Revision of Lecture 3

$X_i, P(X_i)$

$p_{ij}, \, 1 \le i, j \le N$

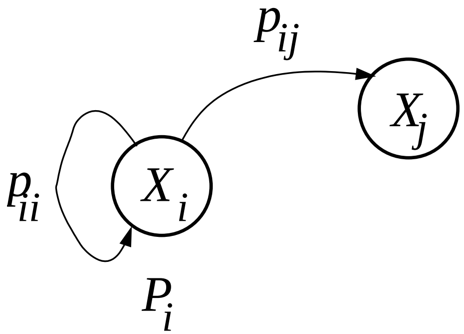
symbol rate R_s (symbols/s)

entropy H (bits/symbol)

information rate:

$R = R_s \cdot H$ (bits/s)

- 1st-order Markov process
  - Prior probability $P_i = P(X_i)$ and conditional probability p_{ij}
  - Symbol (state) entropy $H_i = -\sum_{j=1}^{N} p_{ij} \cdot \log_2 p_{ij}$ (bits/symbol)
  - Source entropy $H = \sum_{i=1}^{N} P_i H_i$ (bits/symbol)



- AR model

$$y(n) = \sum_{k=1}^{M} a_k y(n-k) + \epsilon(n)$$

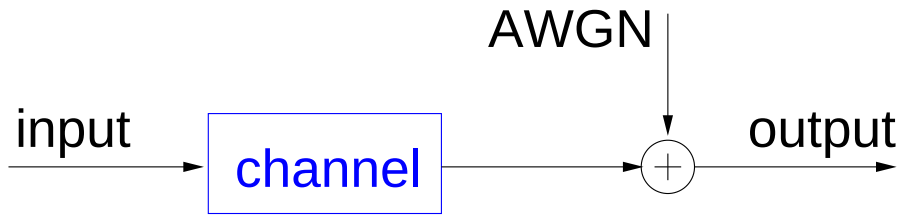
- Run length encoding

What happens to information transmitted through channel?

source (Tx) → channel → destination (Rx)

Information Across Channels

- For conveying information across a transmission channel, a suitable model is:



- The channel itself introduces amplitude and phase distortion, is potentially time-varying, and has a **limited bandwidth** B

- Error-free reception of symbols is additionally impeded by additive white Gaussian noise (AWGN); it severeness is described by the **signal-to-noise ratio** (SNR)

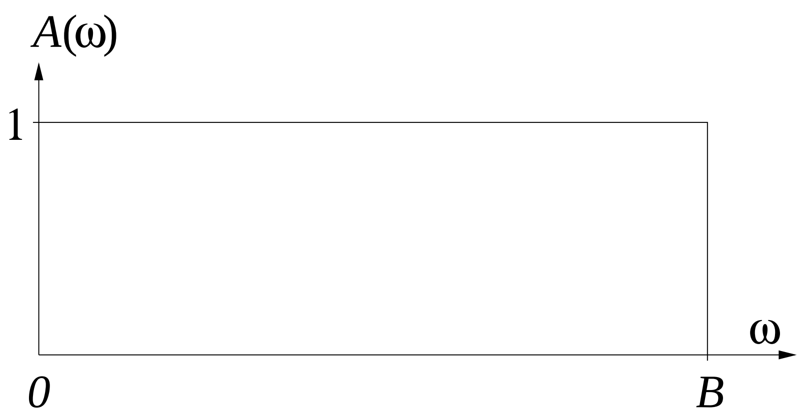
- Therefore, dependent on the above parameters, we are interested in determining the maximum possible error-free information transmission (channel capacity C)

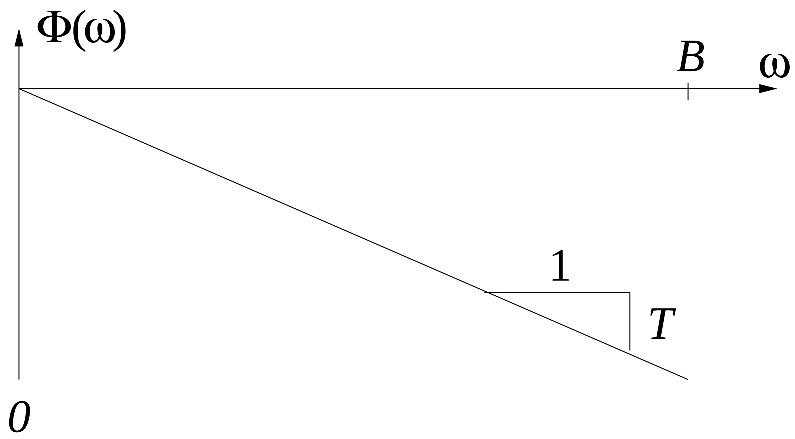
- We will see that C depends on B and SNR

Characteristics of Channel

- The channel can be described by its impulse response $h(t)$ or equivalently its frequency response $H(j\omega) = A(\omega) \cdot e^{j\Phi(\omega)}$ with amplitude response $A(\omega)$ and phase response $\Phi(\omega)$; $h(t)$ and $H(j\omega)$ are Fourier pair

- **Ideal channel** (pure delay): $h(t) = \delta(t - T) \quad \rightarrow \quad A(\omega) = 1, \quad \Phi(\omega) = -\omega T$





  – Flat magnitude and linear phase ($=$ constant group delay $G(\omega) = -\partial\Phi(\omega)/\partial\omega$)
  – The only impairment caused by an ideal channel is AWGN

- **Non-ideal channel**: channel is *dispersive*, causing intersymbol interference

Additive White Gaussian Noise

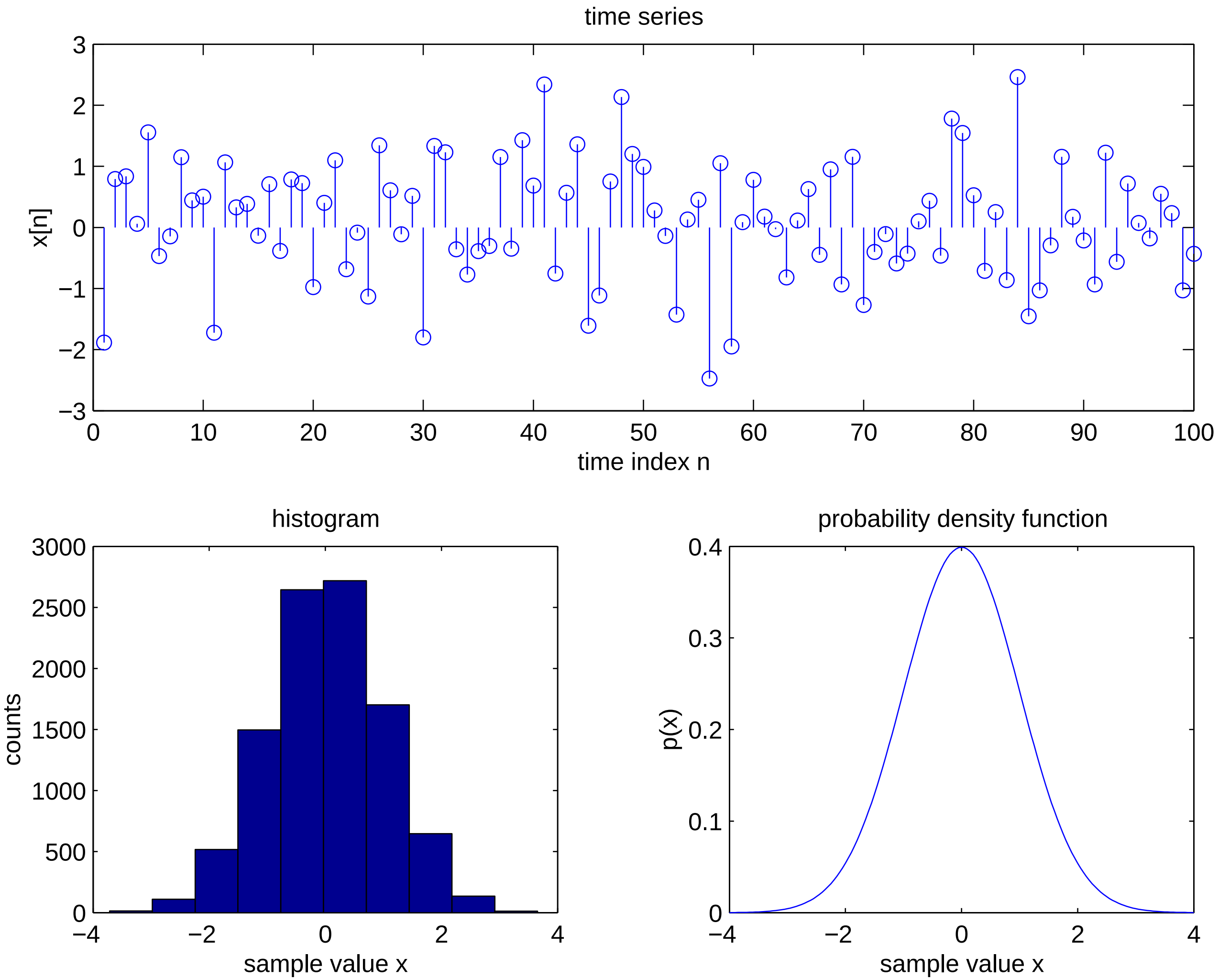
- Noise is uncorrelated with the signal

- Gaussian noise has a bell shaped probability density function (normally distributed)

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)/2\sigma^2}$$

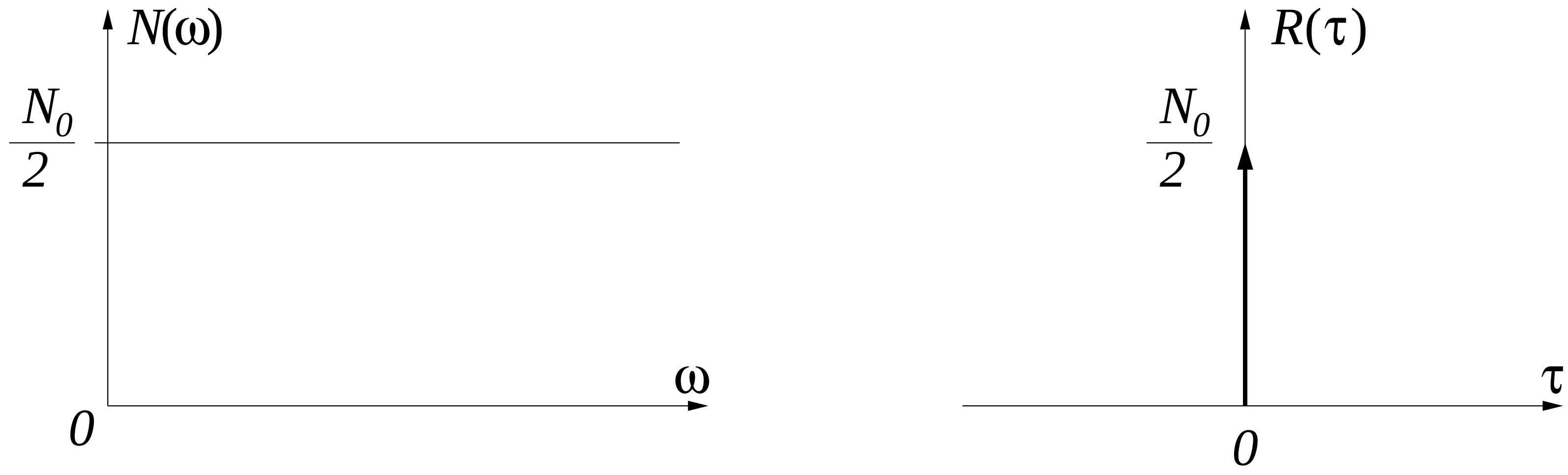
  with mean μ and variance σ^2

- White noise has zero mean, and channel noise is usually modelled as an AWGN



White Noise

- **White noise** is characterised by a **flat** power spectral density function, $N(\omega)$, or equivalently, its **impulse-shaped** auto-correlation function, $R(\tau)$



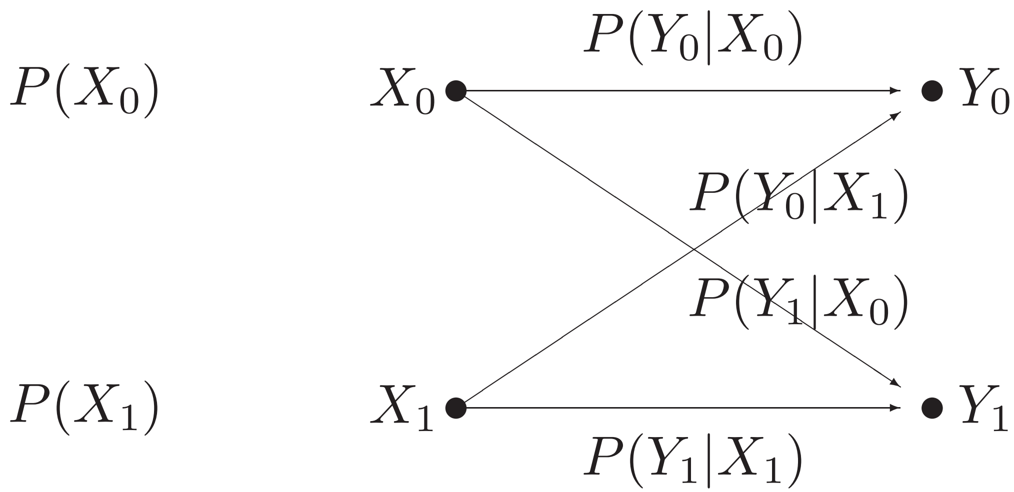
- $N(\omega)$ and $R(\tau)$ are a Fourier pair:

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} N(\omega) e^{j\omega\tau} d\omega \qquad N(\omega) = \int_{-\infty}^{\infty} R(\tau) e^{-j\omega\tau} d\tau$$

- Two-side spectrum is usually used for convenience, and N_0 is the noise power

Binary Symmetric Channel (BSC)

- **BSC** is the simplest model for information transmission via a discrete channel (channel is ideal, no amplitude and phase distortion, only distortion is due to AWGN):



$P(X_i)$: probability of occurrence of X_i at source output, *a priori* probability

$P(Y_j|X_i)$: conditional probability of Rx Y_j given Tx X_i

"Symmetric" refers to $P(Y_1|X_0) = P(Y_0|X_1) = p_e$, p_e being channel error probability

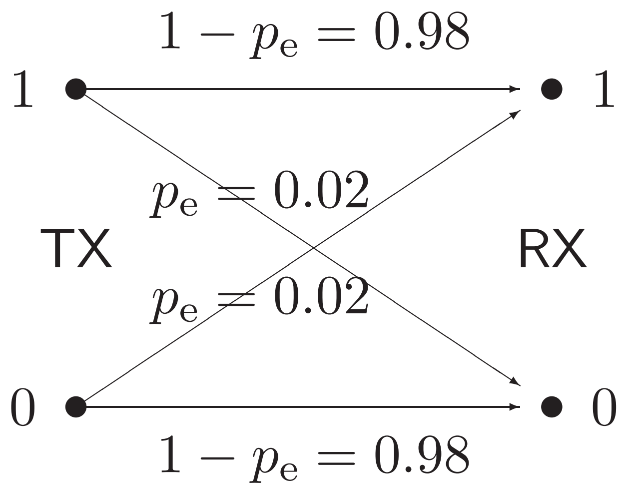
- The joint probability $P(Y_j, X_i)$ (Tx X_i and Rx Y_j) is linked with the conditional probabilities $P(Y_j|X_i)$ by Bayes' rule:

$$
\begin{aligned}
P(Y_j, X_i) &= P(X_i) \cdot P(Y_j|X_i) = P(Y_j) \cdot P(X_i|Y_j) \\
&= P(X_i, Y_j)
\end{aligned}
$$

Binary Symmetric Channel – Example

- Consider a BSC:

$P(X='1') = 0.7$



$P(X='0') = 0.3$

- This has a non-equiprobable source with $P(X='1') = 0.7$ and $P(X='0') = 0.3$: on average, 70% of transmitted bits are '1' and 30% are '0'

- Channel's error probability $p_e = 0.02$: on average, bit error rate is 2%

Binary Symmetric Channel – Example (continue)

- Probability of correct reception: $P_{\text{correct}} = P(Y='1', X='1') + P(Y='0', X='0') = 0.98$, as

$$
\begin{aligned}
P(Y='1', X='1') &= P(X='1') \cdot P(Y='1'|X='1') = 0.7 \cdot 0.98 = 0.686 \\
P(Y='0', X='0') &= P(X='0') \cdot P(Y='0'|X='0') = 0.3 \cdot 0.98 = 0.294
\end{aligned}
$$

- Probability of erroneous reception: $P_{\text{error}} = P(Y='1', X='0') + P(Y='0', X='1') = 0.02$, as

$$
\begin{aligned}
P(Y='1', X='0') &= P(X='0') \cdot P(Y='1'|X='0') = 0.3 \cdot 0.02 = 0.006 \\
P(Y='0', X='1') &= P(X='1') \cdot P(Y='0'|X='1') = 0.7 \cdot 0.02 = 0.014
\end{aligned}
$$

- Total probability of receiving a '1' (or a '0'):

$$
\begin{aligned}
P(Y='1') &= P(X='1') \cdot P(Y='1'|X='1') + P(X='0') \cdot P(Y='1'|X='0') \\
&= 0.7 \cdot 0.98 + 0.3 \cdot 0.02 = 0.692 \\
P(Y='0') &= P(X='0') \cdot P(Y='0'|X='0') + P(X='1') \cdot P(Y='0'|X='1') \\
&= 0.3 \cdot 0.98 + 0.7 \cdot 0.02 = 0.308
\end{aligned}
$$

Mutual Information

- Definition of **mutual information** of X_i and Y_j:

$$I(X_i, Y_j) = \log_2 \frac{P(X_i|Y_j)}{P(X_i)} \qquad \text{(bits)}$$

- Perfect, noiseless channel: $Y_i = X_i$, i.e. $P(X_i|Y_i) = 1$ and

$$I(X_i, Y_i) = \log_2 \frac{1}{P(X_i)}$$

  This is the information of X_i, hence no information is lost in the channel

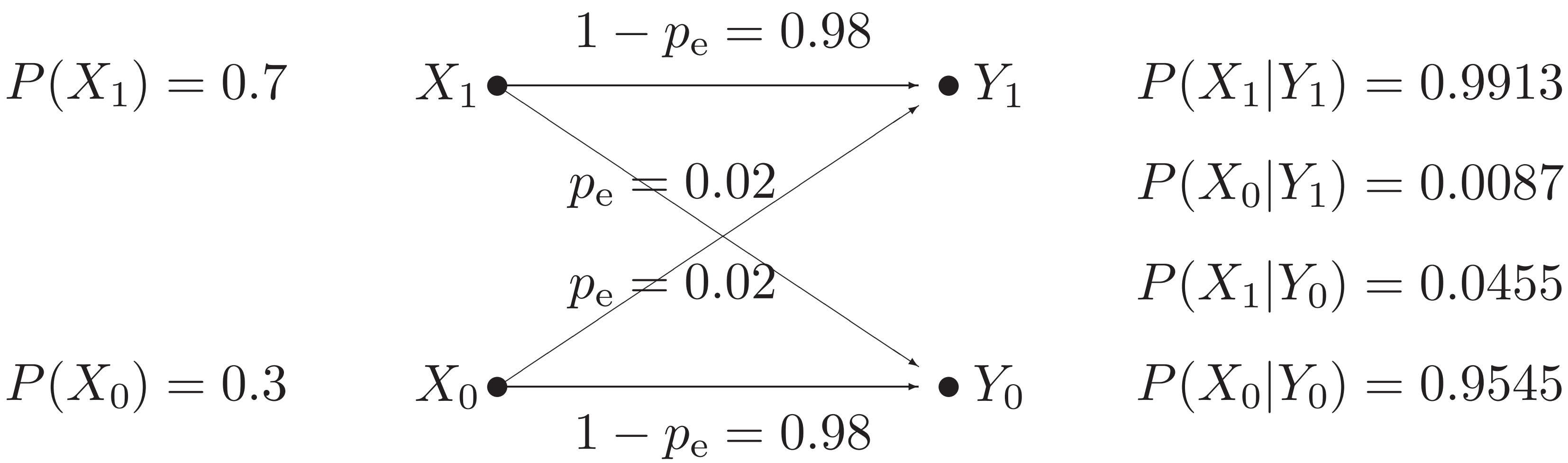
- Extremely noisy channel with error probability $0.5 \rightarrow Y_i$ is independent of X_i, hence

$$P(X_i|Y_i) = \frac{P(X_i, Y_i)}{P(Y_i)} = \frac{P(X_i) \cdot P(Y_i)}{P(Y_i)}$$

  Therefore $I(X_i, Y_i) = \log_2 1 = 0$, meaning all information is lost in the channel

Mutual Information – Example

- Consider the earlier BSC example:



- Here, the mutual information results in:

$$I(X_1, Y_1) = 0.502 \text{ bits} \qquad I(X_0, Y_0) = 1.670 \text{ bits}$$

(source info: $I(X_1) = 0.515$ bits $\qquad I(X_0) = 1.737$ bits)

$$I(X_0, Y_1) = -5.113 \text{ bits} \qquad I(X_1, Y_0) = -3.945 \text{ bits}$$

The negative quantities represent "mis-information"!

Average Mutual Information

- Based on received symbols Y_j given transmitted symbols X_i through a BSC, average mutual information is defined as:

$$\begin{aligned} I(X,Y) &= \sum_i \sum_j P(X_i, Y_j) \cdot I(X_i, Y_j) \\ &= \sum_i \sum_j P(X_i, Y_j) \cdot \log_2 \frac{P(X_i|Y_j)}{P(X_i)} \quad \text{(bits/symbol)} \end{aligned}$$

- This gives the average amount of **source information acquired per received symbol** by the receiver, and should be distinguished form the average source information (entropy $H(X)$)

- Note that due to Bayes:

$$\frac{P(X_i|Y_j)}{P(X_i)} = \frac{P(X_i, Y_j)}{P(X_i) \cdot P(Y_j)} = \frac{P(Y_j|X_i)}{P(Y_j)}$$

Imperfect Channel: Information Loss

- Consider re-arranging the mutual information between transmitted symbol X_i and received symbol Y_j:

$$
\begin{aligned}
I(X_i, Y_j) &= \log_2 \frac{P(X_i|Y_j)}{P(X_i)} = \log_2 \frac{1}{P(X_i)} - \log_2 \frac{1}{P(X_i|Y_j)} \\
&= I(X_i) - I(X_i|Y_j)
\end{aligned}
$$

  $I(X_i, Y_j)$ is the amount of information conveyed to receiver when transmitting X_i and receiving Y_j, $I(X_i)$ is the source information of X_i, and $I(X_i|Y_j)$ can be regarded as the information loss due to the channel

- Therefore,

$$
\underbrace{I(X_i)}_{\text{Source Inf.}} - \underbrace{I(X_i, Y_j)}_{\text{Inf. conveyed to rec.}} = \underbrace{I(X_i|Y_j)}_{\text{Inf. loss}}
$$

- $0 \le I(X_i|Y_j) \le I(X_i)$, see for example the previous cases of $p_e = 0$ and $p_e = 0.5$

Imperfect Channel: Average Mutual Information

- Average mutual information is given by:

$$I(X,Y) = \sum_i \sum_j P(X_i, Y_j) \cdot \log_2 \frac{P(X_i|Y_j)}{P(X_i)} \qquad \text{(bits/symbol)}$$

- But this average conveyed information $I(X,Y)$

$$= \sum_i \sum_j P(X_i, Y_j) \cdot \log_2 \frac{1}{P(X_i)} - \sum_i \sum_j P(X_i, Y_j) \cdot \log_2 \frac{1}{P(X_i|Y_j)}$$

$$= \sum_i \left(\sum_j P(X_i, Y_j) \right) \cdot \log_2 \frac{1}{P(X_i)} - \sum_j P(Y_j) \cdot \left(\sum_i P(X_i|Y_j) \cdot \log_2 \frac{1}{P(X_i|Y_j)} \right)$$

$$= \sum_i P(X_i) \cdot \log_2 \frac{1}{P(X_i)} - \sum_j P(Y_j) \cdot I(X|Y_j) = \underbrace{H(X)}_{\text{av. source info}} - \underbrace{H(X|Y)}_{\text{av. info lost}}$$

- A similar re-arrangement leads to:

$$\underbrace{H(Y)}_{\text{destination entropy}} - \underbrace{I(Y,X)}_{\text{av. conveyed info}} = \underbrace{H(Y|X)}_{\text{error entropy}}$$

Summary

- General consideration for transferring information across channels
- Channel characteristics
- Binary symmetric channel
- Mutual information and average mutual information