Example 1

In a digital communication system, the transmit and receive (pulse shaping) filters have been designed to form a Nyquist system. It is known that the Fourier transform or transfer function of the receive filter is given by

$$G_R(f) = \begin{cases} 1, & |f| \le 1000 \text{ Hz} \\ \sqrt{\frac{-|f|+3000}{2000}}, & 1000 \text{ Hz} < |f| \le 3000 \text{ Hz} \\ 0, & |f| > 3000 \text{ Hz} \end{cases}$$

(a) Determine the transfer function of the transmit filter $G_T(f)$.

(b) Determine the required baseband transmission bandwidth. What is the transmission (symbol) rate of this system? What is the roll-off factor of the system?

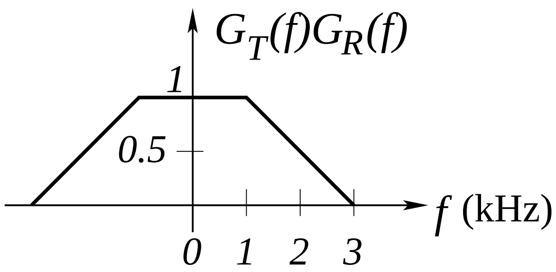
(c) The channel noise is known to be an additive white Gaussian noise (AWGN) with a flat power spectra density (PSD) $\Phi_n(f) = N_o/2$ watts/Hz for all f. Determine the noise power at the receiver output.

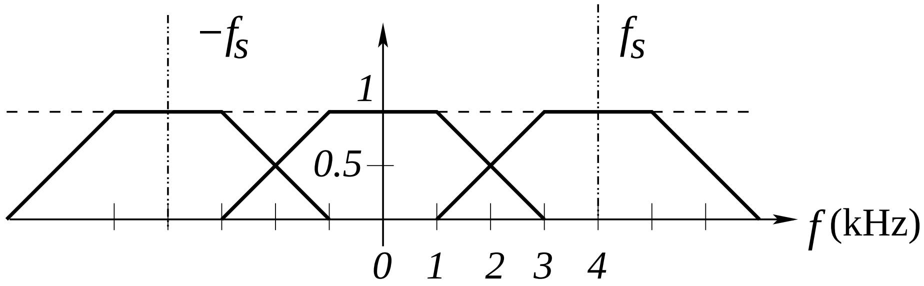
(d) The carrier of this system is 2 MHz. Determine the required passband channel bandwidth and the frequency components of the transmitted radio-frequency (RF) signal.

(1.a) Tx/Rx (pulse shaping) filters:

$$\Phi_R(f) = G_R(f) \cdot G_T(f) = \text{Nyquist system} \quad \text{and} \quad G_T(f) = G_R(f)$$

(1.b) The baseband bandwidth $B = 3$ kHz.



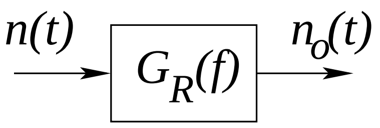


$$\sum_{k=-\infty}^{\infty} \Phi_R(f - kf_s) = \text{constant} \Rightarrow f_s = 4 \ \text{ kHz}$$

The minimum baseband bandwidth for zero ISI is $B_{\min} = \frac{f_s}{2} = 2$ kHz. The roll-off factor

$$\gamma = \frac{B - B_{\min}}{B_{\min}} = \frac{3-2}{2} = 0.5$$

(1.c) Noise passes through $G_R(f)$:

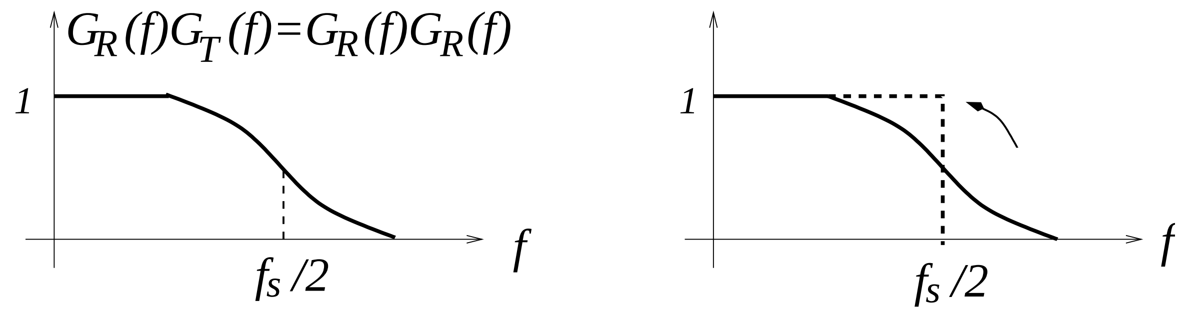


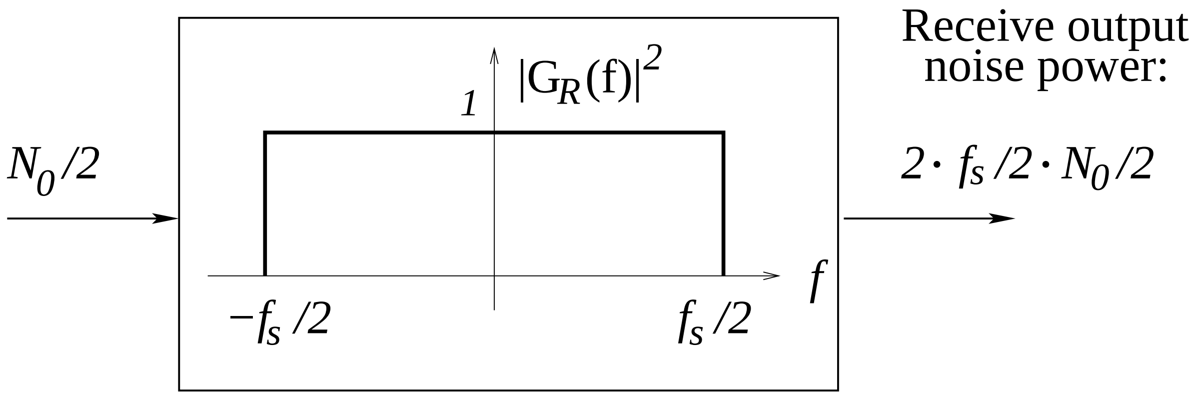
Channel noise PSD $\Phi_n(f) = N_o/2$

Receiver output noise PSD $\Phi_{n_o}(f) = |G_R(f)|^2 \Phi_n(f)$

$$P_{no} = \int_{-\infty}^{\infty} \Phi_{no}(f)df \Rightarrow$$ correct but bad, not fully understand subtle aspects of pulse shaping

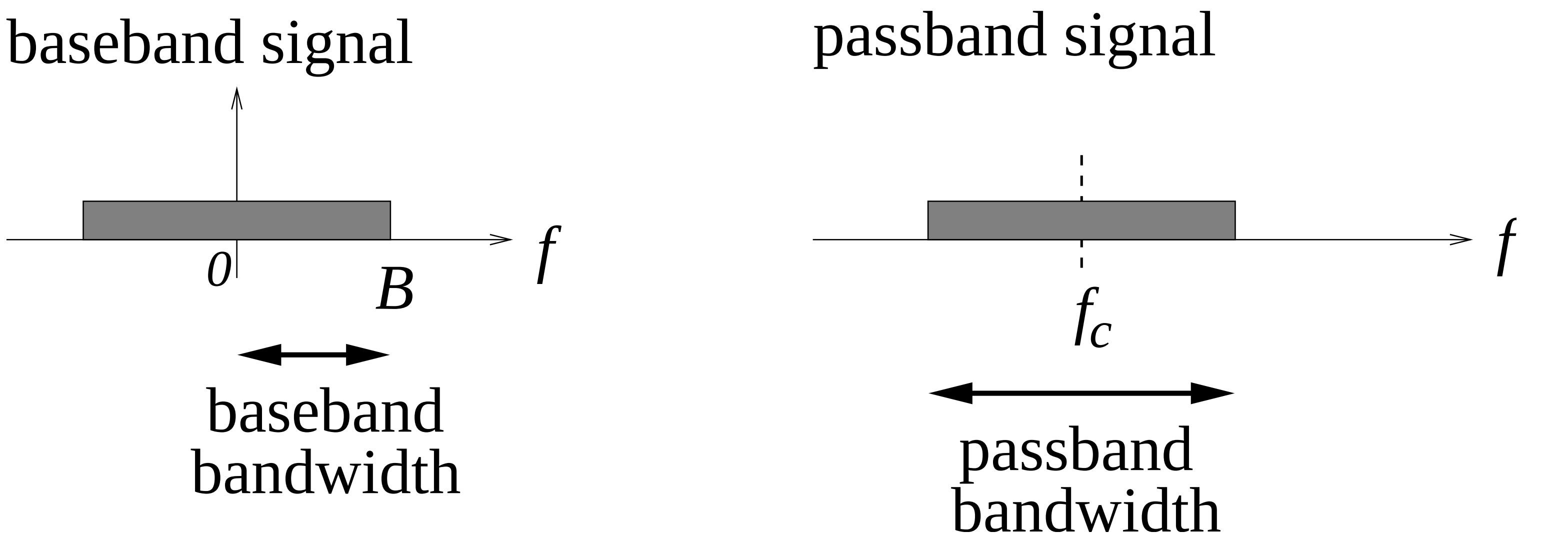
According to Nyquist criterion, combined $G_R(f)G_T(f)$ should be "symmetric" around $\frac{f_s}{2}$:



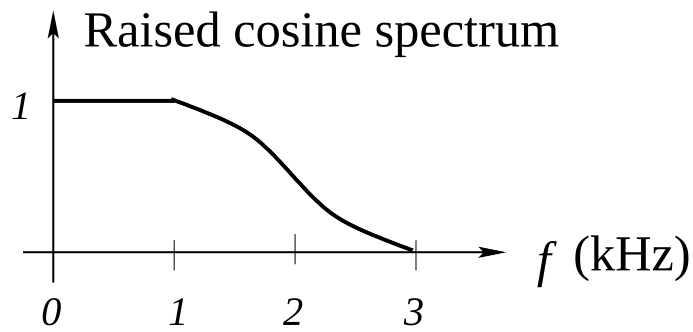


$$P_{no} = 2 \times \frac{f_s}{2} \times \frac{N_0}{2} = f_s \times \frac{N_0}{2} = \frac{N_o}{2} \times 4000 \text{ watts}$$

(1.d) The required passband channel bandwidth $B_p = 2B = 6$ kHz, centered at 2 MHz, and the transmitted RF signal contains frequency components in 1.997 MHz to 2.003 MHz.

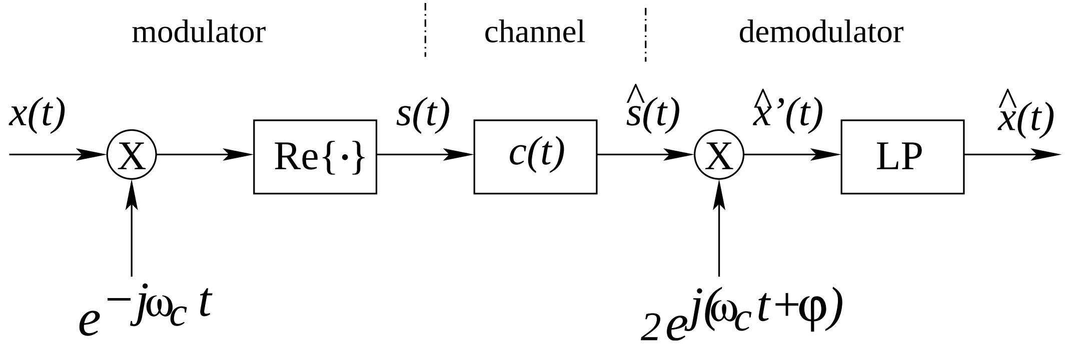


• In Example 1, if $G_R(f)$ is changed to a square-root of raised cosine spectrum. Could you do it?



Example 2

Based on a QAM signal $x(t) = x_i(t) + jx_q(t)$, the transmitted signal in the following figure is given by $s(t) = x_i(t)\cos\omega_c t + x_q(t)\sin\omega_c t$.



(a) With a channel impulse response $c(t) = \delta(t - 0.5T_s)$ where T_s is the symbol period, and a suitably selected lowpass filter LP, show that the receiver output is given by

$$\hat{x}(t) = x(t - 0.5T_s) \cdot e^{j(\varphi + \omega_c 0.5 T_s)}$$

Sketch the magnitude response of the lowpass filter LP.

(b) What is the best value of φ for the demodulator? Name the component in the receiver that is used to lock into this optimal phase offset.

(c) In the receiver, $\hat{x}(t)$ is sampled at $t = kT_s + \tau$ to produce $\hat{x}[k]$. Determine the best value of τ for the sampler. Name the component in the receiver that is used to find this optimal sampling offset.

(2.a) The receiver input is

$$\begin{aligned}\hat{s}(t) &= x_i(t-0.5T_s)\cos\omega_c(t-0.5T_s) + x_q(t-0.5T_s)\sin\omega_c(t-0.5T_s)\\ &= x_i(\tilde{t})\cos\omega_c\tilde{t} + x_q(\tilde{t})\sin\omega_c\tilde{t}\end{aligned}$$

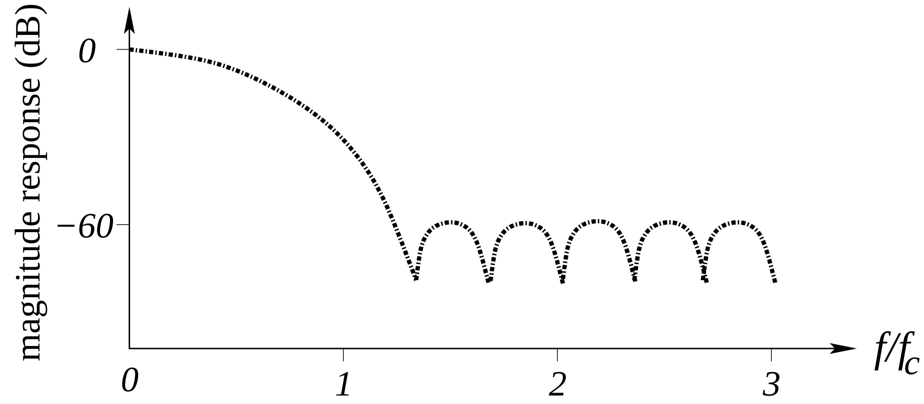
where $\tilde{t} = t - 0.5T_s$.
The demodulator output before LP is

$$\begin{aligned}\hat{x}^{'}(t) &= \hat{s}(t)\cdot 2\cdot e^{j(\omega_c t+\varphi)} = 2\cdot\hat{s}(t)\cdot e^{j\omega_c(t-0.5T_s)}\cdot e^{j(\varphi+\omega_c 0.5T_s)}\\ &= 2\cdot\hat{s}(t)\cdot e^{j\omega_c\tilde{t}}\cdot e^{j(\varphi+\omega_c 0.5T_s)}\\ &= 2\cdot\left(x_i(\tilde{t})\cos\omega_c\tilde{t} + x_q(\tilde{t})\sin\omega_c\tilde{t}\right)\cdot\left(\cos\omega_c\tilde{t} + j\sin\omega_c\tilde{t}\right)\cdot e^{j(\varphi+\omega_c 0.5T_s)}\\ &= \left\{x_i(\tilde{t})\cdot(1+\cos 2\omega_c\tilde{t} + j\sin 2\omega_c\tilde{t})\right.\\ &\quad \left. + jx_q(\tilde{t})\cdot(1-\cos 2\omega_c\tilde{t} - j\sin 2\omega_c\tilde{t})\right\}\cdot e^{j(\varphi+\omega_c 0.5T_s)}\end{aligned}$$

A suitably chosen LP will remove the components modulated at $2\omega_c$

$$\hat{x}(t) = \text{LP}\{\hat{x}^{'}(t)\} = (x_i(\tilde{t}) + jx_q(\tilde{t}))\cdot e^{j(\varphi+\omega_c 0.5T_s)} = x(t-0.5T_s)\cdot e^{j(\varphi+\omega_c 0.5T_s)}$$

Sketch of a suitable LP magnitude response

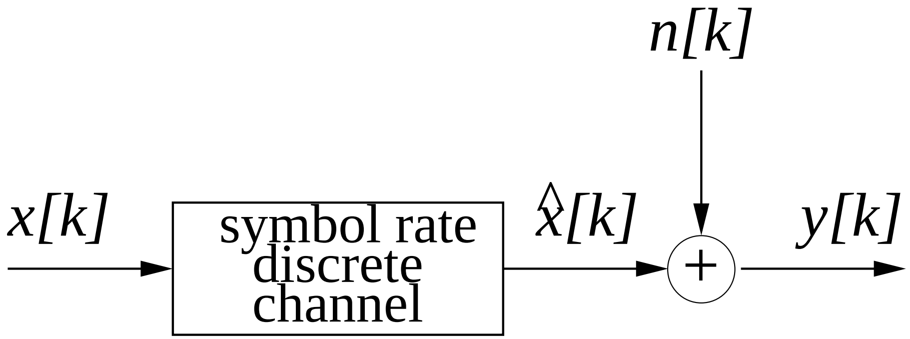


(2.b) The best value for φ is $\varphi + \omega_c 0.5T_s = 0$ or $\varphi = -\omega_c 0.5T_s$. The carrier recovery circuit in the receiver is used to lock into this optimal phase offset.

(2.c) The best value for τ is $kT_s + \tau - 0.5T_s = kT_s$ or $\tau = 0.5T_s$. In this noise-free case, $\hat{x}[k] = x[k]$, the transmitted symbol sequence. The timing recovery (or clock recovery or synchronisation) circuit in the receiver is used to find this optimal sampling offset.

Example 3

Refer to the following symbol-rate discrete channel model



Let $X(z)$ and $Y(z)$ be the z-transforms of the transmitted and received signals $x[k]$ and $y[k]$, respectively. Assume that the z transfer function of the channel is $C(z) = (1 + 0.1z^{-1})$.

(a) In the noise-free case ($n[k] = 0$), determine the zero-forcing (ZF) equaliser's coefficients.

(b) When the first two coefficients are used to approximate this ideal equaliser, determine the resulting residual intersymbol interference (ISI).

(c) Assume that $x[k]$ are independently identically distributed binary phase shift keying (BPSK) symbols, and the channel noise $n[k]$ has a variance of 0.09. Determine the Wiener solution for the two-tap equaliser with a decision delay $k_d = 0$. Compare this minimum mean square error (MMSE) equaliser with the two-tap approximate ZF equaliser given in (b).

(3.a) In the noise-free case, $Y(z) = C(z) \cdot X(z)$. The ZF equaliser $W_{\mathrm{ZF}}(z)$, assuming $k_d = 0$, is given by

$$W_{\mathrm{ZF}}(z) \cdot C(z) = 1$$

or

$$W_{\mathrm{ZF}}(z) = C^{-1}(z) = \frac{1}{1 + 0.1z^{-1}} = \sum_{i=0}^{\infty} \left(-0.1z^{-1}\right)^i = \sum_{i=0}^{\infty} (-0.1)^i z^{-i}$$

Thus, the ZF equalizer is given by

$$f_{\mathrm{ZF}}[k] = \sum_{i=0}^{\infty} (-0.1)^i \cdot y[k-i] = y[k] - 0.1 \cdot y[k-1] + 0.01 \cdot y[k-2] - \cdots$$

(3.b) The ZF equaliser is an infinite duration filter, but it decays rapidly. If $W(z) \approx 1 - 0.1z^{-1}$ is used, then $W(z) \cdot C(z) = (1 - 0.1z^{-1}) \cdot (1 + 0.1z^{-1}) = 1 - 0.01z^{-2}$, and in the noise-free case

$$f[k] = y[k] - 0.1 \cdot y[k-1] = x[k] - 0.01 \cdot x[k-2]$$

This results in a very small residual ISI term: $-0.01 \cdot x[k-2]$.

(3.c) Note that $y[k] = x[k] + 0.1 \cdot x[k-1] + n[k]$, $\mathcal{E}[x^2[k]] = 1$, and $\mathcal{E}[n^2[k]] = 0.09$, we have

$$r_{yy}[0] = \mathcal{E}[y^2[k]] = 1 + 0.01 + 0.09 = 1.1, \quad r_{yy}[1] = \mathcal{E}[y[k] \cdot y[k-1]] = 0.1$$

$$r_{dy}[0] = \mathcal{E}[x[k] \cdot y[k]] = 1, \quad r_{dy}[1] = \mathcal{E}[x[k] \cdot y[k-1]] = 0$$

The MMSE solution

$$\left[\begin{array}{c} w_{0,\text{opt}} \\ w_{1,\text{opt}} \end{array}\right] = \left[\begin{array}{cc} 1.1 & 0.1 \\ 0.1 & 1.1 \end{array}\right]^{-1} \left[\begin{array}{c} 1 \\ 0 \end{array}\right] = \left[\begin{array}{c} \frac{11}{12} \\ -\frac{1}{12} \end{array}\right]$$

The output of this two-tap MMSE equaliser is

$$f[k] = \frac{11}{12} \cdot y[k] - \frac{1}{12} \cdot y[k-1] = \frac{11}{12} \cdot (x[k] + 0.1 \cdot x[k-1] + n[k])$$

$$-\frac{1}{12} \cdot (x[k-1] + 0.1 \cdot x[k-2] + n[k-1])$$

$$= \frac{11}{12} \cdot x[k] + \frac{0.1}{12} x[k-1] - \frac{0.1}{12} \cdot x[k-2] + \frac{11}{12} \cdot n[k] - \frac{1}{12} \cdot n[k-1]$$

while the output of the two-tap approximate ZF equaliser is

$$f[k] = x[k] - 0.01 \cdot x[k-2] + n[k] - 0.1 \cdot n[k-1]$$

The former has a larger residual ISI but a smaller noise power at the equaliser output, compared with the latter.