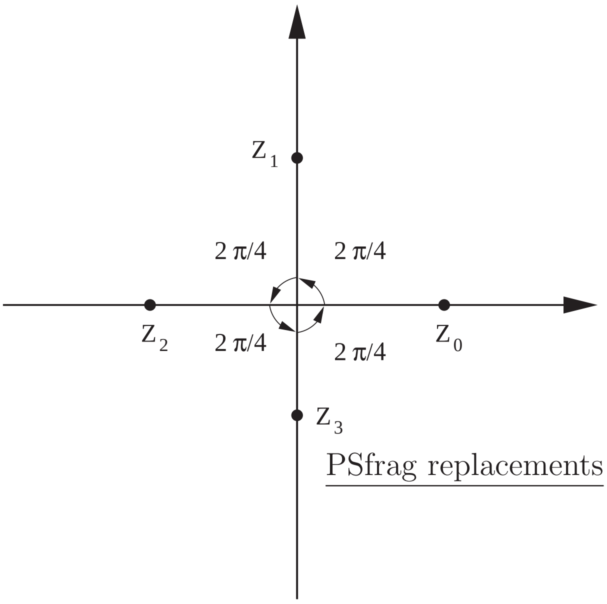
$k = n:$

$k = n + 1:$

If $n = 3$,



If $n = 4$,



There are 3 distinct roots for $z^3 = 1$, 4 distinct roots for $z^4 = 1$ and in general,

$$z^n = 1 \quad \text{has}$$

Note: A general complex polynomial of degree n has n complex roots (counting the roots according to their multiplicity). This result is known as the *Fundamental theorem of Algebra.*

Example 2:

$$z^4 =$$

$$=$$

$$=$$

$$=$$

$$z^4 =$$

Hence: $\quad z =$

This has four distinct roots:

$$z_0 =$$

$$z_1 =$$

$$z_2 =$$

$$z_3 =$$

3 Differential Equations

A differential equation is a functional relationship between a function and its derivatives.

If the function is a function of one variable, then the differential equations involving this function are called Ordinary Differential Equations (ODE's).

If the function is a function of two or more variables, then the differential equations involving this function are called Partial Differential Equations (PDE's).

3.1 Types of Ordinary Differential Equation

The order of a differential equation is the order of the highest derivative in the equation *eg.*

$$\frac{dy}{dx} = \cos x$$

$$\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^3 + y = 0$$

$$\frac{d^3y}{dx^3} + y\frac{dy}{dx} - 4xy = 0$$

An equation is **linear** if y and its derivatives occur as linear terms. The following are examples of linear ODE's:

$$\frac{d^2y}{dx^2} + y\sin x = e^x$$

$$\ln x\frac{d^2y}{dx^2} + e^x\frac{dy}{dx} = x^5\cos x$$

An equation which is not linear is called **non-linear**. The following are examples of non-linear equations

$$\frac{d^2y}{dx^2} + y\frac{dy}{dx} = x^2$$

$$\frac{dy}{dx} + \sin y = 0$$

3.2 Simple Differential Equations

A very simple ODE is given by

$$\frac{dy}{dx} =$$

which may be integrated to give the solution

$$y =$$

Note that we do not get a unique solution. We get a family of solutions depending on c (a 1-parameter family of solutions).

We can obtain an unique solution by specifying a value taken by y, say y_0, for some particular value of x, say x_0, so that $y(x_0) = y_0$. Using this we obtain

$$y_0 =$$

hence

$$y = \int f(x)\ dx =$$

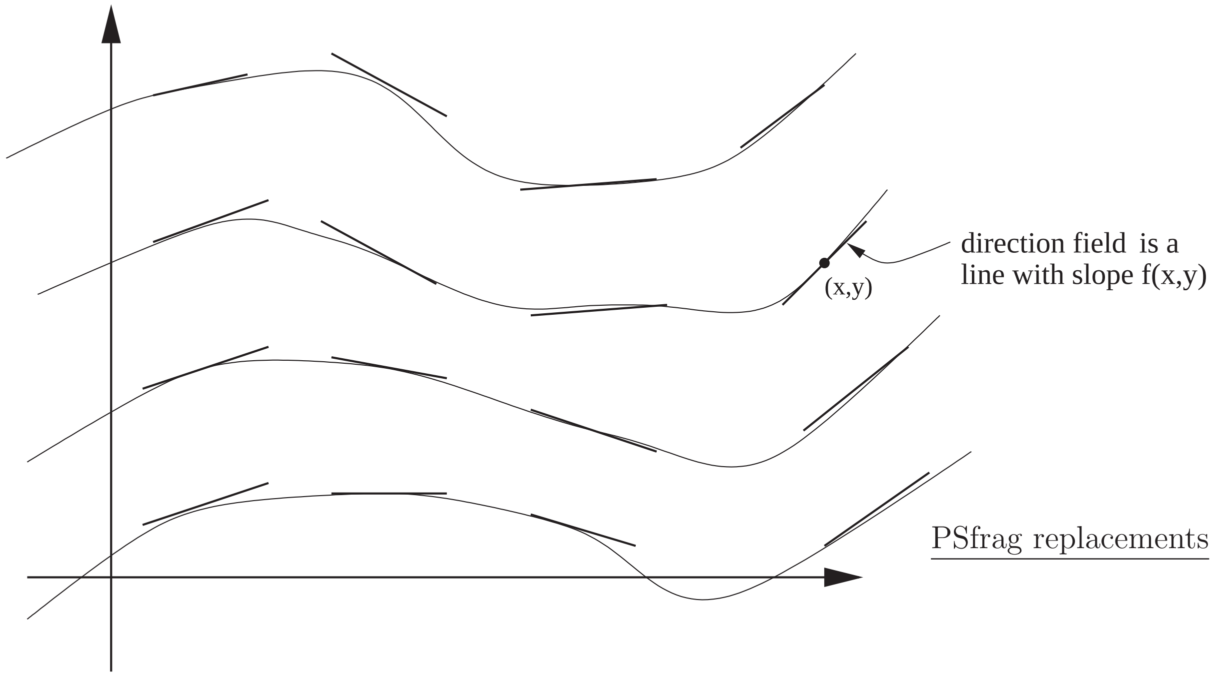
This amounts to having solved the **initial value problem** governed by the two equations

and

A more general differential equation is given by

$$\frac{dy}{dx} = f(x, y)$$

This ODE is one which states that at the point (x, y), the graph of the solution y as a function of x has slope $f(x, y)$.



Thus, in the x, y-plane the tangent to a solution curve through the point (x, y) has slope $f(x, y)$. We can plot short line segments of slope $f(x, y)$ called **direction fields** at every point (x, y) in the x, y-plane. The solution curves are therefore curves tangent to the direction field. Again we can get a unique solution satisfying $y(x_0) = y_0$ by selecting the unique curve through the point (x_0, y_0) that follows the direction field.

There is a systematic method for drawing direction fields, and hence solution curves, by using **isoclines**. Isoclines are curves on which the slope dy/dx is constant, *ie.* curves for which $f(x, y)$ is constant.

Example 1

$$\frac{dy}{dx} = y - x$$

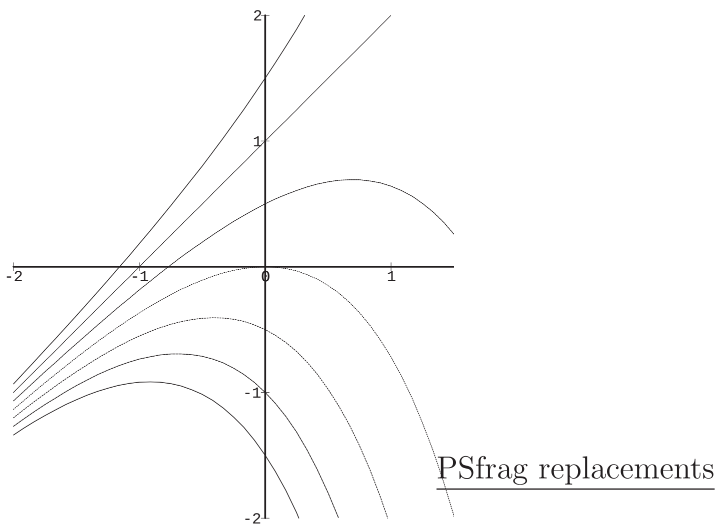
The isoclines are given by

$$\frac{dy}{dx} = f(x, y) = \qquad\qquad =$$

so that

$$y$$

For various fixed values of c, we can now draw the lines $y = x + c$ (the isoclines) and mark on each isocline a short line of slope c (the direction field). Finally we draw curves which have the direction field as tangents (the solution curves).



Example 2

$$\frac{dy}{dx} = x^2 + y^2$$

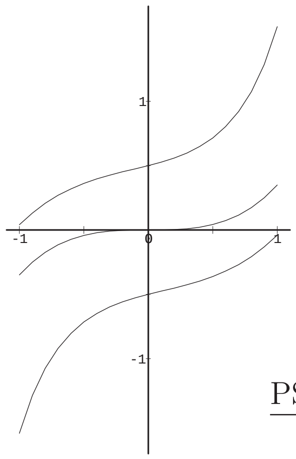
Hence

$$f(x, y) =$$

and the isoclines are given by

$$=$$

Again we draw the isoclines, the direction field and finally the solution curves.



PSfrag replacements

In both of these examples, we see that the differential equation gives a 1-parameter family of curves. Conversely, if we are given a 1-parameter family of curves, we can find a first order ODE with that family as its solution.

Example 1

The following equation describes a family of parabolas

$$y = 4ax^2 \quad \text{where } a \text{ is a free (constant) parameter} \tag{2}$$

Differentiating (2) gives

$$\frac{dy}{dx} = \tag{3}$$

However (3) is not the equation that we want because it contains the arbitrary constant a. We must eliminate a from (2) and (3). Taking their ratio gives:

$$=$$

so that

$$=$$

which is the required equation containing no arbitrary constants.

Example 2

Suppose that we have a family of circles:

$$(x-a)^2 + (y-a)^2 = 2a^2$$

$$\Rightarrow \qquad (4)$$

Differentiating (4) with respect to x, we have

so that

$$(5)$$

Again this is not the required equation as it contains the constant a. However from (4)

$$2a =$$

Substituting into (5) gives

$$= 0$$

and on rearranging we get

$$\frac{dy}{dx} =$$

We can see from these examples that in general a 1-parameter family of curves gives rise to a first order ODE.

3.3 Separation of Variables

Consider a differential equation of the form

$$\frac{dy}{dx} =$$

We can formally separate variable by writing

$$=$$

This has separated the equation into an expression involving x and one involving y, and each side may be integrated to give:

$$\int g(y)dy =$$

Example 1

$$\frac{dy}{dx} = \frac{2y}{x}$$

$$\Rightarrow \quad =$$

$$\Rightarrow \quad =$$

$$\Rightarrow \quad =$$

$$\Rightarrow \quad =$$

$$\Rightarrow \quad =$$

$$\Rightarrow \quad y =$$

Example 2

Solve

$$\frac{dy}{dx} = -\frac{1+y^2}{1+x^2} \quad \text{with} \quad y(0) = -1$$

$$\Rightarrow$$

$$\Rightarrow$$

$$\Rightarrow \quad y =$$

Using the condition $y(0) = -1$ we have

$$\Rightarrow$$

$$\Rightarrow \quad y =$$

Then using

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} \quad \text{with} \quad A = \arctan x, \quad B = \arctan 1$$

gives

$$y =$$

$$\Rightarrow \quad y =$$

3.4 Homogeneous Equations

An ODE is said to be **homogeneous** in x and y if it can be written in the form:

$$\frac{dy}{dx} =$$

We can solve this by introducing a new dependent variable v given by:

$$v =$$

So, $y = vx$ and differentiating this product with respect to x gives:

$$\frac{dy}{dx} =$$

Hence the differential equation becomes

$$\Rightarrow \quad \frac{dv}{dx} =$$

$$\Rightarrow$$

Example 1

$$\frac{dy}{dx} + \frac{x^2 + y^2}{xy} = 0$$

$$\Rightarrow$$

$$\Rightarrow$$

Now let $\quad v = \dfrac{y}{x} \quad$ so that $\quad y = xv \quad$ and $\quad \dfrac{dy}{dx} =$

Substituting in the above ODE gives

$$x\frac{dv}{dx} + v =$$

$$\Rightarrow \quad x\frac{dv}{dx} =$$

$$=$$

$$\Rightarrow \qquad =$$

$$\Rightarrow \qquad =$$

$$\Rightarrow \qquad =$$

$$=$$

$$\Rightarrow \qquad =$$

$$\Rightarrow \qquad =$$

$$\Rightarrow \qquad =$$

$$\Rightarrow \quad y =$$

Example 2

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

$$=$$

Now let $v = \dfrac{y}{x}$ so that $y = xv$

Substituting in the above ODE then gives

$$x\frac{dv}{dx} =$$

$$=$$

$$\Rightarrow$$

$$\Rightarrow$$

$$\Rightarrow$$

$$\Rightarrow$$

$$=$$

$$\Rightarrow$$

$$\Rightarrow$$

$$\Rightarrow \quad y =$$

3.5 First Order Linear Differential Equations

A first order linear differential equation is one that can be written in the form

$$A(x)\frac{dy}{dx} + B(x)y = C(x)$$

By restricting x if necessary, we may consider the case where $A(x) \neq 0$. Then we may divide through by $A(x)$ to obtain

which has the form,

We solve this equation by multiplying both sides by what is called an **integrating factor** $I(x)$

where $I(x)$ must be chosen so that the LHS takes the form of a total derivative, that is

$$I(x)\frac{dy}{dx} + I(x)p(x)y = \tag{6}$$

The ODE then takes the form

which, in principle, we can integrate to give

so that

$$y =$$

The only problem is to determine the integrating factor $I(x)$. However, from (6) we have

$$\Rightarrow$$

$$\Rightarrow$$

$$\Rightarrow$$

$$\Rightarrow$$

$$\Rightarrow \quad I(x) =$$

This is the required formula for the integrating factor.

Note: including a constant of integration c in the exponent would have no effect as it would simply multiply $I(x)$ by a constant e^c.

The general method for solving first order linear ODE's

We now summarise the method for finding an integrating factor and then solving first order linear differential equations.

1. Put the linear equation in standard form:

2. Calculate the integrating factor $I(x)$

3. Multiply standard form by the integrating factor $I(x)$ to get

4. Check that this is the same as

5. Integrate to get

6. Divide by the integrating factor $I(x)$ to get

$$y =$$

This is the general solution.

Example 1

$$x\frac{dy}{dx} + 2y = 8x^2$$

1. Put into standard form

$$=$$

2. Calculate integrating factor

$$p(x) =$$

$$I(x) =$$

$$=$$

$$=$$

$$=$$

3. Multiply standard form by the integrating factor

4. Check that LHS is a derivative

5. Integrate

6. Divide by the integrating factor

Example 2

$$x\frac{dy}{dx} + (3 - x)y + 2 = 0$$

1. Put into standard form

2. Calculate integrating factor

$$\begin{aligned} I(x) &= \\ &= \\ &= \\ &= \end{aligned}$$

3. Multiply standard form by the integrating factor

4. Check LHS is a derivative

$$\begin{aligned} & \frac{d}{dx}(x^3e^{-x}y) \\ \Rightarrow \quad & \frac{d}{dx}(x^3e^{-x}y) \end{aligned}$$

5. Integrate

integrating by parts,

$$=$$

integrating by parts again,

$$=$$

$$=$$

6. Divide by the integrating factor

$$y =$$

3.6 Exact Equations

Consider the differential equation

$$P(x,y) + Q(x,y)\frac{dy}{dx} = 0 \tag{7}$$

and **suppose** that a function $F(x,y)$ can be found such that

$$\frac{\partial F}{\partial x} = P \quad \text{and} \quad \frac{\partial F}{\partial y} = Q \tag{8}$$

then the differential equation (7) can be written as

$$\Rightarrow$$

$$\Rightarrow \tag{9}$$

If a function F satisfying (8) exists then we say the equation is **exact** and the solution is given by equation (9). If we look at equation (8) and differentiate P with respect to y and Q with respect to x we get

Hence if the differential equation is exact we must have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \tag{10}$$

Conversely one can show that if equation (10) holds then it is possible to find an $F(x, y)$ such that equation (8) is satisfied.

Thus the differential equation (7) is exact if

Note that in many of the books on ODE's equation (7) is written formally as

$$P(x, y)\ dx + Q(x, y)\ dy = 0$$

and this equation is defined to be exact if (10) is true.

Example 1

Show that $e^y dx + (xe^y + 2y)dy = 0$ is exact and find the solution.

$$P = \qquad \text{and} \quad Q =$$

Hence

So the equation is exact. We therefore try and find an F such that

$$\frac{\partial F}{\partial x} = \tag{11}$$

$$\frac{\partial F}{\partial y} = \tag{12}$$

Integrating (11) with respect to x treating y as a constant gives

$$F(x, y) = \tag{13}$$

To determine $f(y)$ we differentiate (13) partially with respect to y

$$\frac{\partial F}{\partial y} = \tag{14}$$

Comparing (14) with (12) we see that

$$f'(y) =$$

$$\Rightarrow \quad f(y) =$$

Substituting for $f(y)$ in (13) we get

$$F(x, y) = \tag{15}$$

and the solution to the differential equation is given by

$$=$$

Example 2

Solve the equation

$$3x(xy - 2) + (x^3 + 3y^2)\frac{dy}{dx} = 0$$

$$P = \qquad \text{and} \quad Q =$$

Hence

So the equation is exact. We therefore try and find an F such that

$$\frac{\partial F}{\partial x} = P = \tag{16}$$

$$\frac{\partial F}{\partial y} = Q = \tag{17}$$

Integrating (16) with respect to x while holding y fixed gives

$$F(x, y) = \tag{18}$$

To determine $f(y)$ we differentiate (18) partially with respect to y

$$\frac{\partial F}{\partial y} = \tag{19}$$

Comparing (19) with (17) we see that

$$f'(y) =$$

$$\Rightarrow \quad f(y) =$$

Substituting for $f(y)$ in (18) we get

$$F(x, y) =$$

and the solution to the differential equation is given by

$$=$$

4 Linear ODEs of the Second Order

4.1 Introduction

In this section we shall consider a set of differential equations which occur frequently in situations such as simple harmonic motion, oscillations of a damped spring and linear forcing, Situations such as these can be modeled by

where a, b, c are constants y is the unknown function, and f is a given forcing function. The value of b often represents the damping.

The equation is said to be "second order" because the highest derivative is a two (*nth* order equations have the *nth* derivative as the highest).

The equation is linear, as we shall see that if two functions y_1, y_2 are a solution of the equation then so is any linear combination $\alpha y_1 + \beta y_2$.

The equation is called an ordinary differential equation (ODE) since it contains no partial derivatives, only total ones with respect to one variable, here x.

The equation is said to have constant coefficients since a, b, c are constants.

For that reason such equations are generically called **second order linear ordinary differential equations with constant coefficients**.

The relationship between a, b, c and f can alter the type of solution we obtain and so for simplicity here we only treat the homogeneous, unforced case, i.e., where f=0.

4.2 Homogeneous case: $f = 0$

First we take a homogeneous equation (with $f\,(x) = 0$) and divide through by a (this is allowed as a is non-zero by definition—otherwise the equation would not be second order). This gives

$$\frac{d^2y}{dx^2} + \frac{b}{a}\frac{dy}{dx} + \frac{c}{a}y = 0,$$

or, equivalently,

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0,$$

where $p = b/a$ and $q = c/a$.

We try a solution of the form

This gives

Assuming $y = A\exp(kx)$ is a solution of the homogeneous equation we substitute it into the equation and obtain

The term $\exp(kx)$ can be divided through as it's always non-zero for finite x. This leaves what is called the **auxilliary** equation:

This is a quadratic equation in k and will have two roots k_1, k_2:

We thus have two possible solutions to the original equation, $y = A\exp(k_1x)$ and $y = A\exp(k_2x)$. It is a simple exercise of substitution to show that a linear combination of these two solutions is also a solution:

where A and B are ***any*** two constants. This is, in fact, the most **general** solution possible. In general, a second order linear equation will produce two arbitrary constants. They can only be determined by extra information, called **boundary** or **initial conditions**, as we shall see below.

From the formula for the solution of the auxiliary equation we can see that the nature of the solution of the equation will depend on whether $p^2 > 4q$, $p^2 = 4q$, or $p^2 < 4q$.

4.2.1 Case $p^2 > 4q$: real and distinct roots

When $p^2 > 4q$ the auxiliary equation gives two unequal real solutions and hence the solution of the ODE can be written as above:

Example :

4.2.2 Case $p^2 = 4q$: equal real roots

When $p^2 = 4q$, the roots of the auxiliary equation are real and equal. Let their common value be called k. It is given by

In this particular case, it is not possible to form a general solution with two arbitrary constants, since we could rewrite it in the form

and there would be, effectively, only one arbitrary constant. Thus, we do not have the most general form (which needs two *independent*, but arbitrary, constants).

Let us try a second, independent solution of the form

This gives

and, substituting in the ODE (and dividing through by the common factor e^{kx}),

which holds since both $k^2 + pk + q$ and $2k + p$ are zero if k is a assumed a single root of the auxiliary equation.

Hence, two independent solutions are $y = e^{px/2}$ and $y = xe^{px/2}$, and the general solution is

$$y = (A + Bx)e^{px/2}.$$

Example :

4.2.3 Case $p^2 < 4q$: complex conjugate roots

When $p^2 < 4q$ the auxiliary equation has no real roots. However the complex roots appear as a conjugate pair. The roots will be of the form

Hence the general solution will be

Note that this looks like a complex solution (of our real equation), but this is not necessarily the case. Recall the following relations:

Thus we can write the above general solution as

If we choose A and B to be complex numbers such that $A + B = C$, where C is real, and also so that $i(A - B) = D$ is also real then we obtain the **real** solution

This, in fact, is the *general* real solution, since C and D are arbitrary real numbers. [Note that for each choice of C and D there correspond different $A = \frac{1}{2}(C-iD)$ and $B = \frac{1}{2}(C+iD)$.]

Example :

4.3 Boundary and Initial Conditions

As we have seen above the general solution to a second order linear ODE contains two arbitrary constants, A and B. To obtain a specific solution we must provide two independent bits of information to establish precise values for A and B. These are called boundary conditions, or initial conditions.

Boundary conditions specify the solution y at two *different* positions of the variable x.

Initial conditions specify the solution y and its derivative dy/dx at x=0.

In the homogeneous case the BCs/ICs can be applied immediately once the solution has been found.

Example :

Solve the equation

1st step: look at the auxilliary equation

2nd step: Find two independent solutions

3rd step: Find the general solution

4th step: Use the initial conditions to find A and B

5 Multiple Integration

5.1 Introduction

We have already seen the integral of a function of one variable $y = f(x)$ over an interval:

The geometrical meaning of this symbol is the area of the plane region bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$:

Consider now the meaning of a *double* integral of a function of two variables $f(x, y)$ over a domain D in the xy-plane.

By analogy this is the volume of a three dimensional region S, bounded by the surface $z = f(x, y)$, the xy-plane and the (irregular) cylindrical sides parallel to the z-axis, passing through the boundary of D.

The region D is called the domain of integration. We will represent the volume integral with the symbol

5.2 Properties of the double integral

1. The area of D is given by

At first sight this looks like the volume of an irregular cylinder with base D and height 1, which indeed it is. But we also know the general formula for the volume of a cylinder is:

$$\text{Volume} = \text{area of D} \times \text{height of cylinder}$$

Since the height of the cylinder is 1, the numerical value of the volume (beware units!) is the same as the area of D. Therefore, we can use double integrals to measure the are of regions of the plane.

2. The integral is *linear*:

3. Inequalities are preserved:

4. The triangle inequality holds:

5. Additivity of domains: If $D_1, D_2, D_3, \ldots, D_k$ are nonoverlapping domains in the xy plane on each of which $f(x, y)$ is integrable, then $f(x, y)$ is integrable over the union of these domains: $D = D_1 \cup D_2 \cup D_3 \cup \ldots \cup D_k$.

5.3 Examples of simple integrals

Often multiple integrals can be evaluated by geometrical considerations. The following examples stress this. The technical details of how to evaluate multiple integrals in general cases will be studied later.

Example 1:

$I = \iint\limits_R 3dA$, where R: $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant b$.

Example 2:

$I = \iint\limits_D (\sin x + y^3 + 3)dA$, where D is $x^2 + y^2 \leqslant 1$.

Example 3:

$I = \iint_D \sqrt{1 - x^2 - y^2} dA$, where D is $x^2 + y^2 \leqslant 1$.

5.4 Evaluation of double integrals

5.4.1 Simple and regular domains

The evaluation of an integral is easiest when the domain of integration is said to be *simple.*

y**-simple domain**: bounded by two vertical lines$x = a,\ x = b$ and two continuous graphs $y = c\,(x)\,,\ y = d\,(x)$

x**-simple domain**: bounded by two horizontal lines $y = c,\ y = d$ and two continuous graphs $x = a\,(y)\,,\ x = b\,(y)$

A domain that is not simple but which can be split up into x-simple and y-simple parts is called *regular.* An integral over a regular domain is evaluated as the sum of integrals over each constituent simple domain (section 5.2 property 5.).

Example :

Regular domain into simple domains.

5.4.2 Evaluation of an integral over a simple domain

Suppose that D is y-simple and is bounded by $x = a,\ x = b,\ y = c(x),\ y = d(x)$. The integral

is the volume of the vertical cylinder of base D and bounded at the top by $z = f(x, y)$. Consider a very thin slice of the solid perpendicular to the x-axis at a point x. The value of x does not vary across the slice and y is bounded between

The area of the slice is a one-dimensional integral of the form:

We can think of the cylinder as if it were like loaf of bread formed from (infinitely) many such slices. The volume of the cylinder is then obtained by summing up the contribution of all such thin slices of area $\alpha(x)$ and thickness dx between $x = a$ and $x = b$. Therefore it is given by:

If D is x-simple, we can repeat the same procedure with a slice perpendicular to the y-axis. Therefore we have the following fact:

Fact: If $f(x, y)$ is continuous on the bounded y-simple domain D given by $a \leq x \leq b$ and $c(x) \leq y \leq d(x)$, then

Similarly, if $f(x, y)$ is continuous on the bounded x-simple domain D given by $c \leq y \leq d$ and $a(y) \leq x \leq b(y)$ then

5.4.3 Examples of integral evaluation

These examples stress the importance of drawing the domain of integration in the xy-plane. It gives geometrical insight that can lead to considerable simplification.

Example 1:

Find the volume of the solid lying above the square defined by $0 \leq x \leq 1$, $1 \leq y \leq 2$ and below the plane $z = 4 - x - y$.

The square is both x- and y-simple so the double integral can be evaluated either way: You will get the same answer!

Note: The limits on the inner integral do not depend here on the integration variable of the outer integral. This is because in this example the integration domain D is rectangular (a square). If it were not a rectangle, the inner limits *would* depend on the outer integration variable.

Example 2:

Evaluate $I = \iint\limits_T xydA$, over the triangle T with vertices $(0,0)$, $(1,0)$ and $(1,1)$.

The triangle is both x- and y-simple so again the double integral can be evaluated either way.

Example 3:

Evaluate $I = \iint\limits_{D} e^{y^3}dxdy$, where D is This integral can't be evaluated as it is since we don't know the integral of $\exp\left(y^3\right)$ (if you don't believe this, just try to find it in a table of integrals). We can try to swap the order of integration. The domain of integration D is also y-simple.

5.5 Polar coordinates

The polar coordinate system is an alternative to the rectangular (Cartesian) coordinate system for describing the location of points in a plane. We study it here, since the evaluation of many double and treble integrals can be simplified by changing coordinate systems.

In polar coordinates there is an origin called the *pole* and denoted by O, and a *polar axis* which is a half line extending from O.

The position of a point P in the plane is determined by its polar coordinates (r, θ) where

- r is the distance from O to P
- $\theta$is the angle that the vector OP makes with the polar axis, counterclockwise angles being considered as positive.

NB: polar coordinates are not unique: (r, θ) and $(r, \theta + 2n\pi)$ are the same point when n is an integer.

The conversion formulae that relate Cartesian and polar coordinates are as follows:

An equation in x and y represents a curve in the plane with respect to a Cartesian coordinate system. Similarly, an equation in r and θ generally represents a curve with respect to a polar coordinate system. Some curves have a simpler expression in one system than in the other.

Example 1:

Example 2:

What is shape of $r = 2a\cos\theta$?

This is an example of a case where the transformation to Cartesian coordinates has helped.

Example 3:

Sketch the polar curve $r = a\,(1 - \cos\theta)$ with $a > 0$.

5.6 Double integrals in polar coordinates

As hinted above, some integrals are much simpler to evaluate if they are expressed in polar rather than cartesian coordinates.

Consider the problem of finding the volume V of the solid region lying above the xy-plane and beneath the paraboloid $z = 1 - x^2 - y^2$.

Since the paraboloid intersects the xy-plane in the circle $x^2 + y^2 = 1$, the volume is given in cartesian coordinates by

The integral is a mess. It is much neater to convert to polar coordinates.

The domain of integration becomes:

The integrand becomes

So that

The area element dA is just $dx\,dy$ in Cartesians. In polars it is NOT just $dr\,d\theta$ as can be seen from the following diagrams:

Therefore we have:

In general, whenever you change coordinates in a multidimensional integral you have to be very careful about what dA becomes. (We study a systematic way of working this out in the next section.)

In polars the original integral therefore converts to

Thus, because of the form of the integrand and the shape of the domain of integration, we have achieved a dramatic simplification. Indeed, as the domain of integration in the (r, θ) plane in this example is now a rectangle the order of the integration is irrelevant (but in general may still be important).

Summary: To change to polar coordinates:

1. Express the integrand as a function of r and θ**.**

2. Express the domain of integration in polar coordinates

3. Write the area element as $dA = rdrd\theta$

4. Evaluate the integral

Example 1:

Evaluate the area of the region of the xy-plane enclosed by the curve

$$x^2 + y^2 = 2a\sqrt{x^2 + y^2} + 2ax$$

.

Example 2:

Evaluate the integral

5.7 Changes of variables in double integrals

The transformation of double integrals to polar coordinates is just a special case of a general change of variables formula in double integrals. A transformation of coordinates in a double integral is a set of two functions

that give a relationship between the old variables (x, y) and the new ones (u, v). These functions are "invertible" in the sense that we can describe the domain of integration D in the plane, by either coordinate systems. In the case of polar coordinates we can describe a disk of radius a as

It is possible to prove (and we don't here, and you are not expected to know how to) that if the functions $x(u,v)$, $y(u,v)$ are invertible, then the determinant, called the **Jacobian**

is non-zero.

Moreover, it is possible to prove that the area element in the (u,v) coordinates is given by

We can verify this result in the case of polar coordinates:

The recipe for changing variables in double integral is therefore (cf. polars):

1. Express the integrand as a function of u and v.

2. Write the boundaries of the domain of integration in terms of u and v.

3. Write the area element as $dA = |J(u,v)|\, du dv$.

Example 1:

Use an appropriate change of variables to find the area of an elliptic disk with shape given by $x^2/a^2 + y^2/b^2 \leqslant 1$.

5.8 Triple integrals

Once it is clear how to extend definite integration to two-dimensional domains, the extension to three (or more) dimensions is relatively straightforward.

If $f(x, y, z)$ is a function defined on a domain D in three-dimensional space, then we can calculate the *triple integral*

where dV denotes a small volume element. In Cartesians this takes the form $dV = dx\, dy\, dz$.

This integral can either be interpreted as a hyper-volume (a volume in four dimensions - cf. double integral case) or, more intuitively for example, as the mass of an object D, whose density is $f(x, y, z)$.

The procedure for evaluating the integrals in 3-D is effectively the same as that studied for 2-D: we integrate the three variables in succession.

Example 1:

Let B be a rectangular box $0 \leq x \leq a, \quad 0 \leq y \leq b, \quad 0 \leq z \leq c$. Evaluate the integral $I = \iint\limits_{B}\int (xy^2 + z^3)dV$

Example 2:

If T is the tetrahedron with vertices $(0,0,0)$, $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$, evaluate $\int\iint ydV$.

6 The Gradient, Divergence, and Curl

6.1 Directional Derivatives and the gradient of a scalar function

Given a curve $y = f(x)$ we can work out the gradient (slope) of the curve at each point by differentiating f:

$$\text{Slope} = \frac{df}{dx}$$

It is unique: for a smooth function with no kinks there is only one slope at each point.

Now, suppose you are standing on a hill. What is the slope of the hill at that point? A little thought should tell you that it is *not* unique: It depends on the direction in which you are looking. For example, you could walk around the hill, on a level contour and keeping the same height. In that case the slope is zero in the direction you walk at each point. You could also aim for the summit, in which case you have to go uphill and the gradient will be far from zero!

We therefore must introduce the idea of a **directional derivative**.

Definition: The directional derivative is the slope of a surface $z = f(x, y)$ in a specified direction $\hat{\boldsymbol{n}}$ and is given by:

$$\frac{\partial f}{\partial \hat{\boldsymbol{n}}} = \nabla f.\boldsymbol{n}$$

where $\hat{\boldsymbol{n}}$ is a *unit* vector in the direction of interest and ∇f is the "**gradient** of f", or "grad f" defined as:

$$\nabla f(x, y) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j}.$$

The symbol "∇", is an upside-down Greek capital delta (like a d for derivative) and so is called "del". Note that the gradient, ∇f, is a **vector**.

Example 1:

Evaluate the gradient of the function $f(x, y) = e^{-\left(x^2+y^2\right)}$.

Example 2:

Evaluate the directional derivative of the above function at the point $(x, y) = (1, 1)$ in the directions (a) $\boldsymbol{i} + \boldsymbol{j}$ and (b) $\boldsymbol{i} - \boldsymbol{j}$

(a) $\boldsymbol{n} = \boldsymbol{i} + \boldsymbol{j}$, hence $|\boldsymbol{n}| =$

So that $\hat{\boldsymbol{n}} =$

At $(1, 1)$, $\nabla f =$

So that $\nabla f . \hat{\boldsymbol{n}} =$

(b) $\boldsymbol{n} = \boldsymbol{i} - \boldsymbol{j}$, hence $|\boldsymbol{n}| =$

So that $\hat{\boldsymbol{n}} =$

So that $\nabla f . \hat{\boldsymbol{n}} =$

Example 3:

Evaluate the directional derivative of the above function at $(x, y) = (0, 0)$ in every direction. Why is it the same in every direction?

Properties of grad

1. **Extension to 3-D** : In 3-D the gradient is defined as

$$\nabla f(x, y, z) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

2. **Leibniz rule** The gradient is a derivative and has the same properties of derivatives. e.g.,

$$\nabla (fg) = f\,\nabla g + g\,\nabla f$$

3. The gradient turns a scalar function f into a vector function!

4. Physical examples of the use of gradients include the force $\boldsymbol{F}(x, y, z)$ generated by a potential energy $V(x, y, z)$, which is given by

$$\boldsymbol{F} = -\nabla V.$$

5. The local direction of steepest ascent/descent at any point on a hill $z = f(x, y)$ is in the direction of ∇f. Why? From the definition of the dot-product, if θ is the angle between $\hat{\boldsymbol{n}}$ and the vector ∇f, the directional derivative is given by

$$\frac{\partial f}{\partial \hat{\boldsymbol{n}}} = \nabla f . \hat{\boldsymbol{n}} = |\nabla f| \cos\theta$$

Hence the directional derivative is a maximum when $\cos\theta = 1$ and hence $\theta = 0$ so that $\hat{\boldsymbol{n}}$ is in the same direction as ∇f. Thus ∇f must be the direction of steepest ascent/descent.

Similarly, the level contours along which the direction derivative is zero are perpendicular to ∇f, since there $\theta = \pi/2$ and so $\cos\theta = 0$. The corollary is that ∇f is perpendicular to the level contours.

Example :

Show that the gradient of the function $f(r)$, where r is the radial distance from the origin $r = \sqrt{x^2 + y^2 + z^2}$, is given by

$$\nabla f(r) = \frac{df}{dr}\hat{\boldsymbol{r}}$$

(see later for applications in polar coordinates).

The gradient derivative of a scalar function $f(x, y)$,

$$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}$$

is an example of a **vector field**. We next introduce two derivative operators that can act on vector fields. These are related to the dot and vector products, and are called respectively the "divergence" and the "curl" of $\boldsymbol{F}$. We deal with these in turn.

6.2 Divergence of a vector function

Definition: The "divergence" of a vector function

$$\boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k}$$

is the scalar given by

$$\begin{aligned} \nabla \cdot \boldsymbol{F}(x,y,z) &= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \cdot (F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}) \\ &= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \end{aligned}$$

Note: the divergence of the vector function $\boldsymbol{F}$, or "div$\boldsymbol{F}$", is a scalar. The derivative operators and components of F here interact like the scalar product of two vectors, hence the dot notation between them.

Physical Motivation:

The physical motivation for calculating the divergence of a field $\boldsymbol{F}$ is associated with the *sources* and *sinks* of the field. A *source* is a region of space from which field lines flow outwards, e.g., electric field in the neighbourhood of a positive charge, or fluids streamlines in the neighbourhood of a water source. A *sink* is the region of space where the field lines converge, e.g., the electric field in the neighbourhood of a negative charge, or the fluid streamlines near to the plughole in your bath (assuming it's not clogged up with hair!).

Many physical theories employ the divergence of a vector function. For example the law of mass conservation (= no mass can spontaneously vanish) for a fluid of density $\rho(x, y, z, t)$ with velocity $\boldsymbol{v}(x, y, z, t)$ at each point can be expressed as:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

If you now call $\rho(x, y, z, t)$ the charge density, this is also the equation that expresses the conservation of charge in electromagnetism.

Given a charge density $\rho(x, y, z, t)$, Maxwell's classical theory of electromagnetism states that the associated electric field $\boldsymbol{E}(x, y, z, t)$ is the given by the solution of the divergence equation:

$$\nabla \cdot \boldsymbol{E} = \rho/\epsilon_0$$

If the divergence of a field is zero, then it implies that there is no source or sink (convergence or divergence of it's field lines). Thus if $\boldsymbol{B}$ is the magnetic field another of Maxwells equations is that

$$\nabla \cdot \boldsymbol{B} = 0 \quad \text{everywhere,}$$

which just says that there are no (known) magnetic monopoles (magnetic equivalent of point charges).

In fluid dynamics, a fluid is said to be incompressible if the velocity $\boldsymbol{v}$ satisfies

$$\nabla \cdot \boldsymbol{v} = 0 \quad \text{everywhere.}$$

Example 1:

If the velocity of a fluid is given by $\boldsymbol{v}(x, y, z) = y\boldsymbol{i} - x\boldsymbol{j} + x\cos y\boldsymbol{k}$, show that it is incompressible.

Example 2:

Can the vector function $\boldsymbol{B}(x, y, z) = xy\boldsymbol{i} - xz\boldsymbol{j} + yz\boldsymbol{k}$ represent a real magnetic field?

6.3 Curl of a vector function

Definition: The curl of a vector field

$$\boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k}$$

is another vector field, given by

$$\begin{aligned}
\nabla \times \boldsymbol{F}(x,y,z) &= \left(\frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}\right) \times (F_1\boldsymbol{i} + F_2\boldsymbol{j} + F_3\boldsymbol{k}) \\
&= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\
&= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\boldsymbol{k}
\end{aligned}$$

Note: the curl of the vector function $\boldsymbol{F}$, or "curl$\boldsymbol{F}$", is a vector. The derivative operators and components of F here interact like the vector product of two vectors, hence the $\times$ notation between them. It is easier to remember how to evaluate a curl of a vector function by using the determinant shorthand, provided that when you expand the determinant you always write the differential operator before the component of $\boldsymbol{F}$ it is meant to be differentiating.

Properties of curl:

(i) If g is a scalar function and F a vector field, then

$$\nabla \times (g\boldsymbol{F}) = g(\nabla \times \boldsymbol{F}) + \nabla g \times \boldsymbol{F}$$

(ii) The curl of the gradient of a function is always zero:

$$\nabla \times (\nabla f) = 0$$

(see problem sheet).

Physical Motivation:

The physical motivation for calculating the curl of a field $\boldsymbol{F}$ is to calculate how much the field is "rotating". The vorticity $\boldsymbol{w}(x,y,z)$ of a fluid moving with speed $\boldsymbol{v}(x,y,z)$ is given by

$$\boldsymbol{w} = \nabla \times \boldsymbol{v}$$

and gives an indication how rotational the flow is at that point. A flow with no rotation is called irrotational and has $\boldsymbol{w} = \nabla \times \boldsymbol{v} = \boldsymbol{0}$ everywhere.

In electromagnetism, if a magnetic field $\boldsymbol{B}(x, y, z, t)$ at a point in free space varies in time t, then this induces an electric field $\boldsymbol{E}(x, y, z, t)$ given by the solution of the equation

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$$

Example :

Calculate the vorticity of the shear fluid flow $\boldsymbol{v} = (y, 0, 0)$

Example :

If the electric field at a point in free space is given by $\boldsymbol{E}\,(x, y, z, t) = \sin\,(z)\,\boldsymbol{j}$, find how the associated magnetic field $\boldsymbol{B}$ is changing with time.

7 Curves and Line Integrals

7.1 Introduction

A particle that moves in space describes a *trajectory*, a line or a curve. There are some natural questions that arise:

How is it described?

What is the length of the trajectory?

What work is done by the forces acting on the particle as it moves?

We answer the last question later. Here we focus on the first two. We first consider curves in 2-D before generalising to 3-D.

7.2 Parametric Curves

If a particle moves, its x and y coordinates are functions of time, $x(t)\,,\ y(t)$. As t changes the position of the particle describes a trajectory in space(-time).

Given the functions f and g, then $x = f(t)\,,\ y = g(t)$ are the parametric equations of the trajectory or curve. The parameter t will lie in some interval $a \leq t \leq b$.

Note that a parametric curve has a direction as t increases. Recall that the vector position of the particle on the trajectory is given by

$$\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j}$$

Example 1:

A straight line.

Example 2:

A circle.

Note that this example demonstrates that there is no unique set of parametric equations for a single curve. We can exploit this property to choose the representation that is most suitable to the problem we wish to solve.

Example 3:

An ellipse.

Example 4:

A triangle.

This example shows that it may sometimes be necessary to have different parameterizations for different portions of a curve (usually those with discontinuities in one of their slopes).

7.3 The length of a curve

Recall that the velocity of a particle with position vector $\boldsymbol{r}(t)$ is $\dot{\boldsymbol{r}}(t) \equiv d\boldsymbol{r}/dt$. Suppose a particle is constrained to move along a curve/trajectory with the vector equation $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j}$. The tangent to the curve at time t, which is by definition is the velocity is then *given by*

The distance traveled along the trajectory in the short time dt is then just the speed multiplied by dt. This is approximately

In the limit as $dt \to 0$ this becomes exact. Summing over all these small time intervals we can obtain an expression for the length of the curve as

The term $\left|\frac{d\boldsymbol{r}}{dt}\right| dt$ is represented by the scalar symbol ds, which is called the *arc length element.* Thus

and the length of the curve is represented by

Note that the limits of the integral are the limits of the range of variation of t.

Note also that if the curve has the representation $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j}$ then ds can be written down as follows:

Example 1:

A straight line.

Example 2:

A circle of radius a.

If we had chosen the following alternative parameterisation the answer remains the same!

Example 3:

The length of $y = f(x)$ where $a \le x \le b$.

This curve can be (most easily) parameterised as:

$$\left.\begin{array}{l} x = t \\ y = f(t) \end{array}\right\}\text{where } a \le t \le b \quad \Leftrightarrow \quad \boldsymbol{r}(t) = t\boldsymbol{i} + f(t)\boldsymbol{j}.$$

We can then follow the rules above to obtain the line element of this curve as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt = \sqrt{1 + \left(\frac{df}{dt}\right)^2} dt.$$

For example, consider the curve $y = x^2/2$, $0 \leqslant x \leqslant 1$. Its parametric form is

$$\left.\begin{array}{l} x = t \\ y = f(t) = t^2/2 \end{array}\right\} \quad 0 \le t \le 1.$$

The line element is then

$$ds = \sqrt{\left(\frac{d(t)}{dt}\right)^2 + \left(\frac{d(t^2/2)}{dt}\right)^2} dt = \sqrt{1+t^2} dt.$$

The length of the curve is then given by (see formula sheet).

$$\begin{aligned} L(C) = \int_C ds \quad &= \int_0^1 \sqrt{1+t^2} dt \quad = \left[\tfrac{t}{2}\sqrt{1+t^2} + \tfrac{1}{2}\log\left|t + \sqrt{1+t^2}\right|\right]_{t=0}^{t=1} \\ &= \tfrac{\sqrt{2}}{2} + \tfrac{1}{2}\log(1+\sqrt{2}) \end{aligned}$$

7.4 Line integral of a function

It is possible to extend the concept of the one dimensional integrals you are used to if we make use of the method to evaluate the length of a curve we have just discussed.

An ordinary 1-D integral of a function $f(x)$ over $a \le x \le b$ can be thought of as the area of a straight fence whose height changes from point to point according to $f(x)$. The fence can be thought of as being made up of a sequence of thin fence panels at position x of width dx and height $f(x)$. The area is then the sum of all these infinitesimal fence panels,

$$A = \int_a^b f(x)\, dx \quad .$$

Suppose the fence is no longer straight, but its base is a curve C in the xy-plane given by the position vector $\boldsymbol{r}(t) = f(t)\boldsymbol{i} + g(t)\boldsymbol{j}$. How do we work out the area of this fence? We can repeat the same argument as above and consider the fence as being made up of a sequence of thin fence panels at position $\boldsymbol{r}(t)$ with width now ds, the arc length element, and with corresponding height $f(\boldsymbol{r}(t))$. The corresponding area is then given by

$$A = \int_C f(\mathbf{r}(t))\, ds \quad = \int_{t=a}^{t=b} f(\mathbf{r}(t)) \left| \frac{d\mathbf{r}}{dt} \right| dt \quad .$$

Note that the height function f is a scalar function with a vector input. It must be evaluated on the curve C itself. Thus the value of the integral will, in general, be different for different curves.

Example 1:

Evaluate the integral of the function $f(x, y) = x^2y^2$ over a circle of radius 2, centred at the origin.

The circle has vector parametric equations:

The arc length element is:

The function evaluated on the curve is:

Therefore the integral is:

Example 2:

The curve has vector parametric equations:

The arc length element is:

The function evaluated on the curve is:

Therefore the integral is:

7.5 Curves in three or more dimensions

The above results can be easily generalised to three (or more) dimensions.

A curve C in 3-D can be described by three functions of a parameter:

$$\left.\begin{array}{l} x = x\,(t) \\ y = y\,(t) \\ z = z\,(t) \end{array}\right\} \quad a \leq t \leq b \quad \Leftrightarrow \quad \boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}$$

The vector tangent to the curve is

$$\frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}$$

The arc length element is then

$$ds = \left|\frac{d\boldsymbol{r}}{dt}\right| dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt.$$

The length of the curve C is then

$$L(C) = \int_C ds = \int_{t=a}^{t=b} \left|\frac{d\boldsymbol{r}}{dt}\right| dt.$$

The line integral of a function $f(x, y, z)$ along C is

$$L(C) = \int_C f ds = \int_{t=a}^{t=b} f(x(t), y(t), z(t)) \left|\frac{d\boldsymbol{r}}{dt}\right| dt.$$

Example :

Find the length of the circular helix $\boldsymbol{r}(t) = A\cos t\boldsymbol{i} + A\sin t\boldsymbol{j} + Bt\boldsymbol{k}$ between the points $(A, 0, 0)$ and $(A, 0, 2\pi B)$.

7.6 Line integrals of vector fields

We now have the ingredients to answer the question we asked at the beginning of the chapter: what is the work done by a force acting on a particle that moves on a given trajectory?

We represent the force by a vector field:

$$\boldsymbol{F}(\boldsymbol{r}) \equiv \boldsymbol{F}(x,y,z) = F_1(x,y,z)\boldsymbol{i} + F_2(x,y,z)\boldsymbol{j} + F_3(x,y,z)\boldsymbol{k}$$

We represent the trajectory by the curve C:

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}, \quad a \leqslant t \leqslant b$$

with tangent vector

$$\frac{d\boldsymbol{r}}{dt} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} + \frac{dz}{dt}\boldsymbol{k}, \quad a \leqslant t \leqslant b.$$

With these definitions, the force experienced by the particle will (in general) differ at each point of the trajectory. Thus the force is not constant and so we can't apply the usual formula (see any [old] A-level physics text) of

$$\text{Work done} = \text{Component of force in direction of motion} \times \text{distance moved}$$

However, we can split the trajectory into very small sections of length $d\boldsymbol{r}$ over which the force is approximately constant.

We can then apply the above formula and obtain:

$$dW = \boldsymbol{F}\cdot d\boldsymbol{r} = \boldsymbol{F}\cdot\left(\frac{d\boldsymbol{r}}{dt}dt\right) = \left(\boldsymbol{F}\cdot\frac{d\boldsymbol{r}}{dt}\right)dt$$

The total work done is just the sum of all these small efforts:

$$W = \int\limits_C \boldsymbol{F}\cdot d\boldsymbol{r} = \int\limits_C \boldsymbol{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}dt = \int\limits_{t=a}^{t=b}\left[\boldsymbol{F}(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}\right]dt$$

Note that work is a scalar (it's an energy), but the integrand is formed from the scalar/dot product of two vectors, the force evaluated along the trajectory and the tangent to the trajectory.

We have introduced this type of integral using the concept of work, but it is generally defined for any vector field $\boldsymbol{F}(\boldsymbol{r})$, independently of its physical meaning. If the curve C is closed, this integral is also called the circulation of $\boldsymbol{F}$ around C.

Example 1:

Evaluate the work done by the force field $\boldsymbol{F}(x, y) = -y\boldsymbol{i} + x\boldsymbol{j}$ along:

(a) the semicircle that joins $(1, 0)$ with $(-1, 0)$ in the upper half plane,
(b) the semicircle that joins $(-1, 0)$ with $(1, 0)$ in the upper half plane,
(c) the straight line segment which joins $(1, 0)$ with $(-1, 0)$.

(a) The trajectory $\boldsymbol{r}(t)$ is:

The tangent vector $\frac{d\boldsymbol{r}}{dt}$ is:

The force field on the trajectory is:

The work done is:

(b) The trajectory $\boldsymbol{r}(t)$ is:

The tangent vector $\frac{d\boldsymbol{r}}{dt}$ is:

The force field on the trajectory is:

The work done is:

Conclusion: The line integral of a vector field depends on the direction of the curve. If the direction is changed, the sign of the line integral is changed as well.

(c) The trajectory $\boldsymbol{r}(t)$ is:

The tangent vector $\frac{d\boldsymbol{r}}{dt}$ is:

The force field on the trajectory is:

The work done is:

Conclusion: In general the value of a line integral of a vector function depends on the path, not just the end-points. The exception to these are the so called "conservative" vector fields. Note that the parameterisation of a given trajectory does *not* affect the answer.

8 Surfaces, surface integrals, flux integrals

8.1 Introduction

Knowledge of surfaces are often required in physical problems. Consider, for example:

- Find the mass of a thin sheet of metal.

- Find the electric field generated by a distribution of charges on a surface.

- Find how much water per unit time exits for the mouth of a pipe (flux).

In this section and later we try to answer the following questions:

- How do we describe a surface mathematically.

- How do we find its area.

- How do we integrate functions of physical quantities across a non-flat surface.

- How do we find the flux of physical quantities across it.

8.2 Parametric representation of surfaces

Intuitively a curve is a one-dimensional object. If a point is known to lie on a given curve it can be located by one coordinate, usually the arc length. For a point lying on a two-dimensional surface, we thus require two coordinates, for example the latitude and longditude.

So a 2-D surface is given by a vector

$$\boldsymbol{r}(u,v) = x(u,v)\boldsymbol{i} + y(u,v)\boldsymbol{j} + z(u,v)\boldsymbol{k}$$

whose components are functions of two scalar variables u and v which lie in some domain D of the u,v plane. The equivalent parametric equations are:

$$\left.\begin{array}{l} x = x\,(u,v) \\ y = y\,(u,v) \\ z = z\,(u,v) \end{array}\right\} \quad (u,v) \in D.$$

Often, we can simplify many problems by being able to recognize the parametric or vector equations of simple surface.

Example 1:

The surface $\boldsymbol{r}(u,v) = a\cos u\boldsymbol{i} + a\sin u\boldsymbol{j} + v\boldsymbol{k}$, $0 \leqslant u \leqslant 2\pi,\ 0 \leqslant v \leqslant 5$, represents the surface of a cylinder of radius a and height 5, symmetric about the z-axis.

Clue: A slice perpendicular to the z-axis, where $v=$ constant, gives the parametric/vector equations of a circle of radius a.

Example 2:

The surface $\boldsymbol{r} = v\cos u\boldsymbol{i} + v\sin u\boldsymbol{j} + v\boldsymbol{k}$, $0 \leqslant u \leqslant 2\pi,\ 0 \leqslant v \leqslant 5$, represents a cone of height 5, symmetric about the z-axis.

Clue: A slice perpendicular to the z-axis, where $v=$ constant, gives the parametric/vector equations of a circle of radius v.

Example 3:

The surface

$$\boldsymbol{r}(,v) = a\cos u \sin v\boldsymbol{i} + a\sin u \sin v\boldsymbol{j} + a\cos v\boldsymbol{k}, \quad 0 \leqslant v \leqslant \pi,\ 0 \leqslant u \leqslant 2\pi,$$

is the surface of a sphere, centred on the origin, of radius a.

Clue: The coordinates $x\,(u,v)^2 + y\,(u,v)^2 + z\,(u,v)^2 = a^2$.

Example 4:

The surface $z = f\,(x,y)$ with $(x,y) \in D$ can be represented vectorially as $\boldsymbol{r}(u,v) = x\boldsymbol{i} + y\boldsymbol{j} + f(x,y)\boldsymbol{k}$.

8.3 Area of a surface

To compute the area of a surface $\boldsymbol{r}(u,v)$, $(u,v) \in D$, we divide D up into small rectangles such that the actual surface also becomes a patchwork of curvilinear subdivisions of area dS the *surface area element* (note the capital S).

The area of the surface is then just the sum of all these infinitesimally small patches

$$\text{Surface Area} = \iint\limits_S dS$$

The problem is to find a way of representing the dS patches in terms of the parameters u and v. Consider a node on the lattice in the uv-plane, with uv-coordinates (u_i, v_j)

• The vertical lines $u = u_i$ (a constant) are transformed by $\boldsymbol{r}(u,v)$ into a curve in 3-D space, $\boldsymbol{r}(u_i, v)$.

• The vector between two of these lines is $\Delta u \left.\frac{\partial \boldsymbol{r}}{\partial u}\right|_{u=u_i}$. (Note the partial derivative as $\boldsymbol{r}$ is a function of 2 variables.)

• The horizontal lines $v = v_j$ (a constant) are transformed by $\boldsymbol{r}(u, v)$ into a curve in 3-D space, $\boldsymbol{r}(u, v_i)$.

• The vector between two of these lines is $\Delta v \left.\frac{\partial \boldsymbol{r}}{\partial v}\right|_{v=v_j}$.

Suppose that the sides of the small rectangle in the uv-plane are du and dv. The area is then $dudv$. This rectangle is transformed by $\boldsymbol{r}(u, v)$ into a parallelogram.

The sides of the parallelogram are vector line elements. They are given by the vector line elements of each of these sets of curves. From the curves section above these are (see diagram)

$$\frac{\partial \boldsymbol{r}}{\partial u} du \quad \text{and} \quad \frac{\partial \boldsymbol{r}}{\partial v} dv.$$

Now from the vectors section we know that if a parallelogram has vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ for its sides the area is $|\boldsymbol{a} \times \boldsymbol{b}|$. Thus the area of the parallelogram (area element) is given by

$$dS = \left| \frac{\partial \boldsymbol{r}}{\partial u} du \times \frac{\partial \boldsymbol{r}}{\partial v} dv \right| = \left| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right| dudv.$$

Hence the area of the surface can be written as a double integral over the domain D in the uv-plane:

$$A = \iint_S dS = \iint_D \left| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right| dudv.$$

Example 1:

Find the surface area of the vertical portion of the cylinder of radius a, height h, centred on the z-axis, with the base in the xy-plane.

(a) Geometry:

(b) Integration:

Example 2:

Find the surface area of a hemisphere of radius a.

(a) Geometry:

(b) Integration:

8.4 Surface integral of a function

Consider a thin sheet of metal given by the position vector $\boldsymbol{r}(u, v)$ but which has variable density $f(\boldsymbol{r})$. What is its mass?

Suppose that the metal sheet can be thought of as being made up of elemental patchwork parallelograms of area dS, with dS so small that the density is approximately constant over the parallelogram. Then the mass of the thin sheet is just the sum of the masses of these patchwork pieces, or

$$M = \iint\limits_S f dS = \iint\limits_S f(\boldsymbol{r}(u, v)) dS = \iint\limits_D f(\boldsymbol{r}(u, v)) \left| \frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v} \right| du dv.$$

Example 1:

Find the mass of the thin-walled cylinder of radius a and height h if the density is given by $f(x, y, z) = z(x^2 + y^2)$.

Example 2:

Find the moment of inertia about the z-axis of the parametric surface

$$x = 2uv, \quad y = u^2 - v^2, \quad z = u^2 + v^2, \quad 0 \leqslant u^2 + v^2 \leqslant 1, \quad v \geqslant 0.$$

8.5 Surface integrals of vector functions: flux integrals

The functions we have integrated over surfaces so far have all been scalars. Just as in the line integrals chapter, it is also possible to evaluate integrals of vector fields over a surface. Such integrals are called *flux integrals*.

Physical examples of flux integrals include:

•The flow of water out of a pipe, where the flux integral is over the velocity field of the water and the surface of the pipe mouth. The result is the total volume of water out of the pipe in unit time.

•A version of Faraday's law, where the flux integral is over the magnetic field density through a surface. The rate of change with time of the value of this integral is the total circulation of the electric field around the line-boundary of the surface.

A flux integral takes the following form:

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$$

where

• S is a surface $\boldsymbol{r}(u, v) \in D$.

• $d\boldsymbol{S}$ is an *oriented* surface element: one which has a magnitude $|d\boldsymbol{S}| = dS$ *and* a direction which is one of the normals to the surface (there are two, since an orientable surface has two sides).

• $\boldsymbol{F}$ is the vector field, evaluated on the surface of integration.

• $\boldsymbol{F} \cdot d\boldsymbol{S}$ is the component of flux normal to the surface.

From §8.3 we already know that

$$\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}$$

is a vector normal to the surface. Thus all we have to do is remove the modulus signs from the definition of the surface element dS §6.4 to get the oriented surface element $d\boldsymbol{S}$:

$$d\boldsymbol{S} = \left(\frac{\partial \boldsymbol{r}}{\partial u} \times \frac{\partial \boldsymbol{r}}{\partial v}\right) dudv \quad \text{or} \quad d\boldsymbol{S} = \left(\frac{\partial \boldsymbol{r}}{\partial v} \times \frac{\partial \boldsymbol{r}}{\partial u}\right) dudv$$

depending on whether you want to evaluate the flux "coming out of" or "going in to" the surface.

Thus in a more explicit form the above flux integral becomes,

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D \boldsymbol{F}(\boldsymbol{r}(u,v))\cdot d\boldsymbol{S} = \iint_D \boldsymbol{F}(\boldsymbol{r}(u,v))\cdot\left(\frac{\partial \boldsymbol{r}}{\partial u}\times\frac{\partial \boldsymbol{r}}{\partial v}\right)dudv$$

As the surface has two normals it is important to check that you pick the correct one as the following example shows. The example also shows the standard way in which you evaluate these integrals.

Example 1:

Find the flux of

$$\boldsymbol{F}(x,y,z) = \frac{2x\boldsymbol{i}}{x^2+y^2} + \frac{2y\boldsymbol{j}}{x^2+y^2} + \boldsymbol{k}$$

downwards through the surface S defined parametrically by

$$\boldsymbol{r}(u,v) = u\cos v\boldsymbol{i} + u\sin v\boldsymbol{j} + u^2\boldsymbol{k}, \quad 0 \leqslant u \leqslant 1, \quad 0 \leqslant v \leqslant 2\pi.$$

It is not necessary to be able to visualise the surface to evaluate a flux integral, but for this introductory example see the diagram.

- First calculate the surface element $d\boldsymbol{S}$:

$$d\boldsymbol{S} = \left(\frac{\partial \boldsymbol{r}}{\partial u}\times\frac{\partial \boldsymbol{r}}{\partial v}\right)dudv \quad \text{or} \quad d\boldsymbol{S} = \left(\frac{\partial \boldsymbol{r}}{\partial v}\times\frac{\partial \boldsymbol{r}}{\partial u}\right)dudv$$

$$\begin{aligned}\frac{\partial \boldsymbol{r}}{\partial u} &= \cos v\boldsymbol{i} + \sin v\boldsymbol{j} + 2u\boldsymbol{k}\\ \frac{\partial \boldsymbol{r}}{\partial v} &= -u\sin v\boldsymbol{i} + u\cos v\boldsymbol{j}\end{aligned}$$

$$\begin{aligned}\frac{\partial \boldsymbol{r}}{\partial u}\times\frac{\partial \boldsymbol{r}}{\partial v} &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \cos v & \sin v & 2u\\ -u\sin v & u\cos v & 0\end{vmatrix}\\ &= u\cos^v\boldsymbol{k} + u\sin^2 v\boldsymbol{k} - 2u^2\sin v\boldsymbol{j} - 2u^2\cos v\boldsymbol{i}\\ &= -2u^2\cos v\boldsymbol{i} - 2u^2\sin v\boldsymbol{j} + u\boldsymbol{k}\end{aligned}$$

Now since $u \geqslant 0$ on the surface (see parameter range above) the $\boldsymbol{k}$-component of this normal is also positive and so is actually pointing *upwards*. This means we have to reverse the orientation of $d\boldsymbol{S}$ to obtain the downwards normal for the downwards flux. This can be achieved by setting

$$\begin{aligned}d\boldsymbol{S} &= \left(\frac{\partial \boldsymbol{r}}{\partial v}\times\frac{\partial \boldsymbol{r}}{\partial u}\right)dudv\\ &= -\left(\frac{\partial \boldsymbol{r}}{\partial u}\times\frac{\partial \boldsymbol{r}}{\partial v}\right)dudv\\ &= (+2u^2\cos v\boldsymbol{i} + 2u^2\sin v\boldsymbol{j} - u\boldsymbol{k})dudv\end{aligned}$$

- Evaluate the field $\boldsymbol{F}$ on the surface:

$$\begin{aligned}\boldsymbol{F}(\boldsymbol{r}(u,v)) &= \frac{2x\boldsymbol{i}}{x^2+y^2} + \frac{2y\boldsymbol{j}}{x^2+y^2} + \boldsymbol{k} \\ &= \frac{2u\cos v\boldsymbol{i}}{u^2\cos^2 v + u^2\sin^2 v} + \frac{2u\sin v\boldsymbol{j}}{u^2\cos^2 v + u^2\sin^2 v} + \boldsymbol{k} \\ &= \frac{2\cos v}{u}\boldsymbol{i} + \frac{2\sin v}{u}\boldsymbol{j} + \boldsymbol{k}\end{aligned}$$

Evaluate the elemental flux $\boldsymbol{F}\cdot d\boldsymbol{S}$ on the surface:

$$\begin{aligned}\boldsymbol{F}\cdot d\boldsymbol{S} &= \left(\frac{2\cos v}{u}\boldsymbol{i} + \frac{2\sin v}{u}\boldsymbol{j} + \boldsymbol{k}\right)\cdot(+2u^2\cos v\boldsymbol{i} + 2u^2\sin v\boldsymbol{j} - u\boldsymbol{k})dudv \\ &= (4u\cos^2 v + 4u\sin^2 v - u)dudv \\ &= (4u-u)dudv \\ &= 3ududv\end{aligned}$$

- Compute the integral:

$$\begin{aligned}\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} &= \iint_D \boldsymbol{F}(\boldsymbol{r}(u,v))\cdot\left(\frac{\partial\boldsymbol{r}}{\partial v}\times\frac{\partial\boldsymbol{r}}{\partial u}\right)dudv \\ &= \int_{u=0}^{1}\int_{v=0}^{2\pi} 3ududv \\ &= 3\left[\frac{u^2}{2}\right]_{u=0}^{1}[v]_{v=0}^{2\pi} \\ &= 3\pi\end{aligned}$$

8.6 Relationship between multiple, flux and line integrals

There are several key relationships between all the types of integrals we have studied in this course. They are encompassed in the following results which are stated here without any (required) proof. Both of the following theorems are used extensively in physical situations such as fluid dynamics, or electromagnetism.

8.6.1 The divergence theorem

Let V be a volume bounded by a simple closed surface S and let $\boldsymbol{F}$ be a continuously differentiable vector field defined in V and S. Then:

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint\int_V \nabla\cdot\boldsymbol{F}dV$$

In other words the flux integral of a vector field $\boldsymbol{F}$ is equal to the volume integral of the divergence of $\boldsymbol{F}$. This relationship can sometimes be used to evaluate flux integrals by relating them to simpler volume integrals and vice versa.

8.6.2 Stokes's Theorem

Let C be an oriented closed curve and S be any oriented surface with C as a boundary. Suppose that the orientations are such that when we look through S in the direction of its normal C is traversed clockwise. Then for a continuously differentiable vector field $\boldsymbol{F}$ we have

$$\int_C \boldsymbol{F} \cdot d\boldsymbol{r} = \iint\limits_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$$

In other words, the line integral of the vector field $\boldsymbol{F}$ around C is equal to the flux integral of the curl of $\boldsymbol{F}$ through any surface bounded by C and vice versa. From a practical point of view a major importance of this result lies in the idea that *any* S can be used and so you can chose one that simplifies the evaluation of the integral.

Historical note: This theorem has nothing to do with Stokes! It was thought up by Lord Kelvin at the University of Glasgow in the 19^{th} century. He wrote to his friend Stokes at Cambridge and said, "Hey look at this great result". Stokes immediately set it as a question in the Cambridge final degree examinations. This was the first time the result had appeared publicly and so Cambridge typically attributed it to one of their own rather than the true author!