UNIVERSITY OF SOUTHAMPTON
SCHOOL OF PHYSICS AND ASTRONOMY

PHYS6005 Cosmology : Problem Sheet 1 (6 October 2016)

Note: solutions of the problems imply use of expressions and numbers given in the lectures even when not explicitly recalled

A. Units of measurement and the cosmic distance ladder

- Using that $1AU \simeq 150 \times 10^6\,\mathrm{km}$, show that $1\,\mathrm{pc} \simeq 3 \times 10^{16}\,\mathrm{m}$ (1 significant digit);

  From the definition of parsec, one has

  $$1\,\mathrm{pc} \times 1\mathrm{arcsec}(rad) = 1\,\mathrm{AU} \simeq 1.5 \times 10^{11}\,\mathrm{m}\,.$$

  Since $1\mathrm{arcsec}(rad) = \pi/(180 \times 3600)$ *one finds*

  $$1\,\mathrm{pc} \simeq 1.5 \times 10^{11}\,\mathrm{m} \times 180 \times 3600/\pi \simeq 3 \times 10^{16}\,\mathrm{m}\,.$$

- Show that $1\,\mathrm{ly} \simeq 1 \times 10^{16}\,\mathrm{m}$ (1 significant digit).

  The speed of light is given by $c \simeq 3 \times 10^8\,m\,s^{-1}$ *and* $1\,\mathrm{yr} = 365 \times 24 \times 3600 \simeq 3.15 \times 10^7\,s$. *Therefore,*

  $$1\,\mathrm{ly} = c \cdot \mathrm{yr} \simeq 9.5 \times 10^{16}\,\mathrm{m} \simeq 1 \times 10^{16}\,\mathrm{m}\,.$$

- The Crab Nebula is 6500 ly away from the Earth. Is it inside or outside the Milky Way Galaxy (approximated as a sphere)? Why?

  It is necessarily inside the Galaxy because the diameter of the Milky Way is about $25\,\mathrm{kpc}$ *and the solar system is at a distance of* $\sim 8.5\,\mathrm{kpc}$ *from the galactic center and therefore, an object* $6500\,ly \simeq 2.2\,\mathrm{kpc}$ *away from the Earth, can be only located inside our Galaxy.*

- The mass of a nucleon (a proton or a neutron) is approximately given by $M_N = 1.7 \times 10^{-27}\,\mathrm{kg}$ (two significant digits). Give the equivalent number in electronvolts in the natural system.

Inverting the equation (2.31), one finds in the natural system ($c = 1$)

$$1\,\mathrm{kg} \simeq 1.8^{-1}\,10^{36}\,\mathrm{eV}$$

and, therefore,

$$M_N \simeq \frac{1.7}{1.8}\,10^{9}\,\mathrm{eV} \simeq 950\,\mathrm{MeV}\,.$$

- The typical mass of a galaxy is given by $10^{12}\,M_{\odot}$. Assuming that the nucleons contribute 1/5 to the mass of a galaxy, estimate the typical number of nucleons in a galaxy.

  The mass of a galaxy in form of nucleons will be then given by $M_G^{\mathrm{nucleons}} = 2 \times 10^{11}\,M_{\odot}$. *Therefore, using the equation (2.4) for* $M_{\odot}$*, one finds* $M_G^{\mathrm{Nucleons}} \simeq 4 \times 10^{41}\,\mathrm{kg}$ *and from this, using the value* $M_N \simeq 1.7 \times 10^{-27}\,\mathrm{kg}$ *given in the previous problem, one obtains*

$$\#_{\mathrm{Nucleons}} = \frac{M_G^{\mathrm{Nucleons}}}{M_N} \simeq 2 \times 10^{68}\,.$$

- Knowing that the Newton constant is given by $G = 6.672\times 10^{-11}\,\mathrm{m}^3\,\mathrm{kg}^{-1}\,\mathrm{s}^{-2}$, show that $M_P\,c^2 \simeq 1.2 \times 10^{19}\,\mathrm{GeV}$ (two significant digits).

  Starting from

$$1\,\mathrm{eV} \simeq 1.6 \times 10^{-19}\,\mathrm{J} = 1.6 \times 10^{-19}\,\mathrm{kg}\,\mathrm{m}^2\,\mathrm{s}^{-2}\,,$$

we can write

$$G = 6.672 \times 1.6 \times 10^{-30}\,\mathrm{eV}^{-1}\,\mathrm{m}^5\,\mathrm{s}^{-4}$$

Remember moreover that

$$\hbar\,c \simeq 200\,\mathrm{MeV\,fm} = 200 \times 10^{6}\,\mathrm{eV}\,10^{-15}\,\mathrm{m}\,.$$

Therefore,

$$M_P^2\,c^4 = \frac{\hbar\,c^5}{G} = \frac{20 \times 81}{6.672 \times 1.6}\,10^{54}\,\mathrm{eV}^2\,.$$

and finally

$$M_P\,c^2 \simeq 1.2 \times 10^{19}\,\mathrm{GeV}\,.$$

B. The cosmological principle and the expanding Universe

- Using $H_0 \simeq 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ give an estimation (with 2 significant digits) of H_0^{-1} in Gyr ($1\mathrm{Gyr} \equiv 10^9\,\mathrm{yr}$).

  Using $1\,\mathrm{Mpc} \simeq 3 \times 10^{22}$*m (cf. eq. (2.3)) and* $1\,\mathrm{yr} = 365 \times 24 \times 3600\,\mathrm{s} \simeq 3.15 \times 10^7$*s, one obtains*

$$H_0^{-1} \simeq \frac{3 \times 10^{22}}{70 \times 10^3}\,\mathrm{s} \simeq \frac{(3/7) \times 10^{18}}{3.15 \times 10^7}\,\mathrm{yr} \simeq 14\,\mathrm{Gyr}\,.$$

- Starting from the Hubble's law, give an estimation (1 significant digit) of the so called *Hubble radius* R_H, that particular value of the distance r of a galaxy from us, above which the (Hubble) recession velocity is higher than c, the speed of light.

  The Hubble's law reads $v = H_0\,d$ *and therefore imposing* $v = c$ *one finds the so called Hubble's radius,*

$$d = R_H \equiv c\,H_0^{-1}\,.$$

  Using $H_0^{-1} = (3/7)\,10^{18}\,\mathrm{s} \simeq 4 \times 10^{17}\,\mathrm{s}$ *(see previous problem) and* $c \simeq 3 \times 10^8\,m/s$*, one obtains*

$$R_H \simeq (3 \times 10^8)\,(4 \times 10^{17})\,\mathrm{m} = 12 \times 10^{25}\,\mathrm{m} \simeq 4\,\mathrm{Gpc}\,,$$

  where $1\,\mathrm{Gpc} = 10^3\,\mathrm{Mpc}$.

- The velocity of a galaxy with respect to us is the sum of the recession Hubble velocity plus the so called *peculiar velocity*. Maximum values of the peculiar velocities are about 1000 km/s. Assume as a mean value for the Hubble's constant $H_0 = 72\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. How big should the galaxy distances be in order to determine the Hubble constant with a precision lower than 10%?

  Taking into account the peculiar velocities, the velocity of a Galaxy i *is given by* $\vec{v}_i = H_0\,\vec{r} + \vec{v}_{p,i}$*. The measured value of* H_0 *is therefore given by*

$$H_0^{\mathrm{measured}} = \frac{|\vec{v}_i|}{d} = \bar{H}_0 + \Delta H_0\,,$$

where $d = |\vec{r}|$ *and* $\Delta H_0 = |\vec{v}_{p,i}|/d$. *Imposing* $\Delta H_0/\bar{H}_0 \leq 0.1$ *one finds*

$$d \geq \frac{10\,\mathrm{max}[|\vec{v}_{p,i}|]}{\bar{H}_0} \simeq 140\,\mathrm{Mpc}\,.$$

C. Cosmic Microwave Background radiation

- From the Eq. (2.39) and Eq. (2.45) verify the numerical estimations Eq. (2.40) and Eq. (2.46) respectively.

  Using the Eq. (2.6) for k_B *and the Eq. (2.8) for* $\hbar\, c$ *(converting* fm *in* *cm), we can write:*

$$\begin{aligned} n_{\gamma,0} &= \frac{2\,\zeta(3)}{\pi^2}\,\frac{(k_B\,T_0)^3}{(\hbar\,c)^3} \\ &\simeq \frac{2\,\zeta(3)}{\pi^2}\,\frac{(0.86\times 2.725)^3}{8}\,\frac{10^{-12}}{(10^2\times 10^6\times 10^{-13})^3}\,\frac{\mathrm{eV}^3}{\mathrm{cm}^3\,\mathrm{eV}^3} \\ &\simeq 0.39\times 10^3\,\mathrm{cm}^{-3} \simeq 400\,\mathrm{cm}^{-3}\,. \end{aligned}$$

$$\begin{aligned} \varepsilon_{\gamma,0} &= \frac{\pi^2}{15}\,\frac{(k_B\,T_0)^4}{(\hbar\,c)^3} \\ &\simeq \frac{\pi^2}{15}\,\frac{(0.86\times 2.725)^4}{8}\,\frac{10^{-16}}{(10^2\times 10^6\times 10^{-13})^3}\,\frac{\mathrm{eV}^4}{\mathrm{cm}^3\,\mathrm{eV}^3} \\ &\simeq 0.25\,\mathrm{eV}\,\mathrm{cm}^{-3} \\ &\simeq 0.25\,\mathrm{MeV}\,\mathrm{m}^{-3}\,. \end{aligned}$$

  A more accurate result (3 significant digits) is given by the Eqs. (11.2) and (11.1) respectively.

- Derive an expression for the frequency, as a function of temperature, corresponding to the peak of the Planckian distribution;

  The Planckian distribution (or the Planckian spectrum) at a generic temperature T *is given, in the natural system, by*

$$\varepsilon_\gamma(\nu) = 16\pi^2\,\frac{\nu^3}{e^{\frac{2\,\pi\,\nu}{T}} - 1}\,.$$

  We have to find the peak of this distribution corresponding to that value ν_{peak} *such that* $[d\varepsilon_\gamma(\nu)/d\nu]_{\nu=\nu_{\mathrm{peak}}} = 0$. *This condition is equivalent to*

$\{d[\ln \varepsilon_\gamma(\nu)]/d\nu\}_{\nu=\nu_{\rm peak}} = 0$, *but the latter is much more convenient and it easily yields*

$$\frac{3}{\nu_{\rm peak}} - \frac{2\pi}{T}\,\frac{e^{\frac{2\,\pi\,\nu}{T}}}{e^{\frac{2\,\pi\,\nu}{T}}-1}\Bigg|_{\nu=\nu_{\rm peak}} = 0\,, \qquad (1)$$

where we used the properties of the logarithm $\ln(a\,b) = \ln a + \ln b$ *and* $\ln a^n = n\,\ln a$. *This is equivalent to*

$$x_{\rm peak} = 3\,(1 - e^{-x_{\rm peak}})\,, \qquad \left(x \equiv \frac{2\pi\nu}{T}\right)\,.$$

At this point we <u>*guess*</u> *that*

$$e^{x_{\rm peak}} \gg 1\,.$$

If this is true than the eq. (1) simplifies into

$$\frac{3}{\nu_{\rm peak}} = \frac{2\pi}{T}\,,$$

and, therefore,

$$\nu_{\rm peak} \simeq \frac{3}{2\pi}\,T \simeq 0.47\,T\,.$$

Notice that in the SI, replacing back $\hbar$ *and* k_B*, this implies*

$$h\,\nu_{\rm peak} \simeq 3\,k_B\,T\,.$$

The guess was then correct, since $e^3 \gg 1$ *and therefore the result is a good approximated result (1 significant digit). A better estimation of* $x_{\rm peak}$ *(2 significant digits), is obtained going back to the eq. (1), plugging the approximated result*

$$e^{x_{\rm peak}} \simeq e^3\,,$$

without (this time) neglecting 1 *and in this way one finds a more precise result*

$$h\,\nu_{\rm peak} \simeq 2.85\,k_B\,T\,.$$

One can continue iterating this procedure and check that the series converges producing a better and better approximation.

- Calculate the corresponding energy of photons in the CMB with a frequency equal to the maximum of the Planckian distribution using $T = 2.7\,K$.

$$E_\gamma^{\rm peak} = h\,\nu_{\rm peak} \simeq 2.85 \cdot (0.86 \times 10^{-4}\,{\rm eV\,K^{-1}}) \cdot 2.7\,{\rm K} \simeq 6.6 \times 10^{-4}\,{\rm eV}\,.$$

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PHYS6005 Cosmology : Problem Sheet 2 (13 October 2016)

Note: solutions of the problems imply use of expressions and numbers given in the lectures even when not explicitly recalled

A. On Newtonian Cosmology

1. Suppose you are in an infinitely large static universe in which the number density of stars is $n = 10^9\,\mathrm{Mpc}^{-3}$ and the average stellar radius is equal to the Sun's radius $R_\odot = 3 \times 10^8\,\mathrm{m}$. How far, on average, could you see in any direction before your line of sight struck a star?

   A star at distance R will subtend a portion of solid angle given by

$$\Omega_\star(R) \equiv \frac{\pi\, R_\star^2}{4\pi\, R^2} .$$

   Therefore, a shell of stars at distance R and thickness dR, neglecting the possibility that these stars overlap with the stars at a closer distance, will subtend a portion of solid angle given by

$$d\Omega_\star(R) \simeq \Omega_\star(R)\, dn(R) ,$$

   where

$$dn(R) = n\, 4\pi\, R^2\, dR ,$$

   is the (infinitesimal) number of stars in the shell. Therefore, integrating on R, the total solid angle subtended by all the stars within a distance R is given by

$$\Omega(R) \simeq \int_0^R d\Omega_\star(R) = \pi\, n\, R_\star^2 \int_0^R dR = \pi\, n\, R_\star^2\, R .$$

   The average distance where an arbitrary line of sight would struck a star is then given by that particular value of R, $R_{\rm max}$, at which the stars cover half of the total solid angle given by 2π and therefore, immediately one gets,

$$R_{\rm max} \simeq \frac{2}{n\, R_\star^2} \sim 2 \times 10^{25}\,\mathrm{pc} = 2 \times 10^{16}\,\mathrm{Gpc} .$$

Notice that this is a huge distance. If one takes into account that the stars can overlap with each other, this result would receive a correction and would be even higher. However, it is not difficult to show that it would be just a correction and therefore this estimation gives the correct order-of-magnitude wise result.

2. What would be the size of the Universe in the Milne-Mc Crea model with $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ and the highest observed redshift is $z = 0.1$?

   Starting from the Hubble's law,

$$z = \frac{H_0\, d}{c}\,,$$

   one finds

$$d = \frac{c\,z}{H_0} \simeq \frac{0.1 \times 3 \times 10^8\,\mathrm{m\,s^{-1}}}{70 \times 10^3\,\mathrm{m\,s^{-1}}}\,\mathrm{Mpc} \simeq 0.4\,\mathrm{Gpc}\,.$$

3. Estimate (1 significant digit) the scale factor corresponding to the time when the universe was at room temperature.

   In the MM model we have seen that the equilibrium of CMBR can be understood assuming that during the expansion the photon gas undergoes an adiabatic quesi-static expansion respecting the first law of thermodynamics and from these assumption one deduces easily $a \propto T^{-1}$. *Since at room temperature* $T \sim 30°C \simeq 300\,\mathrm{K}$ *and since today* $T_0 \sim 3\,\mathrm{K}$, *then* $a(T \sim 300\,\mathrm{K}) \simeq a_0\, T_0/T \simeq 10^{-2}\, a_0$.

4. A poor man's estimate of the mass density in the Universe today could be based on the typical galaxy mass and galaxy separation $\rho_0 = M_{\mathrm{gal}}/\mathrm{Mpc}^3$. Evaluate this expression in $\mathrm{kg/m^3}$.

   The typical mass of a galaxy is given by $10^{12}\, M_\odot \simeq 2 \times 10^{42}\,\mathrm{kg}$ *and therefore*

$$\rho_0 = \frac{M_{\mathrm{gal}}}{\mathrm{Mpc}^3} \simeq \frac{2 \times 10^{42}\,\mathrm{kg}}{(3 \times 10^{22}\,\mathrm{m})^3} \simeq 7.4 \times 10^{-26}\,\frac{\mathrm{kg}}{\mathrm{m}^3}\,.$$

5. If $H_0 = 70\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, and given the density found in the previous problem, what would be the ultimate fate of the Universe in a Milne-Mc Crea model, would it expand forever or reach a maximum size and then start a contraction phase?

In the Milne-McCrea model the expansion of the Universe is described by the following equation,

$$H^2 - \frac{8\,\pi\,G}{3}\,\rho = -\frac{k\,c^2}{a^2\,R_0^2}\,.$$

that corresponds to a special case of the Friedmann equation (zero pressure). The sign of the right-hand side (the total energy) is constant and therefore the same has to be for the left-hand side. Therefore, if we determine at the present the sign of the left-hand side this will tell us the sign of the total energy and therefore the fate of the expansion: if the total energy is positive (or vanishing! But this is a very special case requiring very fine tuned conditions) then the Universe will expand forever, if it is negative it will reach a maximum expansion and then it will start to collapse. If we express everything at the present we obtain

$$H_0^2 - \frac{8\,\pi\,G}{3}\,\rho_0 = -\frac{k\,c^2}{R_0^2}\,.$$

It is then clear that we have to determine whether ρ_0 *is higher or lower than*

$$\rho_{\mathrm{c},0} \equiv \frac{3\,H_0^2}{8\,\pi\,G}$$

that is the so called critical energy density at the present time. The value of ρ_0 *is assumed to be that one estimated in the previous problem,*

$$\rho_0 \simeq 7.4\times 10^{-26}\,\frac{\mathrm{kg}}{\mathrm{m}^3}\,.$$

We have therefore just to calculate $\rho_{\mathrm{c},0}$ *and compare it with* ρ_0. *The Newton's constant is given by* $G \simeq 6.67\times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ *and therefore*

$$\begin{aligned}\rho_{\mathrm{c},0} &\simeq \frac{3\,(70\times 10^3)^2}{8\,\pi\,6.67\times 10^{-11}}\,\frac{\mathrm{m^2\,s^{-2}\,kg}}{\mathrm{m^3\,s^{-2}\,Mpc^2}}\\ &\simeq \frac{3\times 49}{8\,\pi\,6.67\times 9}\times\frac{10^8}{10^{11}\,10^{44}}\,\frac{\mathrm{kg}}{\mathrm{m}^3}\\ &\simeq 1\times 10^{-26}\,\frac{\mathrm{kg}}{\mathrm{m}^3}\,.\end{aligned}$$

Therefore, we can conclude that, since $\rho_0 > \rho_{c,0}$, the Universe will start to collapse at some point. However, we will see that this result is not correct for various reasons and ρ_0 is actually much lower than the value estimated in the previous problem. This calculation can be regarded as a first rough attempt to give an answer to a very important problem and a first order-of-magnitude wise estimation of ρ_0 that will prove to be approximately one order-of-magnitude larger than the correct value.

B. On Classical Mechanics

1. Prove that the scalar product in a rotated reference frame is indeed a (3)-scalar. Hint: You have basically to prove that $\vec{x}' \cdot \vec{x}' = \vec{x} \cdot (R^T R)\, \vec{x}$.

   The demonstration proceeds by direct inspection starting from the definition of scalar product:

$$\begin{aligned} \vec{x}' \cdot \vec{x}' &= \sum_i (R\,\vec{x})_i\,(R\,\vec{x})_i = \sum_i (\sum_k R_{ik}\,x_k)\,(\sum_l R_{il}\,x_l) \\ &= \sum_{k,l} x_k\,(\sum_i R_{ik}\,R_{il})\,x_l = \sum_{k,l} x_k\,(R^T\,R)_{kl}\,x_l \\ &= \sum_k x_k \sum_l \delta_{kl}\,x_l = \sum_k x_k\,x_k \\ &= \vec{x}\cdot\vec{x}\,. \end{aligned} \tag{2}$$

   Clearly the demonstration was working also for the scalar product of two different generic vectors $\vec{x}_1$ and $\vec{x}_2$. Notice that we used the orthogonality of the rotation matrix $R^T\,R = I$, that in components can be written as $\sum_i R_{ik}\,R_{il} = \delta_{kl}$.

2. Given a vector $\vec{a}$ and a tensor T, prove that the three quantities

$$b_i = \sum_k T_{ik}\,a_k\,,$$

   are indeed the three components of a 3-vector $\vec{b}$.
   Hint: You should make use of the orthogonality condition $R^T\,R = I$ written in terms of the entries: $\sum_k R_{ki}\,R_{kj} = \delta_{ij}$.

   It is first convenient to express the thesis in equations. We have to demonstrate that, given the definition of $\vec{b}$, its component transform as

$$b'_i = \sum_l R_{il}\,b_l = \sum_l R_{il} \sum_m (T_{lm}\,a_m)\,.$$

Starting from the definition of $\vec{b}$ the following steps lead to the thesis:

$$\begin{aligned} b'_i &= \sum_k T'_{ik}\, a'_k \\ &= \sum_k \left(\sum_{l,m} R_{il}\, R_{km}\, T_{lm}\right) \left(\sum_n R_{kn}\, a_n\right) \\ &= \sum_{l,m,n} R_{il}\, T_{lm}\, a_n \sum_k R_{km}\, R_{kn} \\ &= \sum_{l,m,n} R_{il}\, T_{lm}\, a_n\, \delta_{mn} \\ &= \sum_l R_{il} \sum_m T_{lm}\, a_m \\ &= \sum_l R_{il}\, b_l\,. \end{aligned}$$

Notice that we made use of the orthogonality condition $R^T R = I$ when we wrote $\sum_k R_{km}\, R_{kn} = \delta_{mn}$.

3. Prove that given two vectors $\vec{a}$ and $\vec{b}$, their direct product $T_{ij} = a_i\, b_j$ is a tensor.

   It is quite simple starting from the definition:

$$\begin{aligned} T'_{ij} &= a'_i\, b'_j \\ &= \left(\sum_k R_{ik}\, a_k\right) \left(\sum_l R_{il}\, b_l\right) \\ &= \sum_{k,l} R_{ik}\, R_{jl}\, a_k\, b_l \\ &= \sum_{k,l} R_{ik}\, R_{jl}\, T_{kl}\,, \end{aligned}$$

   where the last line shows that T_{ij} is indeed a tensor.

4. Prove that the cross product components transform as a vector but under parity transformation, defined as that discrete transformation such that the 3-coordinate vector flips (i.e. $\vec{x}' = -\vec{x}$), its three components do not change sign showing that it is not a true (or polar) vector but what is called a pseudo- (or axial-)vector.

Let us start from the definition of cross product of two vectors $\vec{a}$ and $\vec{b}$,

$$(\vec{a} \times \vec{b})_k = a_i\, b_j - a_j\, b_i\,,$$

with (i, j, k) different from each other and in cyclic order. Let us now perform a rotation and in this case, since $\vec{a}$ and $\vec{b}$ are vectors, one has

$$\begin{aligned}(\vec{a} \times \vec{b})'_k &= R_{il}\, R_{jm} a_l\, b_m - R_{jm}\, R_{il} a_m\, b_l \\ &= R_{il}\, R_{jm}(a_l\, b_m - a_m\, b_l)\,,\end{aligned}$$

showing that it transforms as the components of a tensor and, therefore, it is indeed a tensor.

We can also show that at the same time the three independent components of the cross product transform as an axial vector. We can use the fully antisymmetric (or Levi-Civita) matrix to express the cross product components as

$$(\vec{a} \times \vec{b})_k = \varepsilon_{kjl}\, a_j\, b_l\,.$$

Starting from this expression and starting from the definition of rotation transformation, we can then write:

$$\begin{aligned}(\vec{a} \times \vec{b})'_k &= \varepsilon_{kjl}\, a'_j\, b'_l \\ &= \varepsilon_{kjl}\, R_{jm}\, R_{ln}\, a_m\, b_n \\ &= \varepsilon_{kjl}\, R_{jm}\, R_{ln}\, a_m\, b_n \\ &= \varepsilon_{ijl}\, \delta_{ki}\, R_{jm}\, R_{ln}\, a_m\, b_n \\ &= \varepsilon_{ijl}\, R_{ks}\, R_{is}\, R_{jm}\, R_{ln}\, a_m\, b_n \\ &= R_{ks}\, (\varepsilon_{ijl}\, R_{is}\, R_{jm}\, R_{ln})\, a_m\, b_n \\ &= R_{ks}\, \varepsilon_{smn}\, \varepsilon_{ijl}\, R_{i1}\, R_{j2}\, R_{l3})\, a_m\, b_n \\ &= \det(R)\, R_{ks} \varepsilon_{smn}\, a_m\, b_n \\ &= R_{ks}\, \varepsilon_{smn}\, a_m\, b_n \\ &= R_{ks}\, (\vec{a} \times \vec{b})_s\,,\end{aligned}$$

where we used

$$R_{ks}\, \varepsilon_{ijl}\, R_{is}\, R_{jm}\, R_{ln}\, a_m\, b_n = R_{ks}\, \varepsilon_{smn}\, \varepsilon_{ijl}\, R_{i1}\, R_{j2}\, R_{l3}\, a_m\, b_n$$

(it is true because of orthgonality, notice that there is a sum on m, n, s and terms with two of these three indexes equal vanish because of orthogonality) and

$$\varepsilon_{ijl}\, R_{i1}\, R_{j2}\, R_{l3} = \det(R) = 1\,.$$

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PHYS6005 Cosmology : Problem Sheet 3 (20 October 2016)

Note: solutions of the problems imply use of expressions and numbers given in the lectures even when not explicitly recalled

A. On Special Relativity

- Consider the Lorentz transformation in the plane $x^0 - x$ written in the form given by the eq. (4.28). Show that this can be recast as a usual rotation in the plane $ix^0 - x$ (where $i\,x^0 = i\,c\,t$ is the imaginary time) with a complex angle ψ'. How is ψ' related to ψ?

  Let us start from the Lorentz transformation written in the form

$$\begin{aligned} x'^0 &= x^0 \cosh\psi + x \sinh\psi \\ x' &= x^0 \sinh\psi + x \cosh\psi\,. \end{aligned}$$

  Let us define the new (imaginary) variable $z^0 \equiv i\,x^0$. Multiplying both sides of the first equation by i, the equations become

$$\begin{aligned} z'^0 &= z^0 \cosh\psi + i\,x \sinh\psi \\ x' &= -i\,z^0 \sinh\psi + x \cosh\psi\,. \end{aligned}$$

  If we introduce $\psi' \equiv -i\,\psi$, so that $\psi \equiv i\,\psi'$ one has the following relations between trigonometric and hyperbolic functions (easy to derive from the definitions)

$$\begin{aligned} \cosh\psi &= \cos\psi' \\ \sinh\psi &= -i\,\sin\psi'\,, \end{aligned} \tag{3}$$

  One obtains finally

$$\begin{aligned} z'^0 &= z^0 \cos\psi' + x \sin\psi' \\ x' &= -\,z^0 \sin\psi' + x \cos\psi'\,, \end{aligned}$$

  corresponding to a usual rotation though with a complex angle.

- In class we derived the dilation time directly from the invariance of ds and introducing the proper time. In particular the life time τ of a particle measured in flight is $\tau = \gamma\tau_0$, where τ_0 is the life-time at rest. Re-derive the same result from the Lorentz transformations (cf. eqs. (4.31), (4.32)) first considering the particle decaying at rest in $\mathcal{S}'$, the Lorentz boosted reference frame along the x-axis with origin in vt, and then considering the same particle decaying at rest in $\mathcal{S}$ (hint: place the particle in the origin of the two reference frames).

  Let us start from the case when the particle decays at rest in $\mathcal{S}'$. In order to simplify the problem we can place the particle in the origin of $\mathcal{S}'$ so that $x^{1'} = 0$ and this implies, from the second equation,

  $$x^1 = \beta\, x^0 \,.$$

  Moreover, for simplicity, we can set the origin of time when the particle is created in a way that $t'_{\rm in} = 0$ and $t'_{\rm fin} = \tau_0$ and correspondingly $t_{\rm in} = 0$ and $t_{\rm fin} = \tau$. Therefore, the first equation gives

  $$c\,\tau_0 = \gamma\,(c\,\tau - c\,\beta^2\,\tau)$$

  and therefore

  $$\tau = (1-\beta^2)\,\frac{\tau_0}{\gamma} = \gamma\,\tau_0 \,.$$

  If the particle decays at rest in $\mathcal{S}$, it is even easier. This time the particle can be placed in the origin of $\mathcal{S}$, so that $x^1 = 0$. Again we can set the origin of time when the particle is produced, $t_{\rm in} = x^0_{\rm in} = 0$ and this implies also that $x^{0'}_{\rm in} = 0$. The particle decays in $\mathcal{S}$ at $x^0_{\rm fin} = c\,\tau$ corresponding to $x^{0'}_{\rm fin} = c\,\tau$. Therefore, the first equation immediately gives

  $$c\tau = \gamma\,c\,\tau_0 \Rightarrow \tau = c\,\tau_0 \,.$$

  Notice that if in the first case we would have used the inverse Lorentz transformations, the problem would have been (more easily) solved exactly as in this second case using the Lorentz transformations.

  This problem confirms that the Lorentz transformations respect the principle of relativity.

- Show that in the partial derivative $\partial/\partial x^\mu \equiv \partial_\mu$, the index μ is a covariant index (hint: remember that $A_\mu B^\mu$ is a scalar and then identify

A_μ with the partial derivative and B^μ with a properly chosen 4-vector with index up).

Choosing as $A_\mu = \partial_\mu$ and $B^\mu = x^\mu$, one has $\partial_\mu x^\mu = 1$ that is clearly a scalar. Therefore, the partial derivative of a function transforms like a covariant 4-vector.

- From the invariance of the interval, show that for a general Lorentz transformation the matrix Λ associated to the transformation has to satisfy

$$\Lambda^T \eta \Lambda = \eta .$$

Consider the infinitesimal interval

$$ds^2 = \eta_{\mu\nu} \, dx^\mu \, dx^\nu . \tag{4}$$

After a general Lorentz transformation this would transform as

$$ds^{2\,\prime} = \eta_{\mu\nu} \; dx^{\mu\,\prime} \; dx^{\nu\,\prime} ,$$

with $dx^{\mu\,\prime} = \Lambda^\mu{}_\alpha \, dx^\alpha$ and $dx^{\nu\,\prime} = \Lambda^\nu{}_\beta \, dx^\beta$, so that we can write

$$ds^{'2} = \eta_{\alpha\beta} \, \Lambda^\alpha{}_\mu \, \Lambda^\beta{}_\nu \, dx^\mu \, dx^\nu .$$

Imposing the invariance of the interval, $ds^{'2} = ds^2$ and comparing with Eq. (4) one finds the condition

$$\eta_{\alpha\beta} \, \Lambda^\alpha{}_\mu \, \Lambda^\beta{}_\nu = \eta_{\mu\nu} ,$$

that can be also written as

$$\Lambda^\alpha{}_\mu \, \eta_{\alpha\beta} \, \Lambda^\beta{}_\nu = \eta_{\mu\nu} ,$$

also equivalent to write

$$(\Lambda^T \eta \Lambda)_{\mu\nu} = \eta_{\mu\nu} ,$$

or also $\Lambda^T \eta \Lambda = \eta$.

B. Coordinate transformations in the absence of gravitational fields

1. Consider the change of spatial coordinate transformation from cartesian coordinates to spherical coordinates. Write down the metric tensor $g_{\mu\nu}$ in spherical coordinates $(x'^0, x'^1, x'^2, x'^3) = (ct, r, \theta, \phi)$
(Hint: you should look at the Eq. (4.84)).

   We have found that in spherical coordinates the Minkowsky metrics becomes
$$ds^2 = dt^2 - (dr^2 + r^2 [d\theta^2 + \sin^2\theta \, d\phi^2]) \, .$$
   In general, the metric tensor $g_{\mu\nu}$ can be extracted from the metric considering that
$$ds^2 = g_{\alpha\beta} \, dx'^\alpha \, dx'^\beta \, .$$
   Therefore, in our special case, one can see immediately that the metric tensor has to be diagonal (there are no mixed terms) and more specifically has to be given by
$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{pmatrix} .$$

2. Consider now a coordinate transformation from an inertial system $\mathcal{S}$ of cartesian coordinates (t, x, y, z) to a non-inertial system of coordinates (t, x', y', z') in uniform rotation with respect to $\mathcal{S}$ with angular frequency ω and whose rotation axis $\hat{z}'$ coincides with $\hat{z}$ (therefore $z' = z$). Write down the coordinate transformations. Express the metric ds^2 in this new system and write down the metric tensor. Notice that the time coordinate is assumed to be the same in the two reference frames ($t' = t$). In order to find the correct g_{00} you have to take into account the dilation time effect: an observer at rest in the rotating system at distance $r'^2 = x'^2 + y'^2$ from the origin will have a speed $v = \omega \, r'$. If you do things correctly, you should obtain:
$$ds^2 = (c^2 - v^2) \, dt^2 - dx'^2 - dy'^2 - dz^2 - 2 \, x' \, \omega \, dy' \, dt + 2 \, y' \, \omega \, dx' \, dt \, .$$
   From $t' = t$ and $z' = z$ one simply has $dt' = dt$ and $dz' = dz$. The x and y coordinates are connected to the x' and y' by a rotation transformation

with angle $\theta(t) = \omega\, t$ *given by*

$$x = x' \cos\theta - y' \sin\theta \quad (5)$$
$$y = y' \cos\theta + x' \sin\theta\,. \quad (6)$$

Differentiating one obtains

$$dx = dx' \cos\theta - dy' \sin\theta - \omega\, dt'(y' \cos\theta + x' \sin\theta) \quad (7)$$
$$dy = dy' \cos\theta + dx' \sin\theta + \omega\, dt'(x' \cos\theta - y' \sin\theta)\,. \quad (8)$$

From these equations, after some algebraic passages, one finds

$$dx^2 + dy^2 = (dx')^2 + (dy')^2 + v^2\, dt^2 - 2\, y'\omega\, dx'\, dt + 2\, x'\omega\, dy'\, dt\,.$$

where $v^2 = \omega^2\,(x^2 + y^2)$. *Therefore, summing everything together,*

$$ds^2 = (c^2 - v^2)\, dt^{'2} - dx^{'2} - dy^{'2} - dz^{'2} - 2\, x'\, \omega\, dy'\, dt + 2\, y'\, \omega\, dx'\, dt\,.$$

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PHYS6005 Cosmology : Problem Sheet 4 (27 October 2016)

Note: solutions of the problems imply use of expressions and numbers given in the lectures even when not explicitly recalled

A. On General Relativity

1. In the paper *The foundation of the General Theory of Relativity*, Einstein, at the end of Section 1, makes an important observation about intervals of time measured in Special Relativity: which one?

   He notices that for two clocks at rest in the system of reference, for two selected position of the hands there always corresponds an interval of time of definite length independent of place and of time. Basically time flows in the same way in any point and at any time. For example, the life time of a decaying elementary particle measured at rest is constant in time and space.

2. What is the starting motivation of Einstein for an extension of the "Postulate of Relativity" (section 2)?

   It is the famous 'Mach Principle'. In Newtonian physics and in Special Relativity there is a privileged class of inertial reference frames. The laws of physics are different whether one performs an experiment at rest or with an angular rotation in the reference frame. Einstein thinks that in a more 'general theory' such a difference should disappear: 'The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion'. Therefore, the observed differences should be only ascribed to the fact that one is not including the whole Universe in the experiment and therefore the 'experimental device' is rotating with respect to the rest of the Universe.

3. What is the transformation law of a generic 4-vector A^μ (with contra-variant index) in Special Relativity? And in General Relativity?

   In Special Relativity a contra-variant 4-vector A^μ *is an object that under LT transforms like the* $4-$*coordinate vector, i.e.*

   $$A'^\mu = \Lambda^\mu_{\ \nu} \, A^\nu \, .$$

   By definition the 4*-position coordinate vector is a 4-vector.*

B. On the Geometry of a 2-dim universe

1. Suppose you are an ant living on a the surface of a sphere with radius R. An object with size $d\lambda \ll R$ is at distance ℓ from you (all distance are measured on the surface of the sphere). What angular width $d\psi$ will you measure for the object ? Explain the behavior of $d\psi$ as $\ell \to \pi R$.

   Let us indicate with $\bar{r}$ the distance from the z axis and with θ the polar angle. Then one has

   $$\ell = R\,\theta\,, \qquad \bar{r} = R\,\sin\theta = R\sin\left(\frac{\ell}{R}\right)\,. \tag{9}$$

   The angular size is given by

   $$d\psi = \frac{d\lambda}{\bar{r}} \tag{10}$$

   Therefore, one finds

   $$d\psi = \frac{d\lambda}{R\sin\left(\frac{\ell}{R}\right)}\,. \tag{11}$$

   Notice that in the limit $\ell \to \pi R$ it looks like if this expression for $d\psi$ diverges. However, in this limit the object has necessarily to shrink to a point located in the antipodal point with respect to the observer and therefore $dl \to 0$. The antipodal point is very special because there is an infinity of geodesics connecting two antipodal points. Therefore an object in the antipodal point is seen by any direction and therefore $d\psi \to 2\pi$ in this special limit. Therefore, there is no actual divergence.

2. Show that if you draw a circle of radius ℓ on a spherical surface, the circle's circumference is going to be $C = 2\,\pi\,R\,\sin(\ell/R)$. Imagine one can measure distances with an error ± 1m, how long a circle would you have to draw on the Earth's surface ($R = 6371\,$km) to convince yourself that the Earth is spherical rather than flat?

   In the previous problem we found

   $$d\psi = \frac{d\lambda}{R\sin\left(\frac{\ell}{R}\right)} \tag{12}$$

   The circle corresponds to take $d\psi = 2\pi$ and $d\lambda = C$ and therefore

   $$C = 2\pi\,R\sin\left(\frac{\ell}{R}\right) \tag{13}$$

The flat case corresponds to the limit $\ell/R \to 0$ and therefore one has $C_{\rm flat} = 2\pi\,\ell$. One can say that the space is curved if $C_{\rm flat} - C > 0$. If it were not for the error of measurement, it would be possible to show that the space is not flat for an arbitrary value of C but since there is an error one has to impose that the difference $C_{\rm flat} - C > 0$ is large enough. Notice that the error on this difference is given by the sum of the errors on $C_{\rm flat}$ and C. The reason is that one has to measure independently both the distance ℓ giving $C_{\rm flat}$ and the circle's circumference C. Therefore, one has

$$\Delta(C_{\rm flat} - C) = \Delta C_{\rm flat} + \Delta\, C\,, \tag{14}$$

where $\Delta C_{\rm flat} = 2\pi\,\Delta\ell = 2\pi, {\rm m}$ and $\Delta C = 1m$. Notice that the two errors are not the same since in the first case one measures r while in the second case one measures directly C.

One has therefore to impose the condition

$$\bar{C}_{\rm flat} - \bar{C} > (2\pi + 1)\,{\rm m}\,, \tag{15}$$

where the barred quantities are the mean values. This yields

$$\bar{C}_{\rm flat}\left(1 - \frac{\sin x}{x}\right) > (2\pi + 1)\,{\rm m}\,, \tag{16}$$

where $x \equiv \ell/R$. Taylor expanding $\sin x \simeq x - x^3/6$, one finds finally the condition

$$x^3 > \frac{3\,(2\pi + 1){\rm m}}{2\,\pi\,R}\,, \tag{17}$$

that numerically gives $\ell \gtrsim 66\,{\rm km}$ and $\bar{C}_{\rm flat} \simeq \bar{C} \gtrsim 415\,{\rm km}$.

C. Geometry of the Universe

1. Consider the 3-dim spatial metric

$$d\ell^2 = \frac{d\bar{r}^2}{1 - k\,\bar{r}^2/R^2} + \bar{r}^2\,d\Omega^2\,.$$

Knowing that with the coordinate transformation $\bar{r} = S_k(x)$ the metric gets expressed as

$$d\ell^2 = dx^2 + S_k^2(x)\,d\Omega^2\,,$$

find $S_k(x)$ for $k = +1, 0, -1$. What is the physical meaning of the new coordinate x?

Using the change of variable $\bar{r} = S_k(x)$ the metric becomes

$$d\ell^2 = \frac{S_k'^2(x)\, dx^2}{1 - k\, S_k(x)^2/R^2} + S_k(x)^2\, d\Omega^2 \,,$$

where $S_k'(x) \equiv dS_k/dx$. We have, therefore, to impose the condition

$$\frac{dS_k}{\sqrt{1 - k\,(S_k(x)/R)^2}} = dx \,.$$

Using as a variable S_k/R and integrating between 0 and x (in dx') we find

$$\int_{S_k(0)/R}^{S_k(x)/R} \frac{d(S_k/R)}{\sqrt{1 - k\,(S_k/R)^2}} = \frac{x}{R} \,.$$

For $k = 0$ the solution is straightforward and one finds $S_0(x) = x$.

For $k = +1$ one can change variable in the integral, $S_+(x)/R = \sin y$, and in this way one easily finds

$$S_+(x) = R \sin \frac{x}{R} \,.$$

For $k = -1$ there is an analogous change of variable $S_-(x)/R = \sinh y$, and in this way one easily finds

$$S_-(x) = R \sinh \frac{x}{R} \,.$$

Notice that x is the distance of an object from the origin. In the expanding Universe it would correspond to the proper distance (see Eq.(6.19)).

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PHYS6005 Cosmology : Problem Sheet 5 (3 November 2016)

Note: solutions of the problems imply use of expressions and numbers given in the lectures even when not explicitly recalled

On the Fundamental equations of Friedmann Cosmology

- Derive the fluid equation combining the Friedmann equation with the acceleration equation;

  Let us start from the Friedmann equation:

  $$H^2 = \frac{8\,\pi\,G}{3\,c^2}\,\varepsilon - \frac{k\,c^2}{a^2\,R_0^2}\,.$$

  Considering that $H \equiv \dot{a}/a$ *and multiplying both LH and RH side by* a^2 *we obtain:*

  $$\dot{a}^2 = \frac{8\,\pi\,G}{3\,c^2}\,\varepsilon\,a^2 - \frac{k\,c^2}{R_0^2}\,.$$

  Differentiating with respect to time and dividing by $2\,a^2$ *we then obtain*

  $$\frac{\dot{a}}{a^2}\,\ddot{a} = \frac{4\,\pi\,G}{3\,c^2}\,\left(\frac{\dot{a}}{a}\,\varepsilon + \dot{\varepsilon}\right)\,.$$

  Finally using the acceleration equation in the LH side and dividing both sides by $4\pi\,G/(3\,c^2)$ *one obtains:*

  $$-\frac{\dot{a}}{a}\,(\varepsilon + 3\,p) = -\frac{\dot{a}}{a}\,(2\varepsilon + \dot{\varepsilon})\,,$$

  and from this one one easily arrives to the fluid equation

  $$\dot{\varepsilon} = -3\,\frac{\dot{a}}{a}\,(\varepsilon + p)\,.$$

- Derive the Friedmann equation combining the fluid equation with the acceleration equation.

Starting from the acceleration equation and multiplying both sides by $2\,\dot{a}$ *and considering that* $2\dot{a}\,\ddot{a} = d(\dot{a}^2)/dt$ *one has*

$$\frac{d(\dot{a}^2)}{dt} = -\frac{8\,\pi\,G}{3\,c^2}\,(\varepsilon + 3\,p)\,a\,\dot{a}\,.$$

Considering that $\varepsilon + 3p = 3(\varepsilon + p) - 2\varepsilon$ *and that from the fluid equation one has*

$$3\,(\varepsilon + p) = -\dot{\varepsilon}\,\frac{a}{\dot{a}}\,,$$

one can then write

$$\frac{d(\dot{a}^2)}{dt} = \frac{8\,\pi\,G}{3\,c^2}\,\left(\dot{\varepsilon}\,a^2 + 2\varepsilon\,a\,\dot{a}\right)\,.$$

Finally, considering that $(\dot{\varepsilon}\,a^2 + 2\varepsilon\,a\,\dot{a}) = (\varepsilon\,\dot{a^2})$*, integrating on time and dividing by* a^2 *one finds*

$$\frac{\dot{a}^2}{a^2} = \frac{8\,\pi\,G}{3\,c^2} + \frac{const}{a^2}\,,$$

that is the Friedmann equation when one identifies the constant with $-k\,c^2/R_0^2$*.*

Critical energy density, cosmological redshift, physical and co-moving length

- Show that the expression

$$\varepsilon_{\rm c}(t) \equiv \frac{3\,H^2(t)\,c^2}{8\,\pi\,G}\,,$$

for the critical energy density is dimensionally correct.

Since $G\,m^2/r$ *is the gravitational potential energy, then* $[G] = [E\,l\,m^{-2}]$ *and therefore*

$$[\varepsilon_c] = \left[\frac{m^2\,c^2}{E\,l\,t^2}\right] = \left[\frac{m^2\,v^4}{E\,l\,v^2\,t^{-2}}\right] = \left[\frac{E}{l^3}\right]\,.$$

that is indeed an energy density. More elegantly, in the natural system one has $[E] = [l^{-1}]$ *and therefore* $[\varepsilon] = [l^{-4}]$ *in the l.h.s. In the r.h.s.* $c = 1$, $[G] = [l^2]$, $[H] = [l^{-1}]$ *and therefore* $[H^2/G] = [l^{-4}]$.

- The proper distance between two far galaxies having exactly the same cosmological redshift $z = 1$ is 10 Mpc at present. What was the value of the proper distance at the time of emission?

  The Galaxies are sufficiently far that their reciprocal gravitational interaction can be neglected. They can therefore be considered approximately at rest in the co-moving system so that their co-moving distance is constant. In this case, from the general expression

  $$\lambda_{\rm ph}(t) = a(t)\ \lambda_{(0)}(t)\,,$$

  one has $\lambda_{(0)}(t) = d_{\rm pr}^{(0)}(t_0) = const$, *while* $\lambda_{\rm ph}(t) = d_{\rm pr}(t) \propto a(t)$. *By definition* $z = a(t_{\rm em})^{-1} - 1$ *and therefore* $a(t_{\rm em}) = 0.5$ *so that the proper distance was half of the current distance, i.e. 5 Mpc.*

- The physical size of our Galaxy is approximately 25 Kpc. What is its comoving size at present? What was its comoving size when $a(t) = 0.2$?

  At present physical sizes and co-moving sizes coincide because we the scale factor is normalised in such a way that $a_0 = 1$.

  Notice that the question implies that our Galaxy formed already at $a = 0.2$, *corresponding to* $z = 4$, *and that did not evolve significantly from then being a stable system. Therefore, its physical size did not change significantly from that time and* $\lambda_{\rm ph} = const$. *This means that its co-moving size was 5 times bigger at the time* t *when* $a(t) = 0.2$. *This example shows how a physically stable system does not expand tracking the 'Hubble's flow' and in this way its co-moving size decreases with time (i.e. it was bigger in past).*

- Consider a freely propagating proton with energy $E = 2.814\,{\rm GeV}$ at a (past) time t such that $a(t) = 1/2$. Using $m_p\, c^2 = 0.938\,{\rm GeV}$, what is its energy at the present time t_0?

  The energy of the proton at the present is given by

  $$E_0 = \sqrt{m_p^2\, c^4 + c^2\, |\vec{p}|_0^2}\,.$$

  During the expansion the momentum scales like $|\vec{p}| \propto a^{-1}$ *and therefore one has*

  $$|\vec{p}|_0 = |\vec{p}|(t)\, \frac{a(t)}{a_0} = \frac{1}{2}\, |\vec{p}|(t)\,.$$

On the other hand one has $c^2\,|\vec{p}|(t)^2 = E^2(t) - m_p^2\,c^4$. Therefore, one finally finds

$$E_0 = \sqrt{m_p^2\,c^4 + \frac{1}{4}\,(E^2(t) - m_p^2\,c^4)} \simeq 1.625\,\mathrm{GeV}\,.$$

- Find the numerical value of $\varepsilon_{c,0}$, the critical energy density at present, in $\mathrm{GeV\,m^{-3}}$, expressing it in terms of the parameter h defined as $h \equiv H_0/(100\,\mathrm{km\,s^{-1}\,Mpc^{-1}})$ and using $M_\mathrm{P}\,c^2 = 1.2\times 10^{19}\,\mathrm{GeV}$.

  The critical energy density at present is given by

$$\varepsilon_{\mathrm{c},0} \equiv \frac{3\,c^2\,H_0^2}{8\,\pi\,G} \simeq 1.1\,h^2\,10^4\,\mathrm{eV\,cm^{-3}}\,,$$

  Using $G = \hbar\,c/M_P^2$ one obtains

$$\varepsilon_{\mathrm{c},0} = \frac{3}{8\pi}\,\frac{H_0^2\,(M_\mathrm{P}\,c^2)^2}{c^2\,(\hbar\,c)}\,.$$

  Plugging all numbers for the different quantities, in eV *and* cm *one obtains first*

$$\varepsilon_{\mathrm{c},0} = \frac{3}{8\pi}\,h^2\,\frac{10^{14}\,(\mathrm{cm\,s^{-1}})^2}{(3\times 10^{24}\,\mathrm{cm})^2}\,\frac{(1.2\times 10^{28})^2\,\mathrm{eV}^2}{(3\times 10^{10})^2\,(\mathrm{cm\,s^{-1}})^2\,(200\times 10^6\times 10^{-13})\,\mathrm{eV\,cm}}\,.$$

  and then finally

$$\varepsilon_{\mathrm{c},0} = \frac{3}{8\pi}\,\frac{1.2^2}{9\times 9\times 2}\,h^2\,\frac{10^{14}\times 10^{56}\times 10^5}{10^{48}\times 10^{20}} \simeq 1.1\,h^2{\times}10^4\,\mathrm{eV\,cm^{-3}} = 11\,h^2\,\mathrm{GeV\,m^{-3}}.$$

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PHYS6005 Cosmology : Problem Sheet 6 (10 November 2016)

Note: solutions of the problems imply use of expressions and numbers given in the lectures even when not explicitly recalled

Simple cosmological models

- Knowing that the energy density of starlight radiation is negligible compared to the energy density of the CMBR, calculate the value of $\Omega_{R,0}$ in terms of h using $T = 2.725\,\mathrm{K}$ and the value of $\varepsilon_{\mathrm{c},0}$ found in the previous problem.

  Using
  $$\varepsilon_{\gamma,0} = \frac{\pi^2}{15}\,\frac{(k_B\,T_0)^4}{(\hbar\,c)^3} \simeq 0.26\,\mathrm{MeV\,m^{-3}}\,,$$
  one immediately finds
  $$\Omega_{R,0} = \frac{\varepsilon_{\gamma,0}}{\varepsilon_{c,0}} \simeq 0.3\times 10^{-4}\,h^{-2}\,.$$

- Assuming that all matter is in the form of galaxies and using as a number density for galaxies $n_G = 1\,\mathrm{Mpc^{-3}}$ and as a mean galaxy mass $M_G = 10^{12}\,M_{\odot}$, calculate $\Omega_{M,0}$ in terms of h.

  Under the given assumptions, matter energy density will be given by
  $$\varepsilon_{M,0} = n_G\,M_G\,c^2 = 10^{12}\,M_{\odot}\,c^2\,\mathrm{Mpc^{-3}}\,.$$
  The solar mass is given by $M_{\odot}\,c^2 \simeq 2\times 10^{30}\,\mathrm{kg}\,c^2 \simeq 1.1\times 10^{66}\,\mathrm{eV}$. *Remembering moreover that* $1\,\mathrm{Mpc} \simeq 3\times 10^{24}\,\mathrm{cm}$, *one finds*
  $$\varepsilon_{M,0} \simeq 4\times 10^{4}\,\mathrm{eV\,cm^{-3}}\,,$$
  and therefore
  $$\Omega_{M,0} = \frac{\varepsilon_{M,0}}{\varepsilon_{c,0}} \simeq 3.6/h^2 \simeq 7\,.$$
  Notice that this is much bigger than what is actually found because of the presence of huge voids among cluster of galaxies such that the mean galaxy density is actually much lower than $1\,\mathrm{Mpc^{-3}}$ *that is the typical value in the cluster of galaxies.*

- What is the value of the redshift $z_{\rm eq}$ at the time of the matter-radiation equality using the results of the previous problems? Give an estimate of $t_{\rm eq}$ (in years) in terms of h.

  Using
  $$z_{\rm eq} = \frac{\Omega_{M,0}}{\Omega_{R,0}} - 1 \simeq 10^5 \simeq a_{\rm eq}^{-1}\,.$$
  From this result and using the matter dominate regime behaviour
  $$a(t) = \left(\frac{t}{t_0}\right)^{\frac{2}{3}}\,,$$
  one finds
  $$t_{\rm eq} \simeq t_0\,(a_{\rm eq})^{3/2}\,.$$
  Using $t_0 \simeq (2/3)\,H_0^{-1} \simeq 9\,{\rm Gyr}$ *one finally finds*
  $$t_{\rm eq} \simeq 300\,{\rm yr}\,,$$
  much smaller than the value $t_{\rm eq} \simeq 50,000\,{\rm yr}$ *that will be found later on using more correct values and more information but still giving an initial rough estimation.*

- Consider one-fluid cosmological models with $p = w\,\varepsilon$ and $w = {\rm const}$. Find the condition on w in order to have an accelerated expansion.

  From the acceleration equation
  $$\ddot{a} = -\frac{4\,\pi\,G}{3\,c^2}\,(\varepsilon + 3\,p)\,a\,,$$
  one immediately finds the condition $w < -1/3$.

- For flat models with $p/\varepsilon = w > -1$ we have found $\varepsilon(t) \propto t^{-2}$ independently of w. This result has been derived in three steps: first finding a solution $\varepsilon(a)$ from the fluid equation, then finding a solution $a(t)$ from the Friedmann equation and then finally plugging back $a(t)$ in $\varepsilon(a)$. Show that it can be derived directly from the Friedmann equation just knowing that $a(t)$ scales like a power-law (i.e. $a(t) \propto t^n$).

  For flat models the Friedmann equation can be written as
  $$H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2 = H_0^2\,\varepsilon(a)\,.$$
  Since $a(t) \propto t^n$*, where* $n = 2/[3(1+w)]$*, then* $\dot{a}/a = d\ln a/dt \propto 1/t$ *and therefore* $\varepsilon \propto t^{-2}$ *independently of* n.

- In the introduction of the paper *On the curvature of space*, Friedman discusses two models that he claims will be derived as two special cases within his more general new class of models. What are these two models?

  The two models are the static Einstein model and the de Sitter model.

  What is the new remarkable feature that is generally present in his models and missing in these two special cases?

  The new feature is the possibility that curvature depends on time.

A special cosmological model

Consider a one-fluid model with an equation of state $p = w\,\varepsilon$ and $w = -5/3$.

- Find the solution $\varepsilon = \varepsilon(a)$;

  For a fluid with an equation of state $p = w\,\varepsilon$ and w constant we have seen that, from the fluid equation, one has

$$\varepsilon(a) = \frac{\varepsilon_0}{a^{3\,(1+w)}}\,.$$

  Therefore, simply, for $w = -5/3$ one finds $\varepsilon(a) = \varepsilon_0\,a^2$.

- Is there a time when $a(t) = 0$ (i.e. does the model have a singularity?)

  One can introduce an effective potential and this is given by

$$V(a) = -\Omega_0\;a^4\,.$$

  while the effective energy is, as usual, given by $E_0 = (1 - \Omega_0)$. Therefore, for open models with $E_0 > 0$ (i.e. $\Omega_0 < 1$) there is a singularity, while for $E_0 < 0$ there is not. Also, for flat models there is not a singularity, since $a(t) = 0$ only for $t \to -\infty$.

- Consider the flat case and define $t_0 = H_0^{-1}$; What is the value $a(t = 0)$? What is the value $a(t = 2\,t_0)$? What happens at $t > 2\,t_0$? ;

  In the flat case the Friedmann equation becomes

$$\dot{a}^2 = H_0^2\,a^4 \to \dot{a} = +H_0\,a^2\,.$$

  We can, therefore, write

$$\frac{da}{a^2} = H_0\,dt\,.$$

Integrating between 0 *and a generic time* t *we find*

$$a(0)^{-1} = H_0\, t + a(t)^{-1}\,.$$

Imposing $t_0\, H_0 = 1$, *and since* $a_0 = 1$, *one finds then* $a(0) = 1/2$. *This implies that at* $t = 2\, t_0 = 2\, H_0^{-1}$ *one has* $a(t) = \infty$.

- Give an analytical expression for $a(t)$;

  Plugging $a_0 = 1/2$ *in the previous expression one finds*

  $$a(t) = \frac{1}{2 - t\, H_0}\,.$$

  This expression summarises all results found so far. At $t \to -\infty$ *there is a singularity that, therefore, is only an asymptotic limit that can be only indefinitely approached but never realised. The interesting thing is that at* $t = 2\, H_0^{-1}$ *the Universe really blows up, there is a sort of 'wall' (in the literature this is called as the 'Big Rip'). Notice that for* $t > 2\, H_0^{-1}$ *one has unphysical negative values of* a *that should be disregarded.*

- Plot $a(t)$;

  See figure 1.

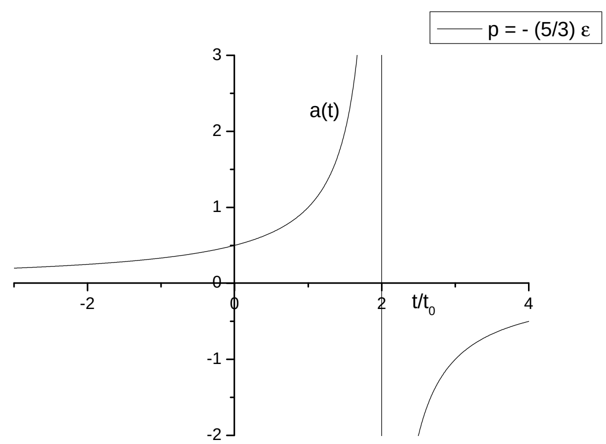

Figure 1: Plot of $a(t)$ for a model with $p = -(5/3)\,\varepsilon$. There is no singularity (it is only indefinitely approached for $t \to -\infty$) but there is a 'wall' at $t = 2\,t_0$ where the scale factor $a \to \infty$: the Universe seems to blow up there. Negative values are unphysical, they are shown just for completeness.

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PHYS6005 Cosmology : Problem Sheet 7 (17 November 2016)

Note: solutions of the problems imply use of expressions and numbers given in the lectures even when not explicitly recalled

On the cosmological constant

1. What is the dimensionality of the cosmological constant Λ ?

   It can be seen in different ways, for example very simply, from the Friedmann equation (7.7), one has $[\Lambda] = [H^2] = [t^{-2}]$.

2. Find the value of the matter energy density ε_M (in $\mathrm{MeV/m^3}$) for an Einstein Universe with radius $R_E = 1\,\mathrm{Gpc}$.

   From the equation for the Einstein radius (cf. eq. (7.15)) one finds

$$\varepsilon_M = \frac{M_{\mathrm{P}}^2\, c^4}{4\,\pi\,\hbar\, c\, R_E^2}\,, \tag{18}$$

   where we used the eq. (2.10) to express G *in terms of* M_P. *Using the eqs. (2.10), (2.8) and (2.3) one finds then*

$$\varepsilon_M \simeq 64,000\,\mathrm{MeV}\,m^{-3}\,. \tag{19}$$

3. Find the corresponding value of the cosmological constant Λ (in yr^{-2}).

   From the eq. (7.15) one simply finds

$$\Lambda = \frac{c^2}{R_E^2} \simeq 10^{-19}\,\mathrm{yr}^{-2}\,. \tag{20}$$

On Lemaitre models

1. Consider the expression

$$V(a) = V_{RM\Lambda}(a) \equiv -a^2\left(\frac{\Omega_{R,0}}{a^4} + \frac{\Omega_{M,0}}{a^3} + \Omega_{\Lambda,0}\right) \tag{21}$$

for the effective potential in a Lemaitre model with an admixture of radiation, matter and a fluid with $p_\Lambda = -\varepsilon_\Lambda$ corresponding to a cosmological constant in the Einstein equations. Find an expression for $V_{\rm max}$ and for a_E.

The peak marks a transition between a regime where the dark energy contribution is negligible to a regime where dark energy dominates. There are two possibilities: either there is a matter-dominated regime and in this case the Λ-dominate regime takes place directly after the radiation dominated regime and there the Universe never experiences a matter-dominated regime or there is a matter dominated regime between the matter and the Λ dominated regime. As discussed in class, this second option is verified under the condition

$$\Omega_{M,0} \gtrsim \Omega_{R,0}^{3/4}\,\Omega_{\Lambda,0}^{1/4}$$

that is very well verified considering that $\Omega_{R,0} \simeq 10^{-4}$ and that $\Omega_{M,0} \sim \Omega_{\Lambda,0}$.

Let us then start considering this much more plausible situation. In this case the radiation term can be neglected and imposing $V'(a) = 0$ one finds

$$a_E = \left(\frac{\Omega_{M,0}}{2\,\Omega_{\Lambda,0}}\right)^{1/3} .$$

The maximum of the potential is then given by

$$V_{\rm max} = V(a_E) = -3\,a_E^2\,\Omega_{\Lambda,0}\,.$$

(observation) One can verify that if one imposes that the Universe 'stops' on the top of the potential, then $a_E = 1$ ($R_{\rm peak} = R_0 = const$) and one recovers all the expressions found in the Einstein's model.

Let us, for completeness, also consider the other option (excluded by the observations !) where $\Omega_{M,0} < \Omega_{R,0}^{3/4}\,\Omega_{\Lambda,0}^{1/4}$ so that the matter dominated regime never occurs. In this case one can neglect the matter contribution and repeating the procedure one first finds

$$a_E = \left(\frac{\Omega_{R,0}}{\Omega_{\Lambda,0}}\right)^{1/4} .$$

and then

$$V_{\rm max} = V(a_E) = -2\,a_E^2\,\Omega_{\Lambda,0}\,.$$

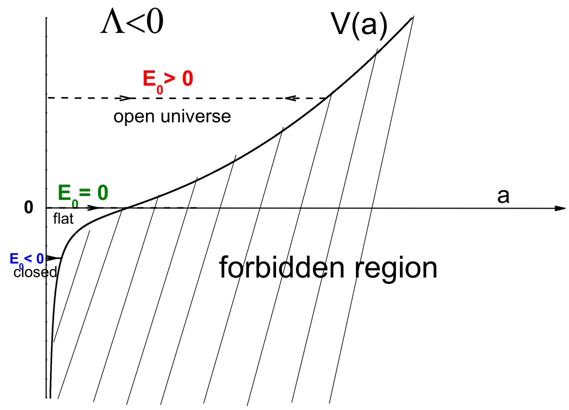

Figure 2: Effective potential for a Lemaitre model with a negative cosmological constant. The fate of the Universe is to start collapsing at some point, independently of the geometry given by the sign of E_0.

2. What kind of cosmological models would one obtain if instead of a positive cosmological constant term one would consider a negative one? Would models with a big bounce be possible in this case? Describe the situation graphically showing the dynamics in a plot where the scale factor is on the x axis and the effective energy on the y-axis.

   The potential would not have a peak but would grow monotonically to infinite (see figure). Only re-collapsing models would be possible (the trajectory unavoidably hits the potential at some point) and there cannot be a 'big bounce'.

The age of the Universe

1. Solve the integral

$$t_0\, H_0 = f_M(\Omega_0) \equiv \int_0^1 \frac{da}{\sqrt{\frac{\Omega_0}{a} + (1-\Omega_0)}}\,,$$

that gives the age of the Universe for a matter Universe with arbitrary Ω_0, deriving the analytic expressions

$$f_M(\Omega_0 < 1) = \frac{1}{1-\Omega_0} - \frac{\Omega_0 \, \mathrm{arcsinh}\left[\sqrt{\frac{1-\Omega_0}{\Omega_0}}\right]}{(1-\Omega_0)^{3/2}}$$

and

$$f_M(\Omega_0 > 1) = \frac{\Omega_0 \arcsin\left[\sqrt{\frac{\Omega_0-1}{\Omega_0}}\right]}{(\Omega_0-1)^{3/2}} - \frac{1}{\Omega_0 - 1} \, .$$

The integral

$$t_0 \, H_0 = f_M(\Omega_0) \equiv \int_0^1 \frac{da}{\sqrt{\frac{\Omega_0}{a} + (1-\Omega_0)}} \tag{22}$$

can be solved analogously to the integral for $f_{M\Lambda}(\Omega_0 = 1)$*. Let us start from the case* $\Omega_0 < 1$*. The first step is the change of variable* $a = x^2$ *so that the integral becomes*

$$f_M(\Omega_0) = \frac{2}{\sqrt{\Omega_0}} \int_0^1 \frac{x^2 \, dx}{\sqrt{1 + x^2 \, \frac{1-\Omega_0}{\Omega_0}}}$$

The second step is the change of variable

$$y^2 = x^2 \, \frac{1-\Omega_0}{\Omega_0}$$

and in this way one obtains

$$f_M(\Omega_0) = \frac{2}{\sqrt{\Omega_0}} \left(\frac{\Omega_0}{1-\Omega_0}\right)^{\frac{3}{2}} \int_0^{\sqrt{\frac{1-\Omega_0}{\Omega_0}}} \frac{y^2 \, dy}{\sqrt{1+y^2}} \, .$$

The last change of variable is $y = \sinh\theta$ *so that*

$$f_M(\Omega_0) = \frac{2}{\sqrt{\Omega_0}} \left(\frac{\Omega_0}{1-\Omega_0}\right)^{\frac{3}{2}} \int_0^{\sinh^{-1}\left(\sqrt{\frac{1-\Omega_0}{\Omega_0}}\right)} \sinh^2\theta \, d\theta \, .$$

This integral can be solved noticing that

$$\sinh^2\theta = \frac{1}{2} \cosh(2\theta) - \frac{1}{2} \, .$$

In this way the integral splits into two pieces. The second is trivially solved. The first integral can be also solved noticing that $\cosh(2\theta)d\theta = d[\sinh(2\theta)]/2$ *and that* $\sinh[2\sinh^{-1}\sqrt{(1-\Omega_0)/\Omega_0}] = 2\sqrt{1-\Omega_0}/\Omega_0$. *In this way one easily arrives to the solution.*

The solution for $\Omega_0 > 1$ *can be obtained in an analogous way simply that this time the changes of variables to be done are*

$$y^2 = x^2\,\frac{\Omega_0 - 1}{\Omega_0}$$

and $y = \sin\theta$.

2. Consider a flat Universe with matter and Λ. What is the value of $\Omega_{\Lambda,0}$ corresponding to $t_0 = H_0^{-1}$ (two significant digits).
(Hint: You can find it with an analytical procedure by iterations starting from an initial approximate value that you can read out from figure 8.2.)

   We have to impose

$$f_{M\Lambda}(\Omega_{\Lambda,0}) = \frac{2}{3\sqrt{\Omega_{\Lambda,0}}}\,\ln\left[\frac{1+\sqrt{\Omega_{\Lambda,0}}}{\sqrt{1-\Omega_{\Lambda,0}}}\right] = 1\,.$$

   This is also equivalent to solve

$$\ln\left(\frac{1+x}{1-x}\right) = 3\,x\,,$$

   where $x = \sqrt{\Omega_{\Lambda,0}}$ *and one finds* $\Omega_{\Lambda,0} = 0.74$. *This is exactly the value that is found by the observations as we will see. Therefore, our Universe at present, seems to realise quite exactly (but accidentally !) the condition* $t_0 = H_0^{-1}$.

3. Consider an expanding, positively curved Universe containing only a cosmological constant ($\Omega_0 = \Omega_{\Lambda,0} > 1$). Show that such a Universe underwent a "Big Bounce" at a value of the scale factor given by

$$a_{\rm bounce} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2}\,. \qquad (23)$$

   Show that in this model the scale factor depends on time as

$$a(t) = a_{\rm bounce}\cosh[\sqrt{\Omega_0}\,H_0\,(t - t_{\rm bounce})]\,, \qquad (24)$$

where $t_{\rm bounce}$ is the time at which the Big Bounce occurred. What is the time elapsed since the Big Bounce, $t_0 - t_{\rm bounce}$, expressed in terms of Ω_0 and H_0?

The Friedmann equation specialises in this case into

$$\dot{a}^2(t) = H_0^2\, a^2(t)\, \Omega_{\Lambda,0} + H_0^2\,(1 - \Omega_0)\,.$$

The condition for the bounce is $\dot{a} = 0$ and therefore from the Friedmann equation

$$a^2_{\rm bounce}\, \Omega_{\Lambda,0} = \Omega_{\Lambda,0} - 1$$

implying

$$a_{\rm bounce} = \left(\frac{\Omega_0 - 1}{\Omega_0}\right)^{1/2}\,.$$

We can now go back to the Friedmann equation and recast it in the new variable

$$x = \frac{a}{a_{\rm bound}}\,,$$

so that

$$\frac{\dot{x}^2}{\Omega_0\, H_0^2} = x^2 - 1$$

and from this expression we obtain

$$\sqrt{\Omega_0}\, H_0\, dt = \frac{dx}{\sqrt{x^2 - 1}}\,.$$

Integrating from $x' = 1$ $(a = a_{\rm bounce})$ to x one obtains

$$\sqrt{\Omega_0}\, H_0\,(t - t_{\rm bounce}) = \text{arccosh}(x)\,,$$

and inverting the hyperbolic function one obtains the final result

$$a(t) = a_{\rm bounce}\cosh[\sqrt{\Omega_0}\, H_0\,(t - t_{\rm bounce})]\,.$$

Notice that the time elapsed from the bounce till the present can be expressed as

$$t_0 - t_{\rm bounce} = \frac{\text{arccosh}(\sqrt{\frac{\Omega_0}{\Omega_0 - 1}})}{\sqrt{\Omega_0}\, H_0}\,.$$

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PHYS6005 Cosmology : Problem Sheet 8 (24 November 2016)

Note: solutions of the problems imply use of expressions and numbers given in the lectures even when not explicitly recalled

A) Horizon distance and cosmological redshifts

1. What is the asymptotic limit for ω such that the horizon distance for a fluid with $p = \omega\varepsilon$ (see Eq. (9.6)) is given by $d_H(t_0) = c\,t_0$?

   The horizon distance for a fluid with $p = \omega\varepsilon$ is given by the expression

$$d_H(t_0) = \frac{3(1+\omega)}{(1+3\,\omega)}\,c\,t_0\,.$$

   The only possibility to have $d_H(t_0) = c\,t_0$ is by taking the asymptotic limit $|\omega| \to \infty$.

2. Can you give the physical reason why for $-1 > w > -1/3$ (accelerating models with a singularity) the horizon distance (Eq. (9.6)) is infinite despite the fact that t_0 is finite?

   If we inspect the integral

$$d_{H,0} = c\int_0^1 \frac{da}{a^2\,H}\,,$$

   from a mathematical point of view there is a divergence if for $a \to 0$ one has $H^{-1}/a \to \infty$ (or constant). This is equivalent to say that $a\,H \to 0$ for $a \to 0$ or equivalently that $a\,H$ increases with the scale factor or with time. But $a\,H$ is nothing else that $\dot{a}$ and, therefore, it increases with time if the expansion accelerates. What does that mean physically? The quantity $c\,H^{-1}/a$ is $R_H^{(0)}$, the Hubble radius in comoving scale. If the Hubble radius goes to infinite for $a \to 0$ this means that any co-moving scale (e.g. the distance between two points at rest in the co-moving system) sufficiently back in past, i.e. for a sufficiently small a, was within the Hubble radius. But this means that the expansion velocity between these two points was slower than the speed of light and this implies that a signal sent before the scale went outside the Hubble radius, could have causally connected the two points. Of course notice *that here we are calculating the integral starting from the singularity while, more realistically, we should start from some initial value $a_{\rm in}$. This means that an initial acceleration stage with a sufficiently small $a_{\rm in}$ might have produced a horizon much bigger than our observable Universe. This result will be used when we will discuss Inflation solution to the horizon problem.*

3. Derive the Eq. (9.26) starting from the Eq. (9.25).

   A first (slower) procedure is to solve the 2nd order algebraic equation and Taylor expand the square root at second order in z^2. However a faster procedure is possible considering that if $z \to 0$ then $(t_0 - t_{\rm em}) \to 0$ and therefore the solution has to be necessarily of the form $t_0 - t_{\rm em} = A\,z + B\,z^2 + \mathcal{O}(z^3)$ implying $(t_0 - t_{\rm em})^2 = A^2\,z^2 + \mathcal{O}(z^3)$. From the (9.22) one can then first write simply

$$t_0 - t_{\rm em} = z\,H_0^{-1} - (t_0 - t_{\rm em})^2\,\left(1 + \frac{q_0}{2}\right)\,H_0$$

   implying $A = H_0^{-1}$ and $B = -A^2\,(1 + q_0/2)\,H_0$ so that in conclusion the eq.(9.23) follows,

$$t_0 - t_{\rm em} = z\,H_0^{-1} - z^2\,\left(1 + \frac{q_0}{2}\right)\,H_0^{-1} + \mathcal{O}(z^3)\,.$$

4. Derive the Eq. (9.27) starting from the Eq. (9.26).

   The proper distance at present of an object that emitted light at the time $t_{\rm em}$ is expressed in terms of the lookback time as

$$d_{\rm pr}(t_0) = c\int_{t_{\rm em}}^{t_0}\frac{dt}{a(t)} = c\,(t_0 - t_{\rm em}) + \frac{1}{2}\,c\,H_0\,(t_0 - t_{\rm em})^2 + \mathcal{O}[(t_0 - t_{\rm em})^3]\,.$$

   On the other hand the lookback time is expressed in terms of the redshift as

$$t_0 - t_{\rm em} = H_0^{-1}\,\left[z - z^2\,\left(1 + \frac{q_0}{2}\right)\right] + \mathcal{O}(z^3)\,.$$

   Replacing this in the previous expression, one finds

$$d_{\rm pr}(t_0) = c\,H_0^{-1}\,\left[z - z^2\,\left(1 + \frac{q_0}{2}\right)\right] + \frac{1}{2}\,c\,H_0^{-1}\,\left[z - z^2\,\left(1 + \frac{q_0}{2}\right)\right]^2 + \mathcal{O}(z^3)\,.$$

Grouping terms of the same order one finally finds

$$d_{\rm pr}(t_0) = c\,H_0^{-1}\left[z - z^2\left(\frac{1+q_0}{2}\right)\right] + \mathcal{O}(z^3)\,.$$

5. Given $q_0 = -0.55$ and $H_0 = 70\,{\rm km\,s^{-1}\,Mpc^{-1}}$, what is the value of z such that $d_L = 1\,{\rm Gpc}$?

   We have to impose

   $$d_L(t_0) = 1{\rm Gpc}\,,$$

   including the order z^2 term and therefore

   $$c\,H_0^{-1}\left[z + z^2\left(\frac{1-q_0}{2}\right)\right] = 1\,{\rm Gpc}\,.$$

   Using $q_0 = -0.6$ and $c\,H_0^{-1} \simeq 4\,{\rm Gpc}$, one finds

   $$3.2\,z^2 + 4\,z - 1 = 0\,,$$

   from where $z \simeq (-2+\sqrt{7.2})/3 \simeq 0.21$. Notice that by definition $z \geq 0$ (negative redshift values would be obtained for objects that emit light in the future and are observed now, something that would obviously violate causality). Therefore the negative solution has to be disregarded as non-physical.

B) Observational parameters

1. Consider the two relations Eq. (9.36) and Eq. (9.38) and find the values for $\Omega_{M,0}, \Omega_{\Lambda,0}$ and q_0.

   $\Omega_{M,0} \simeq 0.3, \Omega_{\Lambda,0} \simeq 0.7,$ *and* $q_0 \simeq -0.55$.

2. Suppose to be in a spatially flat universe containing a single component with an equation of state $p = w\,\varepsilon$. What are the current proper distance $d_{\rm pr,0}$ and the luminosity distance d_L as a function of z and w?

   In this case one has

   $$q_0 = \frac{1}{2}\,(1+3\omega)\,.$$

   Plugging this expression in the expressions for the proper distance and the luminosity distance one finds

   $$d_{\rm pr}(t_0) = c\,H_0^{-1}\left[z - \frac{3}{4}\,z^2\,(1+\omega)\right] + \mathcal{O}(z^3)$$

and

$$d_L(t_0) = c\, H_0^{-1}\left[z + z^2\left(\frac{1-3\,\omega}{4}\right)\right] + \mathcal{O}(z^3)\,.$$

3. Let us go back to the Milne-Mc Crea model and inspect in more detail whether it can successfully reproduce the observed cosmological redshifts. Since we observe cosmological redshifts $z \gtrsim 1$, we know that a non-relativistic Doppler effect is ruled out. However, one can still wonder whether a relativistic Doppler effect can successfully reproduce the correct relation between redshift and luminosity distance (Eq. (6.27) on the notes). Show that this relation is in disagreement with the measurement $q_0 \simeq -0.55$.

   In the MM model one would expect to find exactly

   $$z = \frac{H_0\, d}{c}\,,$$

   where z is the cosmological redshift of the source and d its distance (in the MM model one has the usual Euclidean static space and the speed of light is infinite. Therefore, all definition of distances coincide. The cosmological redshift is given by the simple non-relativistic Doppler effect, $z = v/c$, where v is the velocity of the source.

   However, since the model is strictly non-relativistic, in this way one would predict $z \ll 1$ in contradiction with the observation of sources of cosmological redshifts $z \gtrsim 1$.

   One could think to rescue the MM model simply replacing the non-relativistic Doppler effect with the relativistic Doppler effect. In this case we should use the equation (2.17),

   $$z = z(\beta) \equiv \sqrt{\frac{1+\beta}{1-\beta}} - 1\,, \tag{25}$$

   where $\beta \equiv v/c$ and now v (and not z) would respect exactly the Hubble's law in a way that $v = H_0\, d/c$. Now, with this new equation, any arbitrarily large value of z is possible.

   However, as we have seen, one has still to check whether this equation can successfully reproduce the experimental results that, as we have *seen, find that the distance-redshift relation is given, for $z \lesssim 1$, by the equation (9.27),*

$$d(z) = c\, H_0^{-1} \left[z + z^2 \frac{1 - q_0}{2} \right] + \mathcal{O}(z^3)\,,$$

with $q_0 \simeq -0.55$.

We have, therefore, to check whether, expanding first at the second order in β^2 the relation $z(\beta)$ in the equation (25) and then inverting it in a way to get an expression $d(z)$, the 'effective' value of q_0 could be compatible with the experimental one.

An (Taylor) expansion at the second order of $z(\beta)$ gives

$$z(\beta) \simeq \beta + \frac{3}{2}\,\beta^2\,. \tag{26}$$

Inverting this relation (similarly to what we did when we found the lookback time as a function of z) one finds

$$d = c\, H_0^{-1} \left[z - \frac{3}{2}\, z^2 \right]\,, \tag{27}$$

that, compared with the definition of q_0, gives $q_0 = 4$, incompatible with the experimental value $q_0 = -0.55$. Therefore, the MM model, even with this relativistic extension, cannot explain the experimental observations.

C) Expansion in the ΛCDM model

1. Derive the value for the matter-radiation equality time (1 significant digit)

$$t_{\rm eq}^{RM} \simeq 30,000\,{\rm yr}\,.$$

This problem extends and improves the rough estimation of $t_{\rm eq}^{RM} \sim \mathcal{O}(100)$yr made in the ps n. 5 within a simple matter-radiation dominated model. One cannot apply straightforwardly the matter-dominated expression for the scale factor since one has to take into account that the last phase of the Universe expansion is Λ-dominated. Therefore, while in the problem ps n.6 we started from,

$$a_{\rm eq}^{RM} = a_0 \left(\frac{t_{\rm eq}^{\rm RM}}{t_0} \right)^{\frac{2}{3}}\,, \tag{28}$$

this time we have to start from

$$a_{\rm eq}^{RM} = a_{\rm eq}^{M\Lambda} \left(\frac{t_{\rm eq}^{RM}}{t_{\rm eq}^{M\Lambda}} \right)^{\frac{2}{3}} . \tag{29}$$

In this way one easily finds the expression

$$t_{\rm eq}^{RM} = t_{\rm eq}^{M\Lambda} \left(\frac{a_{\rm eq}^{RM}}{a_{\rm eq}^{M\Lambda}} \right)^{\frac{3}{2}} . \tag{30}$$

Using now (see the notes) the results $a_{\rm eq}^{M\Lambda} \simeq 0.7$, $t_{\rm eq}^{M\Lambda} \simeq 10\,{\rm Gyr}$ *and* $a_{\rm eq}^{RM} \simeq 2 \times 10^{-4}$, *one finds finally*

$$t_{\rm eq}^{RM} \simeq 30,000\,{\rm yr} . \tag{31}$$

2. Given the measured values for the parameters of the ΛCDM model, how long will it take for the scale factor to increase of a factor 4 from now?

The observations support a cosmological model, the Λ*CDM model, where the Universe expansion entered a de Sitter expansion stage at* $t = t_{\rm eq}^{M\Lambda} \simeq 10\,{\rm Gyr}$. *Therefore, the scale factor scales like*

$$a(t) = a_{\rm eq}^{M\Lambda}\, e^{H_{\rm eq}^{M\Lambda}(t - t_{\rm eq}^{M\Lambda})} ,$$

where $H_{\rm eq}^{M\Lambda}$ *is the Hubble parameter at the matter-*Λ *equality time and is constant and therefore* $H_{\rm eq}^{M\Lambda} = H_0$. *The scale factor evolution can be therefore easily recast in terms of quantities at present as*

$$a(t) = a_0\, e^{H_0\,(t - t_0)} = e^{H_0\,(t - t_0)} .$$

Hence, imposing

$$a(t_{\rm future}) = 4 ,$$

and using $H_0^{-1} = 12.8\,{\rm Gyr}$ *(cf. eq. (6.56)) one immediately finds*

$$t_{\rm future} - t_0 = H_0^{-1}\, \ln 4 \simeq 18\,{\rm Gyr} .$$

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PHYS6005 Cosmology : Problem Sheet 9 (1 December 2016)

Note: solutions of the problems imply use of expressions and numbers given in the lectures even when not explicitly recalled

A) Dark Matter

1. Derive the numerical expression (Eq. (10.16) in the notes):

$$M_G(R) = \frac{v^2\,R}{G} \simeq 10^{12}\,M_\odot\,\left(\frac{v}{220\,\mathrm{km}\,s^{-1}}\right)^2\,\left(\frac{R}{100\,\mathrm{kpc}}\right) \qquad (R \lesssim R_{\rm halo})\,.$$

2. Derive (order-of-magnitude wise) the expression (Eq. (10.32) in the notes)

$$n_{\rm DM,0} \sim 1\,\mathrm{m}^{-3}\,\left(\frac{\rm GeV}{M_{\rm DM}}\right) \sim 10^{-9}\,n_{\gamma,0}\,\left(\frac{\rm GeV}{M_{\rm DM}}\right)\,.$$

From the definition of Ω_{DM} *we can write*

$$\Omega_{DM,0} = \frac{\varepsilon_{DM,0}}{\varepsilon_{c,0}} = \frac{M_{DM}\,n_{DM,0}}{\varepsilon_{c,0}}\,.$$

Therefore this gives for the ratio $n_{DM,0}/n_{\gamma,0}$*, the quantity we want to calculate, and of* $\Omega_{DM,0}\,h^2$*, the quantity we measure at present, as*

$$\frac{n_{DM,0}}{n_{\gamma,0}} = \frac{\varepsilon_{c,0}}{h^2\,n_{\gamma,0}}\,\frac{\Omega_{DM,0}\,h^2}{M_{DM}}\,.$$

We can now use $n_{\gamma,0} \simeq 400\,\mathrm{cm}^{-3}$*,* $\varepsilon_{c,0}\,h^{-2} \simeq 10^4\,\mathrm{eV}\,\mathrm{cm}^{-3}$ *and* $\Omega_{DM,0}\,h^2 \simeq 0.1$*, to find finally (natural system)*

$$\frac{n_{DM,0}}{n_{\gamma,0}} \simeq 10^{-9}\,\frac{\rm GeV}{M_{DM}}\,.$$

3. Suppose that primordial black holes of mass $10^{-8}\,M_\odot$ make up all the dark matter in the halo of our galaxy. How far away would you expect the nearest such black hole to be?

We have first of all to express the mass of a black hole in eV's. Remember that

$$1\,M_\odot \simeq 2\times 10^{30}\,\mathrm{kg}\,.$$

We need now to convert kg in eV. This can be done just using the equation (2.31) or if one wants to derive it one should first remember that $1eV \simeq 1.6\times 10^{-19}\,J$. and then that $1\,J = kg\,m^2\,s^{-2}$. It then follows that

$$1\,\mathrm{kg} \simeq \frac{1}{1.6}\,10^{19}\,(m/s)^{-2}\,\mathrm{eV}.$$

In the natural system one has $c = 3\times 10^8\,(m/s) = 1$ and therefore one finds

$$1\,kg = \frac{9}{1.6}\,10^{35}\,\mathrm{eV} \simeq 6\times 10^{35}\,\mathrm{eV}\,.$$

Therefore finally the mass of a Black Hole in eV is given by

$$M_{BH} \simeq 1.2\times 10^{58}\,\mathrm{eV} \sim 10^{49}\,\mathrm{GeV}\,.$$

We can now use the expression found in the previous problem in order to calculate $n_{DM,0}$.

$$n_{BH,0} \simeq 10^{-9}\,n_{\gamma,0}\,10^{-49} \simeq 4\times 10^{-56}\,\mathrm{cm}^{-3}\,.$$

This means that the mean distance between two black-holes would be given by (order-of-magnitude) $n_{BH,0}^{-1/3} \simeq 0.3\times 10^{19}\,\mathrm{cm} \sim 1\,\mathrm{pc}$ and this also means that we should expect to find such a black hole within 1pc *from us. Notice that this calculation assumes that the black holes density is homogeneous but in reality Dark Matter partially clumps in galactic halos and therefore one can expect higher densities and lower intra-distance in this case.*

4. How frequently would you expect such a black hole to pass within $1AU$ of the Sun?

   The flux of black holes across our Solar System is given by $\Phi_{BH} = n_{BH}\,v$, where v is the relative speed between the solar system and the black hole. Since the mass of a black hole is huge, the speed of a black hole can be neglected because due to the momentum redshift the black hole lost all of his kinetic energy during the expansion. The speed of the solar system is on the other hand given by $\sim 100km/s$ (order-of-magnitude). The number of black holes per unit of time (frequency) *'colliding' with a sphere of* $1AU$ *centered around the Sun would be therefore given by*

$$\nu_{BH} = n_{DM}\, v\, \pi\, (1AU)^2 \simeq 3 \times 10^{-22}\,\mathrm{s}^{-1} \sim 10^{-14}\,\mathrm{yr}^{-1}.$$

5. Read the paper *Existence and Nature of Dark Matter in the Universe* by V. Trimble and answer the following questions. What are the commonly used probes to show the extension of the Milky Way halo?

   Commonly used probes include RR Lyraes and other field stars, globular clusters and satellite galaxies

   Why are Dwarf Galaxies particularly interesting?

   Because they have the potential for telling us the smallest configuration that can have a dark halo and thereby constraining the minimum particle mass possible in the halos.

   $4 \times 10^{11}\, M_{\odot}$

   Why is the distinction between Hot Dark Matter and Cold Dark Matter important?

   because they give rise to two different scenarios of galaxy formation. In the HDM case (for $M_D M \simeq (10-100)\,\mathrm{eV}$*) super-clusters form first and later fragment into galaxies. In the CDM case galaxies or smaller structures form first and then larger structures build up by gravitational clustering.*

   What is the problem of a Universe with a pure Hot Dark Matter component?

   Galaxies form too late ($z \lesssim 3$) *than observed.*

B) Baryon-to-photon number ratio

Show that the baryon-to-photon number ratio at temperatures $k_B T \ll m_e c^2 \simeq 0.5\,\mathrm{MeV}$ can be expressed as $\eta_{B,0} \simeq 273.6\,\Omega_{B,0}\, h^2 \times 10^{-10}$ (see Eq. (11.11) in the notes), plugging the proper numbers for the involved quantities ($m_p, n_{\gamma,0}, \varepsilon_{\mathrm{c},0}$).

The baryon contribution to the energy density at the present time is in the form of nucleons. Using that the proton and neutron masses are approximately equal, the baryon energy density at the present time can be written as

$\varepsilon_{B,0} = m_N \, n_{B,0}$, where $n_{B,0}$ is the number density of baryons at the present time.

Therefore, using using $\Omega_{B,0} \equiv \varepsilon_{B,0}/\varepsilon_{\rm c,0}$, one can write

$$\eta_{B,0} \simeq \frac{n_{B,0}}{n_{\gamma,0}} = \frac{\varepsilon_{B,0}}{m_N \, n_{\gamma,0}} = \frac{\Omega_{B,0} \, \varepsilon_{\rm c,0}}{m_N \, n_{\gamma,0}} \, .$$

Finally, plugging precise numerical values for the various quantities given ($\varepsilon_{c,0} \simeq 1.1 \, h^2 \times 10^4 \, {\rm eV \, cm^{-3}}$, $m_p \simeq 938 \, {\rm MeV}$, $n_{\gamma,0} \simeq 411 \, {\rm cm^{-3}}$), one finds

$$\eta_{B,0} \simeq \frac{1.1 \, {\rm eV} \times 10^4 \, {\rm cm^{-3}}}{938 \times 10^6 \, {\rm eV} \; \times \; 411 \, {\rm cm^{-3}}} \, \Omega_{B,0} \, h^2 \simeq 285 \times 10^{-10} \, \Omega_{B,0} \, h^2 \, .$$

If one uses even more precise (4 significant figures) values, ($\varepsilon_{c,0} \simeq 1.054 \, h^2 \times 10^4 \, {\rm eV \, cm^{-3}}$, $m_p \simeq 938.3 \, {\rm MeV}$, $n_{\gamma,0} \simeq 410.44 \, {\rm cm^{-3}}$), one finds more precisely $\eta_{B,0} \simeq 273.6 \times 10^{-10} \, \Omega_{B,0} \, h^2$, as in the notes.

UNIVERSITY OF SOUTHAMPTON
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PHYS6005 Cosmology : Problem Sheet 10 (8 December 2016)

Note: solutions of the problems imply use of expressions and values of different quantities given in the lecture notes even when not explicitly recalled

A) Matter-radiation decoupling and recombination

1. Derive the expression

$$t_{\rm dec} = \frac{t_{\rm eq}^{M\Lambda}}{(a_{\rm eq}^{M\Lambda}\, z_{\rm dec})^{\frac{3}{2}}} \simeq 400,000\,{\rm yr}\,,$$

including the numerical estimation.

Since the decoupling occurs after the matter-radiation equality time, one can use the matter-dominated expression for $a(t)$ with $t = t_{\rm dec}$

$$a(t_{\rm dec}) = a_{\rm eq}^{M\Lambda}\left(\frac{t_{\rm dec}}{t_{\rm eq}^{M\Lambda}}\right)^{\frac{2}{3}}$$

and from this one finds easily the result

$$t_{\rm dec} = \frac{t_{\rm eq}^{M\Lambda}}{(a_{\rm eq}^{M\Lambda}\, z_{\rm dec})^{\frac{3}{2}}} \simeq 400,000\,{\rm yr}\,.$$

Notice that instead of starting from $t_{\rm eq}^{M\Lambda}$ and then scale back to $t_{\rm dec}$, one can also start from $t_{\rm eq}$ and then scale forward to $t_{\rm dec}$, obtaining

$$t_{\rm dec} = t_{\rm eq}^{MR}\left(\frac{a_{\rm dec}}{a_{\rm eq}^{RM}}\right)^{\frac{3}{2}} \simeq \frac{t_{\rm eq}^{MR}}{(a_{\rm eq}^{RM}\, z_{\rm dec})^{\frac{3}{2}}}\,,$$

and using $t_{\rm eq}^{RM} \simeq 30,000\,{\rm yr}$, $a_{\rm eq}^{RM} \simeq 1,6 \times 10^{-4}$ (remember that we are not accounting for neutrinos in the radiation component) and $z_{\rm dec} \simeq 1130$ one finds numerically again $t_{\rm dec} \simeq 400,000\,{\rm yr}$. These are both approximate expressions and one cannot go beyond 1 significant figure. In order to get an exact expression for $t_{\rm dec}$, one should basically use the general integral expression (8.13) with $\Omega_0 = 1$. From this expression, neglecting $t_{\rm eq}$, $\Omega_{R,0}$ and $\Omega_{\Lambda,0}$, one can also easily recover the two approximate expressions for $t_{\rm dec}$.

2. How many Thomson scatterings per second occur in the early Universe when $k_B T \simeq 1\,\mathrm{eV}$.

   First of all let us see how many scatterings per photon per second occur when $k_B T \simeq 1\,\mathrm{eV}$. As we have seen the mean free time is given by

   $$\tau = \tau_0\, a^3\,, \quad \text{where} \;\; \tau_0 = 0.2 \times 10^{21}\,\mathrm{s}\,.$$

   When $k_B T \simeq 1\,\mathrm{eV}$ one has $a \simeq (k_B T_0)/1\,\mathrm{eV} \simeq 2 \times 10^{-4}$ and therefore for the interaction rate $\Gamma = \tau^{-1}$ one finds

   $$\Gamma|_{k_B T \simeq 1\,\mathrm{eV}} \simeq 0.6 \times 10^{-9}\, s^{-1}\,.$$

   This is the number of Thomson scatterings per photon per second. One can then also calculate the number of Thomson scatterings per second per unit of volume just multiplying the rate times the photon density $n_\gamma = n_{\gamma,0}\, a^{-3} \simeq 0.5 \times 10^{14}\,\mathrm{cm}^{-3}$. Therefore, if we calculate

   $$(n_\gamma\, \Gamma)|_{k_B T \simeq 1\,\mathrm{eV}} = n_{\gamma,0}\, \Gamma_0\, a^{-6} \simeq 0.3 \times 10^5\,\mathrm{cm}^{-3}\, s^{-1}\,,$$

   gives the number of Thomson scatterings per unit of volume per second.

B) CMB temperature fluctuations

What would be the value of l_1 (the position of the first peak) assuming $\Omega_{R,0} \simeq 5 \times 10^{-3}$, i.e about 100 times greater than the measured value? Assume that this change in $\Omega_{R,0}$ does not affect the value of $z_{\mathrm{dec}} \simeq 1100$.

If $\Omega_{R,0} \simeq 5 \times 10^{-3}$, the value of the scale factor at the matter-radiation equality would be given by

$$a_{\mathrm{eq}}^{\mathrm{RM}} = \frac{\Omega_{R,0}}{\Omega_{M,0}} \simeq 0.02 \simeq 23\, a_{\mathrm{dec}}\,,$$

in a way that decoupling would now occur during the radiation-dominated regime instead than during the matter-dominated regime (having assumed that the value of a_{dec} is not affected).

We have then to re-calculate the value of $l_1 = \pi/\delta\theta$. Since

$$\delta\theta = \frac{d_{\mathrm{sh}}^{(0)}(t_{\mathrm{dec}})}{d_H(t_0)}\,,$$

we have to re-calculate the (comoving) sound-horizon size at decoupling, $d_{\mathrm{sh}}^{(0)}(t_{\mathrm{dec}})$, in the RD regime.

In this case one has to use the eq. (11.22) for the Hubble rate neglecting $\Omega_{M,0}$ *finding*

$$d_{\rm sh}^{(0)}(t_{\rm dec}) = \frac{c_s\, H_0^{-1}\, a_{\rm dec}}{\sqrt{\Omega_{R,0}}}\,.$$

The horizon at the present time can be still calculated using the eq. (11..22) for the Hubble rate but this time neglecting $\Omega_{R,0}$ *and obtaining*

$$d_H(t_0) = \frac{2\,c\,H_0^{-1}}{\sqrt{\Omega_{M,0}}}\,.$$

It can be now noticed that in the notes, where we were using that decoupling occurs in the matter-dominated regime, we could use $\Omega_{M,0} = 1$ *without worrying of the exact value of* $\Omega_{M,0} \simeq 0.265$*, since this is not entering in the final result that, as we can see now, it would be independent on the value of* $\Omega_{M,0}$ *since this would appear both in the numerator and in the denominator. However, now,* $\Omega_{M,0}$ *does not cancel out since we have* $\Omega_{R,0}$ *in the numerator instead of* $\Omega_{M,0}$*.*

From these two equations one then arrives to

$$l_1^{RD} \simeq 2\sqrt{\frac{a_{eq}^{MR}}{a_{\rm dec}}}\, l_1^{MD}\,,$$

where we are indicating with l_1^{MD} *the value obtained in the notes using that decoupling occurs in the matter-dominated regime (as today we know from the observations).*

Since $a_{eq}^{MR} \simeq 25\, a_{\rm dec}$*, one finds* $l_1^{RD} \simeq 10\, l_1^{MD} \simeq 1800$*.*

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PHYS6005 Cosmology : Problem Sheet 11 (15 December 2016)

Early Universe

1. What would be the value of g_R at temperatures $m_\mu c^2 \simeq 100\,\mathrm{MeV} \gtrsim k_B T \gtrsim 1\,\mathrm{Mev}$ (natural system) if right-handed neutrinos and left-handed anti-neutrinos are assumed to exist and are in ultra-relativistic thermal equilibrium ?

   As we discussed in the notes, in the Standard Model there are not right-handed neutrinos and left-handed antineutrinos and for this reason $g_\nu = 1$ instead than 2 as one would expect from $g_X = (2\,S_X + 1)$ considering that the spin of neutrinos is $S = 1/2$ (like for electrons). In this case at temperatures between $1\,MeV$ and much below half of the muon mass ($m_\mu c^2 \simeq 105\,\mathrm{MeV}$), the elementary particles in ultrarelativistic thermal equilibrium are photons, electrons and positrons and 3 species of neutrinos and therefore one obtains

$$g_R(50\,\mathrm{MeV} \gtrsim T \gtrsim 1\,\mathrm{MeV}) = g_\gamma + \frac{7}{8}\,(g_e \times 2 + 3 \times g_\nu \times 2) = 2 + \frac{7}{8} \times (4+6) = \frac{43}{4} = 10.75\,.$$

   If one adds right-handed neutrinos and left-handed antineutrinos then this result modifies simply considering that $g_\nu = 2$ instead than $g_\nu = 1$ and, therefore, one obtains

$$g_R(100\,\mathrm{MeV} \gtrsim T \gtrsim 1\,\mathrm{MeV}) = g_\gamma + \frac{7}{8}\,(g_e \times 2 + 3 \times g_\nu \times 2) = 2 + \frac{7}{8} \times (4+12) = 16\,.$$

   [1]

[1] This result would produce a much higher Helium abundance in BBN than observed and is therefore incompatible with the observations. For this reason we know that in the early Universe there were not thermalized right-handed neutrinos at least not with the same properties of left-handed neutrinos. The most accredited picture for the explanation of the neutrino masses, the so called seesaw mechanism, predicts the existence of right-handed neutrinos but with masses much heavier not only than the ordinary left-handed neutrinos but even than all the other known massive particles (the heaviest known elementary particle is the top quark with a mass $m_t \simeq 174\,\mathrm{GeV}$). Therefore, such heavy right-handed neutrinos predicted by the seesaw mechanism, in any case cannot be in thermal equilibrium below $100\,\mathrm{MeV}$.

2. Re-derive the matter-radiation equality time calculated neglecting the contribution from neutrinos (cf. Eq. (9.46) on the notes), including the contribution from neutrinos and verifying the result $t_{\rm eq}^{RM} \simeq 55,000\,{\rm yr}$ (cf. Eq. (12.21)).

   We have to go back to the expression found in the ps. n. 8 that we write here again,

$$t_{\rm eq}^{RM} = t_{\rm eq}^{M\Lambda} \left(\frac{a_{\rm eq}^{RM}}{a_{\rm eq}^{M\Lambda}}\right)^{\frac{3}{2}} .$$

   Plugging now $a_{\rm eq}^{RM} \simeq 2.3 \times 10^{-4}$ *and still using* $a_{\rm eq}^{ML} \simeq 0.7$ *we find the result given in eq. (12.21), i.e.* $t_{\rm eq}^{RM} \simeq 55,000\,{\rm yr}$*, showing how, because of the effect of neutrinos, the matter-radiation equality is shifted from* $30,000{\rm yr}$ *to* $55,000{\rm yr}$.

Big Bang Nucleosynthesis

- What is the age of the Universe when Deuterium is synthesized at $k_B\,T_{\rm nuc}$? *At* $k_B\,T_{\rm nuc} \simeq 0.065\,{\rm MeV}$ *Deuterium nuclei finally get synthesized. All neutrons very quickly get trapped into Deuterium nuclei and neutron decays get inhibited. By inverting the time-temperature relation eq. (12.7) with* $g_R \simeq 3.36$ *(cf. eq. (12.18)), one finds that the value* $k_B\,T_{\rm nuc} \simeq 0.065\,{\rm MeV}$ *corresponds approximately to a time* $t_{\rm nuc}$ *that can be calculated from* $T_{\rm nuc}$ *using the useful relation (natural system)*

$$t \simeq \frac{2.4\,{\rm s}}{\sqrt{g_R}} \left(\frac{\rm MeV}{T}\right)^2 ,$$

  finding

$$t_{\rm nuc} \simeq 310\,s\,.$$

  This is the age of the Universe at $T_{\rm nuc}$.

- How would the predicted value of the Helium-4 abundance change if the neutron life time is tripled? *The ratio of the* n_n/n_p *value at the time of nucleosynthesis and at freeze-out (cf. eq. (13.14)) would be now given by*

$$\frac{(n_n/n_p)_{\rm nuc}}{(n_n/n_p)_{\rm fr}} = e^{-\frac{t_{\rm nuc}}{\tau}} = e^{-\frac{310}{3\times 885}} \simeq 0.89\,.$$

  Therefore one would have now

$$(n_n/n_p)_{\rm nuc} \simeq 0.89 \times 0.22 \simeq 0.196$$

and finally for the Helium-4 abundance

$$Y_p = 2\,\frac{(n_n/n_p)_{\rm nuc}}{1+(n_n/n_p)_{\rm nuc}} \simeq 0.33\,,$$

therefore much higher than observed. This problem nicely shows the interplay between cosmology and particle physics.

- How would the predicted value of the Helium-4 abundance change if instead of 3 neutrino species one assumes the existence of 4 neutrino species in ultra-relativistic thermal equilibrium prior the onset of Big Bang Nucleosynthesis (when $T \gtrsim 10\,{\rm MeV}$)? *The number of degrees of freedom at freeze out would increase from* 10.75 *to* $g_R = 2 + (7/8)(4+8) = 12.5$. *The freeze-out temperature* $T_{\rm fr} \propto g_R^{1/6}$ *would then increase by a factor* $(12.5/10.75)^{1/6} \simeq 1.025$ *and therefore it would be* $T_{\rm fr} \simeq 0.85 \times 1.025\,{\rm meV} = 0.87\,{\rm MeV}$. *Plugging this number in the expressions for* $(n_n/n_p)(T_{\rm fr})$ *finally one would find that the primordial Helium abundance would increase to* 0.276.

Inflation

- Assume that at the end of the inflationary stage, at $t_{\rm f}$, the temperature is given by $k_B\,T_{\rm f} = 10^{14}\,{\rm GeV}$. Assume moreover that the number of e-folds is given by $N = 50$. What was the size (radius) of the current observable Universe at the beginning of inflation, i.e. at $t_{\rm i}$, and at the end of inflation, i.e. at $t_{\rm f}$?

  The size of the observable Universe today (the radius) is given by $d_{H,0} \simeq 2\,c\,H_0^{-1} \sim 10\,{\rm Gpc} \sim 10^{27}\,{\rm m}$. *The temperature of the Universe today (of the relic photons) is* $k_B\,T_0 \sim 10^{-4}\,{\rm eV}$. *Assuming that* $k_B\,T \propto 1/a$ *during the whole history of the Universe, one then finds that at the end of Inflation*

$$a_f = a_0\,\frac{k_B\,T_0}{k_B\,T_f} \sim \frac{10^{-4}}{10^{23}} \sim 10^{-27}\,.$$

  This means that the size of the current observable Universe was at the end of Inflation just $10^{-27}\,d_{H,0} \sim 1\,{\rm m}$. *By definition of the number of e-folds the scale factor at the beginning of Inflation was*

$$a_i = a_f\,e^{-N} \sim 10^{-22}\,a_f\,.$$

  This means that the size of the current observable Universe was at the beginning of Inflation $\sim 10^{-22}\,{\rm m}$, *i.e. a microscopic size where quantum* *fluctuations where important. These quantum fluctuations are today regarded as the seeds of the observed large scale structure of the Universe (super-clusters of Galaxies, clusters of Galaxies, Galaxies, ...). Inflation therefore would have acted has an extraordinarily powerful magnifying lens that stretched tiny microscopic scales to cosmological scales.*