SEMESTER 1 (2016-17): FIRST REVIEW PROBLEM SHEET

(POSTED ON FRIDAY 11 NOVEMBER 2016)

Cosmology and the early Universe

Duration: 1 WEEK

This paper contains 5 questions.

Answer **all** questions in **Section A**

To be handed in Faculty Reception (Building 59) indicating your ID number

DEADLINE : Friday 18 November at 2 pm

This test will contribute 5% to the overall mark

The default system of units is SI. You may give expressions in natural units, but you should state when you start to use them and note when you change unit systems. Throughout the paper the scale factor is normalized in such a way that at the present time $a_0 = 1$.

Section A

A1. Consider the value of the Hubble constant $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

(i) Convert this value in hertz (Hz);

Since $1\,\mathrm{Mpc} \simeq 3 \times 10^{19}\,\mathrm{km}$*, one immediately finds* $H_0 \simeq 2.3 \times 10^{-18}\,\mathrm{Hz}$.

[0.5]

(ii) Calculate the present value of the Hubble radius, $R_{H,0} \equiv c\,H_0^{-1}$, in Gpc (2 significant figures).

$$R_{H,0} = \frac{3 \times 10^8\,\mathrm{m\,s^{-1}}}{70 \times 10^3\,\mathrm{m\,s^{-1}}}\,\mathrm{Mpc} \simeq 4.3\,\mathrm{Gpc}\,.$$

[0.5]

[1]

A2. Consider Hubble's law,

$$z = \frac{H_0}{c}\,d_L\,,$$

relating the cosmological redshift z of an astronomical object to its luminosity distance d_L and valid for redshift $z \lesssim 0.1$. How is the redshift defined in terms of the wavelength of the radiation at emission, $\lambda_{\rm em}$, and the observed wavelength, $\lambda_{\rm obs}$? How is the cosmological redshift interpreted within the Milne-McCrea model based on a Newtonian description of the cosmological expansion? How is the cosmological redshift interpreted within the Friedmann models based on

General Relativity and how is it expressed in terms of the scale factor at the time of the emission $a_{\rm em}$? [1]

(bookwork). The cosmological redshift is defined as

$$z \equiv \frac{\lambda_{\rm obs} - \lambda_{\rm em}}{\lambda_{\rm em}} \, .$$

[0.25]

It is interpreted in terms of non relativistic Doppler effect within the Milne-McCrea model

[0.25]

In Friedmann cosmological models the cosmological redshift is interpreted in terms of stretching of the wavelengths due to the expansion of the Universe and it is a consequence of the redshift of momenta.

[0.25]

(seen problem) The cosmological redshift can be expressed as

$$z \equiv \frac{1}{a_{\rm em}} - 1 \, .$$

[0.25]

A3. The physical size of the visible component of our Galaxy at the present time is about $25\,{\rm kpc}$. Consider a future time $t_\star$ such that $a(t_\star) = 3$. What will the approximate physical size of the of the visible component of our Galaxy be at $t_\star$ neglecting any astrophysical evolution? What will the comoving size of the of the visible component of our Galaxy be at $t_\star$? [1]

TURN OVER

Since the Galaxy is a stable bound system (so called virialised system), its physical size is constant and therefore at some future time $t_\star$ it will remain equal to the physical size today, $\lambda_{\rm ph}^{\rm Galaxy}(t_\star) = \lambda_{\rm ph,0}^{\rm Galaxy}$ if any astrophysical evolution can be neglected.

[0.5]

The co-moving size will be then decrease and given by $\lambda_{(0)}^{\rm Galaxy}(t_\star) = \lambda_{\rm phys,0}/a(t_\star) \simeq$ $8.3\,{\rm kpc}$.

[0.5] .

A4. Consider a 4-dim Euclidean space with metric

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$$

and an embedded 3-dim hyperbolic space (also called 3-dim pseudo-sphere) described by the parametric equation

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = -R^2 \,,$$

with R positive. Show that the metric for this space can be written as

$$d\ell^2 = R^2 \left(\frac{dr^2}{1+r^2} + r^2\, d\Omega^2 \right) \,,$$

with r and Ω properly defined. [1]

Starting from

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = -R^2 \,,$$

where x_1*,* x_2*,* x_3 *and* x_4 *are the coordinates in a 4-dim euclidean space with metric given by*

$$d\ell^2 = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2\,,$$

we can remove the x_4 *coordinate writing*

$$x_4^2 = -R^2 - (x_1^2 + x_2^2 + x_3^2)\,,$$

showing that if x_1, x_2, x_3, R *are real, then* x_4 *is necessarily purely imaginary. We can then change variable introducing* x_4' *such that* $x_4 \equiv i\,x_4'$*, obtaining*

$$(x_4')^2 = R^2 + (x_1^2 + x_2^2 + x_3^2)\,.$$

[0.5]

Differentiating one obtains

$$(dx_4')^2 = \frac{(x_1\,dx_1 + x_2\,dx_2 + x_3\,dx_3)^2}{R^2 + x_1^2 + x_2^2 + x_3^2}\,.$$

Using the result see for the transformation from (x_1, x_2, x_3) *to cylindrical polar coordinates (with the replacement* $r \to \bar{r}$*) one also has*

$$dx_1^2 + dx_2^2 + dx_3^2 = d\bar{r}^2 + \bar{r}^2\,d\Omega^2\,,$$

where $d\Omega^2 \equiv [d\theta^2 + \sin^2\theta\,d\phi^2]$*. Using again the trick to calculate all the* dx_i *around* $\phi = 0$ *one obtains*

$$(dx_4')^2 = \frac{\bar{r}^2\,d\bar{r}^2}{R^2 + \bar{r}^2}$$

TURN OVER

and using $(dx_4)^2 = -(dx_4')^2$*, summing to the result obtained for* $dx_1^2 + dx_2^2 + dx_3^2$*, we finally obtain*

$$d\ell^2 = R^2 \left(\frac{dr^2}{1 + r^2} + r^2\, d\Omega^2 \right) ,$$

where we have again defined $r \equiv \bar{r}/R$*.*

[0.5]

A5. Consider a Universe described by an equation of state $p = w\,\varepsilon$ with $w = -2/3$.

(i) Find $\varepsilon(a)$, i.e. how the energy density ε depends on the scale factor a;

(ii) Calculate the age of the Universe in Gigayears using $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$.

[1]

For these kind of fluids the fluid equation can be written as

$$\frac{\dot{\varepsilon}}{\varepsilon} = -3\,(1 + w)\,\frac{\dot{a}}{a}\,,$$

immediately implying

$$\varepsilon(a) = \frac{\varepsilon_0}{a^{3\,(1+w)}}\,,$$

so that in our case one immediately obtains $\varepsilon(a) = \varepsilon_0/a$*, confirming that this model has a singularity for* $a = 0$*.*

[0.5]

If we now again restrict ourselves to a flat Universe ($\Omega_0 = 1$*), then the*

Friedmann equation can be written as

$$\dot{a}(t) = \frac{H_0}{a^{\frac{1+3w}{2}}}$$

and we can write

$$\frac{da^{\frac{3(1+w)}{2}}}{dt} = \frac{3(1+w)}{2} H_0 \,,$$

and, defining as usual the origin of time such that $a(t=0)=0$*, one finds*

$$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3(1+w)}} \,,$$

with the age of the Universe given by

$$t_0 = \frac{2}{3(1+w)} H_0^{-1} \,.$$

so that for $w = -2/3$ *one has*

$$a(t) = \left(\frac{t}{t_0}\right)^2 \,, \quad \text{with} \quad t_0 = 2 H_0^{-1} \simeq 28\,\mathrm{Gyr} \,.$$

[0.5]

END OF PAPER