SEMESTER 1 (2016-17): SECOND REVIEW PROBLEM SHEET

(POSTED ON FRIDAY 9 DECEMBER 2016)

COSMOLOGY AND THE EARLY UNIVERSE

Duration: 1 WEEK

This paper contains 5 questions.

Answer **all** questions in **Section A**

To be handed in Faculty Reception (Building 59) indicating your ID number

DEADLINE : Friday 16 December at 2 pm

This test will contribute 5% to the overall mark

The default system of units is SI. You may give expressions in natural units, but you should state when you start to use them and note when you change unit systems. Throughout the paper the scale factor is normalized in such a way that at the present time $a_0 = 1$.

Section A

A1. Given the present value of the Λ-like fluid contribution to the total energy density parameter, $\Omega_{\Lambda,0} = 0.7$, and using for the Hubble constant $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, calculate (in Gpc) what would be the radius R_E of an Einstein static Universe with the same amount of energy density ε_Λ. [1]

The amount of energy density in Λ would be simply given by $\varepsilon_\Lambda = \Omega_{\Lambda,0}\,\varepsilon_{c,0}$. Since the critical energy density at the present time is given by

$$\varepsilon_{c,0} \equiv \frac{3\,c^2\,H_0^2}{8\,\pi\,G} \simeq 10.54\,h^2\,\mathrm{GeV\,m^{-3}} \simeq 5.2\,\mathrm{GeV\,m^{-3}},$$

then $\varepsilon_\Lambda \simeq 3.6\,\mathrm{GeV\,m^{-3}}$. [0.5]

In the Einstein static model the matter energy density is given by $\varepsilon_M = 2\,\varepsilon_\Lambda \simeq 10.4\,\mathrm{GeV\,m^{-3}}$ and finally the Einstein radius by

$$R_E = \frac{c^2}{\sqrt{4\pi\,G\,\varepsilon_M}} = \frac{M_P\,c^2}{\sqrt{4\,\pi\,\hbar\,c\,\varepsilon_M}} \simeq 3\,\mathrm{Gpc}\,,$$

where we used Eqs. (2.3) and (2.8) on the notes. [0.5]

Alternatively, one can directly plug the expression for the critical energy density into the expression for R_E, using $\varepsilon_M = 2\,\Omega_{\Lambda,0}\,\varepsilon_{c,0}$ [0.5] and in this way one arrives to the much simpler expression

$$R_E = \frac{c\,H_0^{-1}}{\sqrt{3\,\Omega_{\Lambda,0}}} = \frac{R_{H,0}}{\sqrt{3\,\Omega_{\Lambda,0}}} \simeq 3\,\mathrm{Gpc}\,.[0.5]$$

A2. Calculate the lookback time (in years) for a quasar of redshift $z = 10$ in a Einstein-de Sitter model with $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. [1]

There is a wrong way and a correct way to solve this problem. The wrong way would be to try to use the Eq. (9.26) for the lookback time in terms of z. This is a model independent expression but it has been obtained as a Taylor expansion about the present time and is clearly valid only for $z \lesssim 1$, so it cannot be used in this case. The correct way is considering that the problem is specifying the model, the Einstein-de Sitter model, and in this case from the relation between redshift and scale factor at the emission $z = a_{\rm em}^{-1} - 1$ and from the Einstein-de Sitter relation for the scale factor in terms of time, $a_{\rm em} \equiv a(t_{\rm em}) = (t_{\rm em}/t_0)^{2/3}$, one easily obtains the expression for the lookback time in this model valid for any redshift,

$$t_0 - t_{\rm em} = t_0 \left[1 - \frac{1}{(1+z)^{3/2}}\right] .$$

Using $z = 10$ and $t_0 = 2\,H_0^{-1}/3 \simeq 9.6\,{\rm Gyr}$, one then finds $t_0 - t_{\rm em} \simeq 0.97\,t_0 \simeq 9.3\,{\rm Gyr}$.

A3. The Draco galaxy is a dwarf galaxy within the Local Group. Its luminosity is $L = 1.8 \times 10^5\,L_\odot$ and half of its total mass is contained within a sphere of radius $r_h = 120\,{\rm pc}$. The mean square velocity of the red giant stars in the Draco galaxy is $\langle v^2 \rangle = 10^4\,{\rm km}^2\,{\rm s}^{-2}$. Assuming that the Draco galaxy obeys the virial theorem and that its gravitational potential energy is given by $W = -\alpha\,G\,M^2/r_h$ with $\alpha = 0.45$, what is its mass M (in solar mass units)? What is its mass-to-light ratio (in solar units)? [1]

Assuming the validity of the virial theorem, as for the Coma cluster, one would

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find

$$M_{\rm Draco} = \left.\frac{r_h\,\langle v^2\rangle}{\alpha\,G}\right|_{\rm Draco} \simeq 6\times 10^8\,M_\odot\,,$$

and

$$\left.\left\langle \frac{M}{L}\right\rangle\right|_{\rm Draco} \simeq 3300\,\frac{M_\odot}{L_\odot}\,.$$

A4. Consider the deceleration parameter $q(t) \equiv -\ddot{a}(t)/[H^2(t)\,a(t)]$ at an arbitrary time t. Assuming that the Universe expansion is described by a flat Lemaitre model with $\Omega_{M,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$, what is the asymptotic value of $q(t)$ for $t \to \infty$? [1]

For a Lemaitre model the acceleration parameter can be written as

$$q(t) = \frac{1}{2}\sum_{i=R,M,\Lambda}\Omega_i(t)\,(1+3w_i)\,,$$

In the limit $t \to \infty$ *since* $\Omega_M/\Omega_\Lambda \propto a^{-3} \to 0$ *and since* $\Omega = 1 = {\rm const}$*, necessarily* $\Omega_\Lambda \to \Omega = 1$ *and, therefore,* $q(t) \to -1$*, correctly recovering asymptotically the value holding for the de Sitter model.*

A5. Using for the current age of the Universe $t_0 = 14\,{\rm Gyr}$, for the CMB radiation temperature $k_B\,T_0 = 2\times 10^{-4}\,{\rm eV}$, estimate (2 significant figures) the age of the Universe at the recombination time in a flat Lemaitre model with $\Omega_{M,0} = 0.3$ and $\Omega_{\Lambda,0} = 0.7$. [1]

Since the recombination occurs after the matter-radiation equality time, one *can use the matter-dominated expression for* $a(t)$ *with* $t = t_{\rm rec}$

$$a(t_{\rm rec}) = a_{\rm eq}^{M\Lambda} \left(\frac{t_{\rm rec}}{t_{\rm eq}^{M\Lambda}}\right)^{\frac{2}{3}}$$

and from this one finds easily the result

$$t_{\rm rec} = \frac{t_{\rm eq}^{M\Lambda}}{(a_{\rm eq}^{M\Lambda}\, z_{\rm rec})^{\frac{3}{2}}} \, .$$

Since $k_B\, T_{\rm rec} \simeq 0.32\,{\rm eV}$ *one has* $z_{\rm rec} \simeq a_{\rm rec}^{-1} = (k_B\, T_{\rm rec})/(k_B\, T_0) \simeq 1600$*. Moreover* $a_{\rm eq} = (\Omega_{M,0}/\Omega_{\Lambda,0})^{1/3} \simeq 0.75$ *and, therefore,* $(a_{\rm eq}^{M\Lambda}\, z_{\rm rec})^{\frac{3}{2}} \simeq 41,600$*. One has then to calculate*

$$t_{\rm eq}^{M\Lambda} = \frac{2\, H_0^{-1}}{3\, \sqrt{\Omega_{\Lambda,0}}}\, \ln(1+\sqrt{2}) \simeq 9.8\,{\rm Gyr}\, ,$$

having used $H_0^{-1} \simeq t_0$*. From this last result one finally finds* $t_{\rm rec} \simeq 240,000\,{\rm yr}$*.*

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