Example 1

1. A source emits symbols X_i, $1 \le i \le 6$, in the BCD format with probabilities $P(X_i)$ as given in Table 1, at a rate $R_s = 9.6$ kbaud (baud=symbol/second).

   State (i) the information rate and (ii) the data rate of the source.

2. Apply Shannon-Fano coding to the source signal characterised in Table 1. Are there any disadvantages in the resulting code words?

3. What is the original symbol sequence of the Shannon-Fano coded signal 110011110000110101100?

4. What is the data rate of the signal after Shannon-Fano coding? What compression factor has been achieved?

5. Derive the coding efficiency of both the uncoded BCD signal as well as the Shannon-Fano coded signal.

6. Repeat parts 2 to 5 but this time with Huffman coding.

Table 1.

X_i	$P(X_i)$	BCD word
A	0.30	000
B	0.10	001
C	0.02	010
D	0.15	011
E	0.40	100
F	0.03	101

Example 1 - Solution

1. (i) Entropy of source:

$$\begin{aligned} H &= -\sum_{i=1}^{6} P(X_i)\cdot\log_2 P(X_i) = -0.30\cdot\log_2 0.30 - 0.10\cdot\log_2 0.10 - 0.02\cdot\log_2 0.02 \\ &\quad -0.15\cdot\log_2 0.15 - 0.40\cdot\log_2 0.40 - 0.03\cdot\log_2 0.03 \\ &= 0.52109 + 0.33219 + 0.11288 + 0.41054 + 0.52877 + 0.15177 \\ &= 2.05724 \text{ bits/symbol} \end{aligned}$$

Information rate: $R = H\cdot R_s = 2.05724$ [bits/symbol] $\cdot$ 9600 [symbols/s] $= 19750$ [bits/s]

(ii) Data rate $= 3$ [bits/symbol] $\cdot$ 9600 [symbols/s] $= 28800$ [bits/s]

2. Shannon-Fano coding:

X	$P(X)$	I (bits)	steps					code
E	0.4	1.32	0					0
A	0.3	1.74	1	0				10
D	0.15	2.74	1	1	0			110
B	0.1	3.32	1	1	1	0		1110
F	0.03	5.06	1	1	1	1	0	11110
C	0.02	5.64	1	1	1	1	1	11111

Disadvantage: the rare code words have maximum possible length of $q-1=6-1=5$, and a buffer of 5 bit is required.

3. Shannon-Fano encoded sequence: $110|0|11110|0|0|0|110|10|110|0$ = DEFEEEDADE

4. Average code word length:

$$d = 0.4 \cdot 1 + 0.3 \cdot 2 + 0.15 \cdot 3 + 0.1 \cdot 4 + 0.05 \cdot 5 = 2.1 \quad \text{[bits/symbol]}$$

   Data rate:

$$d \cdot R_{\text{s}} = 2.1 \cdot 9600 = 20160 \quad \text{[bits/s]}$$

   Compression factor:

$$\frac{3 \ \text{[bits]}}{d \ \text{[bits]}} = \frac{3}{2.1} = 1.4286$$

5. Coding efficiency before Shannon-Fano:

$$\text{CE} = \frac{\text{information rate}}{\text{data rate}} = \frac{19750}{28800} = 68.58\%$$

   Coding efficiency after Shannon-Fano:

$$\text{CE} = \frac{\text{information rate}}{\text{data rate}} == \frac{19750}{20160} = 97.97\%$$

   Hence Shannon-Fano coding brought the coding efficiency close to 100%.

6. Huffman coding:

X	$P(X)$	steps 1	2	3	4	5	code
E	0.4					1	1
A	0.3				0	0	00
D	0.15			0	1	0	010
B	0.1		0	1	1	0	0110
F	0.03	0	1	1	1	0	01110
C	0.02	1	1	1	1	0	01111

step 1

E	0.40	
A	0.30	
D	0.15	
B	0.10	
F	0.03	0
C	0.02	1

step 2

E	0.40	
A	0.30	
D	0.15	
B	0.10	0
FC	0.05	1

step 3

E	0.40	
A	0.30	
D	0.15	0
BFC	0.15	1

step 4

E	0.40	
A	0.30	0
DBFC	0.30	1

step 5

ADBFC	0.60	0
E	0.40	1

Same disadvantage as Shannon-Fano: the rare code words have maximum possible length of $q - 1 = 6 - 1 = 5$, and a buffer of 5 bit is required.

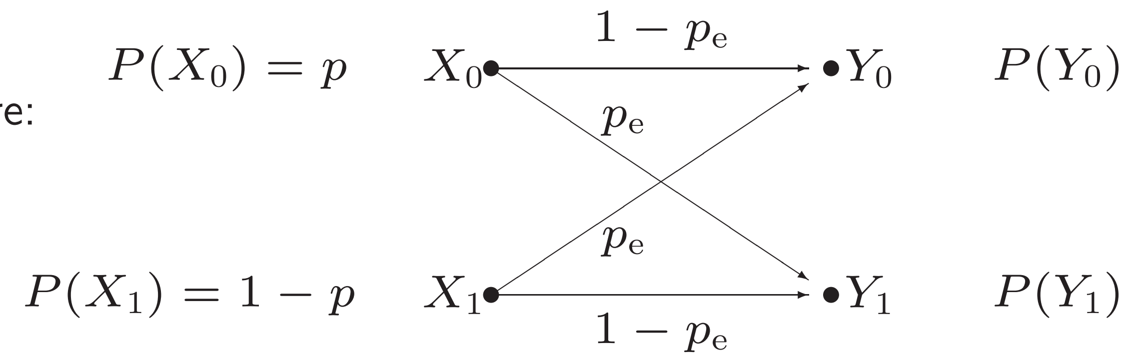
$1|1|00|1|1|1|1|00|00|1|1|010|1|1|00 =$ EEAEEEEAAEEDEEA

The same data rate and the same compression factor achieved as Shannon-Fano coding.

The coding efficiency of the Huffman coding is identical to that of Shannon-Fano coding.

Example 2

1. Considering the binary symmetric channel (BSC) shown in the figure:



   From the definition of mutual information,

$$I(X, Y) = \sum_i \sum_j P(X_i, Y_j) \cdot \log_2 \frac{P(X_i|Y_j)}{P(X_i)} \quad \text{[bits/symbol]}$$

   derive both

   (i) a formula relating $I(X, Y)$, the source entropy $H(X)$, and the average information lost per symbol $H(X|Y)$, and

   (ii) a formula relating $I(X, Y)$, the destination entropy $H(Y)$, and the error entropy $H(Y|X)$.

2. State and justify the relation ($>$,$<$,$=$,$\leq$, or $\geq$) between $H(X|Y)$ and $H(Y|X)$.

3. Considering the BSC in Figure 1, we now have $p = \frac{1}{4}$ and a channel error probability $p_e = \frac{1}{10}$. Calculate all probabilities $P(X_i, Y_j)$ and $P(X_i|Y_j)$, and derive the numerical value for the mutual information $I(X, Y)$.

Example 2 - Solution

1. (i) Relating to source entropy and average information lost:

$$\begin{aligned}
I(X,Y) &= \sum_i \sum_j P(X_i, Y_j) \cdot \log_2 \frac{P(X_i|Y_j)}{P(X_i)} \\
&= \sum_i \sum_j P(X_i, Y_j) \cdot \log_2 \frac{1}{P(X_i)} - \sum_i \sum_j P(X_i, Y_j) \cdot \log_2 \frac{1}{P(X_i|Y_j)} \\
&= \sum_i \left(\sum_j P(X_i, Y_j) \right) \cdot \log_2 \frac{1}{P(X_i)} \\
&\quad - \sum_j P(Y_j) \cdot \left(\sum_i P(X_i|Y_j) \cdot \log_2 \frac{1}{P(X_i|Y_j)} \right) \\
&= \sum_i P(X_i) \cdot \log_2 \frac{1}{P(X_i)} - \sum_j P(Y_j) \cdot I(X|Y_j) = H(X) - H(X|Y)
\end{aligned}$$

(ii) Bayes rule :

$$\frac{P(X_i|Y_j)}{P(X_i)} = \frac{P(X_i, Y_j)}{P(X_i) \cdot P(Y_j)} = \frac{P(Y_j|X_i)}{P(Y_j)}$$

Hence, relating to destination entropy and error entropy:

$$I(X,Y) = \sum_i \sum_j P(X_i,Y_j)\cdot \log_2 \frac{P(Y_j|X_i)}{P(Y_j)} = \sum_i \sum_j P(Y_j,X_i)\cdot \log_2 \frac{1}{P(Y_j)}$$

$$-\sum_i \sum_j P(Y_j,X_i)\cdot \log_2 \frac{1}{P(Y_j|X_i)} = H(Y) - H(Y|X)$$

2. Unless $p_e = 0.5$ or for equiprobable source symbols X, the symbols Y at the destination are more balanced, hence $H(Y) \geq H(X)$. Therefore, $H(Y|X) \geq H(X|Y)$.

3. Joint probabilities:

$$\begin{aligned} P(X_0,Y_0) &= P(X_0)\cdot P(Y_0|X_0) = \frac{1}{4}\cdot\frac{9}{10} = 0.225 \\ P(X_0,Y_1) &= P(X_0)\cdot P(Y_1|X_0) = \frac{1}{4}\cdot\frac{1}{10} = 0.025 \\ P(X_1,Y_0) &= P(X_1)\cdot P(Y_0|X_1) = \frac{3}{4}\cdot\frac{1}{10} = 0.075 \\ P(X_1,Y_1) &= P(X_1)\cdot P(Y_1|X_1) = \frac{3}{4}\cdot\frac{9}{10} = 0.675 \end{aligned}$$

Destination total probabilities:

$$P(Y_0) = P(X_0)\cdot P(Y_0|X_0) + P(X_1)\cdot P(Y_0|X_1) = \frac{1}{4}\cdot\frac{9}{10} + \frac{3}{4}\cdot\frac{1}{10} = 0.3$$

$$P(Y_1) = P(X_0)\cdot P(Y_1|X_0) + P(X_1)\cdot P(Y_1|X_1) = \frac{1}{4}\cdot\frac{1}{10} + \frac{3}{4}\cdot\frac{9}{10} = 0.7$$

Conditional probabilities:

$$\begin{aligned}
P(X_0|Y_0) &= \frac{P(X_0,Y_0)}{P(Y_0)} = \frac{0.225}{0.3} = 0.75\\
P(X_0|Y_1) &= \frac{P(X_0,Y_1)}{P(Y_1)} = \frac{0.025}{0.7} = 0.035714\\
P(X_1|Y_0) &= \frac{P(X_1,Y_0)}{P(Y_0)} = \frac{0.075}{0.3} = 0.25\\
P(X_1|Y_1) &= \frac{P(X_1,Y_1)}{P(Y_1)} = \frac{0.675}{0.7} = 0.964286
\end{aligned}$$

Mutual information:

$$\begin{aligned}
I(X,Y) &= P(X_0,Y_0)\cdot\log_2\frac{P(Y_0|X_0)}{P(Y_0)} + P(X_0,Y_1)\cdot\log_2\frac{P(Y_1|X_0)}{P(Y_1)}\\
&\quad + P(X_1,Y_0)\cdot\log_2\frac{P(Y_0|X_1)}{P(Y_0)} + P(X_1,Y_1)\cdot\log_2\frac{P(Y_1|X_1)}{P(Y_1)}\\
&= 0.3566165 - 0.0701838 - 0.1188721 + 0.2447348 = 0.4122954 \quad \text{[bits/symbol]}
\end{aligned}$$

Example 3

A digital communication system uses a 4-ary signalling scheme. Assume that 4 symbols -$\mathbf{3}$,-$\mathbf{1}$,$\mathbf{1}$,$\mathbf{3}$ are chosen with probabilities $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{8}$, respectively. The channel is an ideal channel with AWGN, the transmission rate is 2 Mbaud (2×10^6 symbols/s), and the channel signal to noise ratio is known to be 15.

1. Determine the source information rate.

2. If you are able to employ some capacity-approaching error-correction coding technique and would like to achieve error-free transmission, what is the minimum channel bandwidth required?

Example 3 - Solution

1. Source entropy:

$$H = 2 \cdot \frac{1}{8} \cdot \log_2 8 + \frac{1}{4} \cdot \log_2 4 + \frac{1}{2} \cdot \log_2 2 = \frac{7}{4} \text{ [bits/symbol]}$$

Source information rate:

$$R = H \cdot R_\mathrm{s} = \frac{7}{4} \cdot 2 \times 10^6 = 3.5 \text{ [Mbits/s]}$$

2. To be able to achieve error-free transmission

$$R \leq C = B \log_2 \left(1 + \frac{S_P}{N_P} \right) \to 3.5 \times 10^6 \leq B \log_2 (1 + 15)$$

Thus

$$B \geq 0.875 \text{ [MHz]}$$

Example 4

A predictive source encoder generates a bit stream, and it is known that the probability of a bit taking the value **0** is $P(\mathbf{0}) = p = 0.95$. The bit stream is then encoded by a run length encoder (RLC) with a codeword length of $n = 5$ bits.

1. Determine the compression ratio of the RLC.

2. Find the encoder input patterns that produce the following encoder output cordwords

$$11111 \quad 11110 \quad 11101 \quad 11100 \quad 11011 \quad \cdots \quad 00001 \quad 00000$$

3. What is the encoder input sequence of the RLC coded signal 110110000011110?

Example 4 - Solution

1. Codeword length after RLC is $n = 5$ bits, and average codeword length d before RLC with $N = 2^n - 1$

$$d = \sum_{l=0}^{N-1} (l+1) \cdot p^l \cdot (1-p) + N \cdot p^N = \frac{1-p^N}{1-p}$$

   Compression ratio
$$\frac{d}{n} = \frac{1-p^N}{n(1-p)} = \frac{1-0.95^{31}}{5 \times 0.05} = 3.1844$$

2. RLC table

$$\begin{aligned}
&\underbrace{00\cdots00000}_{31} \longrightarrow 11111\\
&\underbrace{00\cdots0000}_{30}1 \longrightarrow 11110\\
&\underbrace{00\cdots000}_{29}1 \longrightarrow 11101\\
&\underbrace{00\cdots00}_{28}1 \longrightarrow 11100\\
&\underbrace{00\cdots0}_{27}1 \longrightarrow 11011\\
&\vdots\\
&01 \longrightarrow 00001\\
&1 \longrightarrow 00000
\end{aligned}$$

3. $11011 \mid 00000 \mid 11110 \longleftarrow$ the encoder input sequence

$$\underbrace{00\cdots0}_{27}1\ 1\ \underbrace{00\cdots0000}_{30}1$$