Revision of Lecture 5

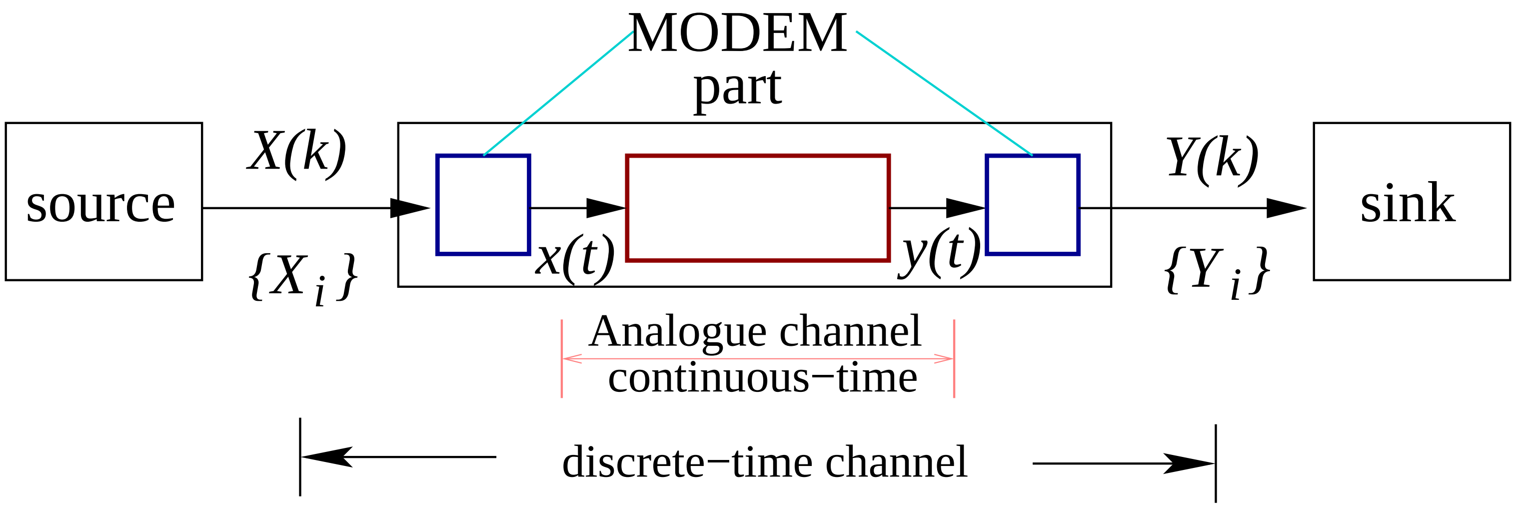
- Information transferring across channels
  - Channel characteristics and binary symmetric channel
  - Average mutual information

- Average mutual information tells us what happens to information transmitted across channel, or it "characterises" channel
  - But average mutual information is a bit too mathematical (too abstract)
  - As an engineer, one would rather characterises channel by its physical quantities, such as **bandwidth**, signal power and noise power or **SNR**

- Also intuitively given source with information rate R, one would like to know if channel is capable of "carrying" the amount of information transferred across it
  - In other word, what is the channel **capacity**?
  - This lecture answers this fundamental question

A Closer Look at Channel

- Schematic of communication system



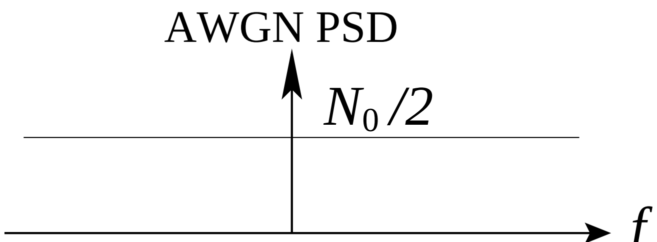
  - Depending which part of system: discrete-time and continuous-time channels
  - We will start with discrete-time channel then continuous-time channel

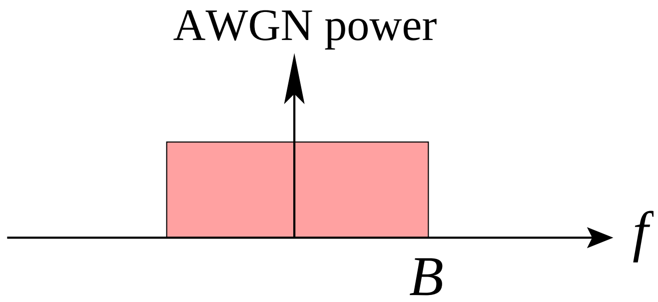
- Source has information rate, we would like to know "capacity" of channel
  - Moreover, we would like to know under what condition, we can achieve error-free transferring information across channel, i.e. no information loss
  - According to Shannon: information rate $\leq$ capacity $\rightarrow$ error-free transfer

Review of Channel Assumptions

- No amplitude or phase distortion by the channel, and the only disturbance is due to additive white Gaussian noise (AWGN), i.e. ideal channel
  - In the simplest case, this can be modelled by a binary symmetric channel (BSC)

- The channel error probability p_e of the BSC depends on the noise power N_P relative to the signal power S_P, i.e. SNR$= S_P/N_P$
  - Hence p_e could be made arbitrarily small by increasing the signal power
  - The channel noise power can be shown to be $N_P = N_0 B$, where $N_0/2$ is power spectral density of the noise and B the channel bandwidth





- Our aim is to determine the **channel capacity** C, the maximum possible error-free information transmission rate across the channel

Channel Capacity for Discrete Channels

- Shannon's channel capacity C is based on the average mutual information (average conveyed information across the channel), and one possible definition is

$$C = \max\{I(X,Y)\} = \max\{H(Y) - H(Y|X)\} \qquad \text{(bits/symbol)}$$

  where $H(Y)$ is the average information per symbol at channel output or destination entropy, and $H(Y|X)$ error entropy

- Let t_i be the symbol duration for X_i and t_{av} be the average time for transmission of a symbol, the channel capacity can also be defined as

$$C = \max\{I(X,Y)/t_{\mathrm{av}}\} \qquad \text{(bits/second)}$$

- C becomes maximum if $H(Y|X) = 0$ (no errors) and the symbols are equiprobable (assuming constant symbol durations t_i)

- Channel capacity can be expressed in either (bits/symbol) or (bits/second)

Channel Capacity: Noise-Free Case

- In noise free case, error entropy $H(Y|X) = 0$ and $I(X,Y) = H(Y) = H(X)$
  - But the entropy of the source is given by:

$$H(X) = -\sum_{i=1}^{q} P(X_i) \log_2 P(X_i) \qquad \text{(bits/symbol)}$$

  - Let t_i be symbol duration for X_i; average time for transmission of a symbol is

$$t_{\text{av}} = \sum_{i=1}^{q} P(X_i) \cdot t_i \qquad \text{(second/symbol)}$$

  - By definition, the channel capacity is $C = \max\{H(X)/t_{av}\}$ (bits/second)

- Assuming constant symbol durations $t_i = T_s$, the maximum or the capacity is obtained for memoryless q-ary source with equiprobable symbols

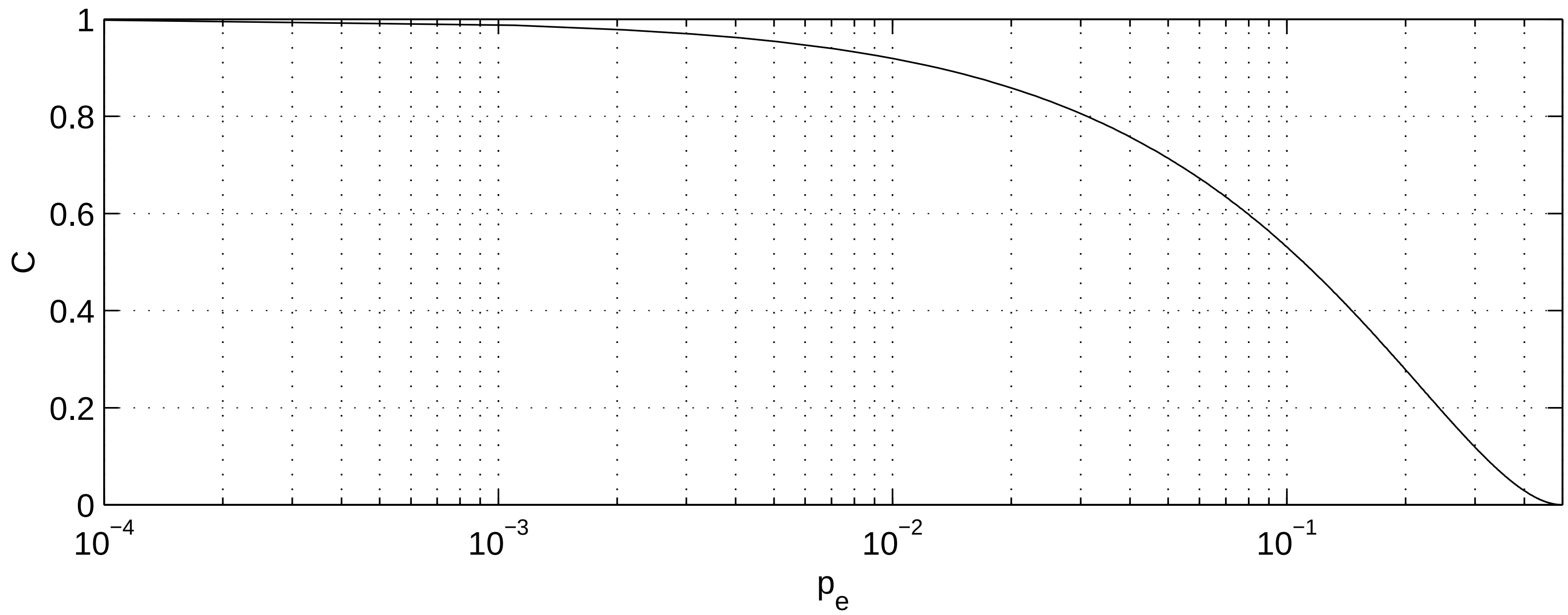
$$C = \log_2 q / T_s$$

  - This is the maximum achievable information transmission rate

Channel Capacity for BSC

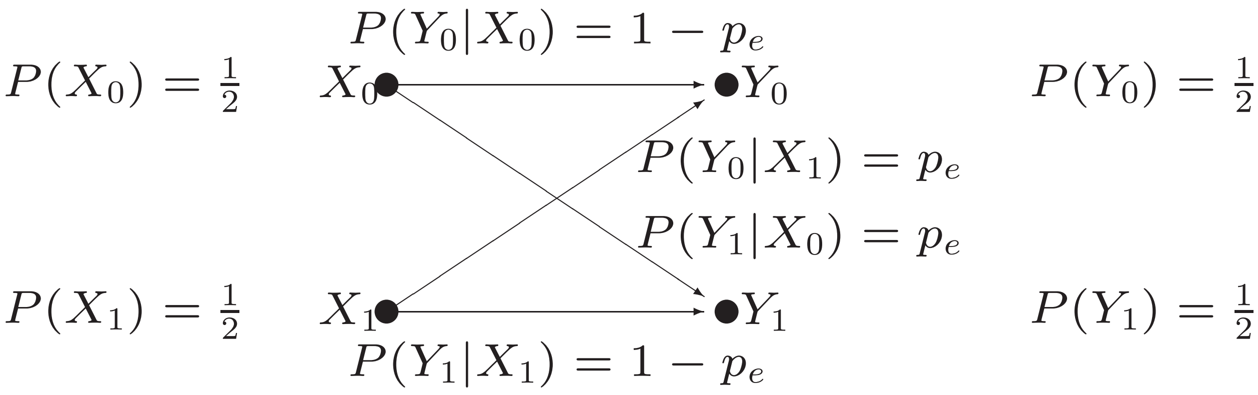
- BSC with equiprobable source symbols $P(X_0) = P(X_1) = 0.5$ and variable channel error probability p_e (due to symmetry of BSC, $P(Y_0) = P(Y_1) = 0.5$)
- The channel capacity C (in **bits/symbol**) is given as

$$C = 1 + (1 - p_\mathrm{e}) \log_2(1 - p_\mathrm{e}) + p_\mathrm{e} \log_2 p_\mathrm{e}$$



If $p_\mathrm{e} = 0.5$ (worst case), $C = 0$; and if $p_\mathrm{e} = 0$ (best case), $C = 1$

Channel Capacity for BSC (Derivation)



$$P(X_0, Y_0) = P(X_0)P(Y_0|X_0) = (1 - p_e)/2, \quad P(X_0, Y_1) = P(X_0)P(Y_1|X_0) = p_e/2$$

$$P(X_1, Y_0) = p_e/2, \quad P(X_1, Y_1) = (1 - p_e)/2$$

$$\begin{aligned} I(X, Y) \quad = \quad & P(X_0, Y_0) \log_2 \frac{P(Y_0|X_0)}{P(Y_0)} + P(X_0, Y_1) \log_2 \frac{P(Y_1|X_0)}{P(Y_1)} + \\ & + P(X_1, Y_0) \log_2 \frac{P(Y_0|X_1)}{P(Y_0)} + P(X_1, Y_1) \log_2 \frac{P(Y_1|X_1)}{P(Y_1)} \end{aligned}$$

$$= \frac{1}{2}(1 - p_e) \log_2 2(1 - p_e) + \frac{1}{2} p_e \log_2 2p_e + \frac{1}{2} p_e \log_2 2p_e + \frac{1}{2}(1 - p_e) \log_2 2(1 - p_e)$$
$$= 1 + (1 - p_e) \log_2 (1 - p_e) + p_e \log_2 p_e \quad \text{(bits/symbol)}$$

Channel Capacity and Channel Coding

- **Shannon's theorem**: If information rate $R \leq C$, there exists a coding technique such that information can be transmitted over the channel with arbitrarily small error probability; if $R > C$, error-free transmission is impossible
  - C is the maximum possible error-free information transmission rate
  - Even in noisy channel, there is no obstruction of reliable transmission, but only a limitation of the rate at which transmission can take place
  - Shannon's theorem does not tell how to construct such a capacity-approaching code
- Most practical channel coding schemes are far from optimal, but capacity-approaching codes exist, e.g. turbo codes and low-density parity check codes
- Practical communication systems are far from near capacity, but recently near-capacity techniques have been developed – iterative turbo detection-decoding

Entropy of Analogue Source

- Entropy of a continuous-valued (analogue) source, where the source output x is described by the PDF $p(x)$, is defined by

$$H(x) = -\int_{-\infty}^{+\infty} p(x) \log_2 p(x) dx$$

- According to Shannon, this entropy attends the maximum for Gaussian PDFs $p(x)$ (equivalent to equiprobable symbols in the discrete case)

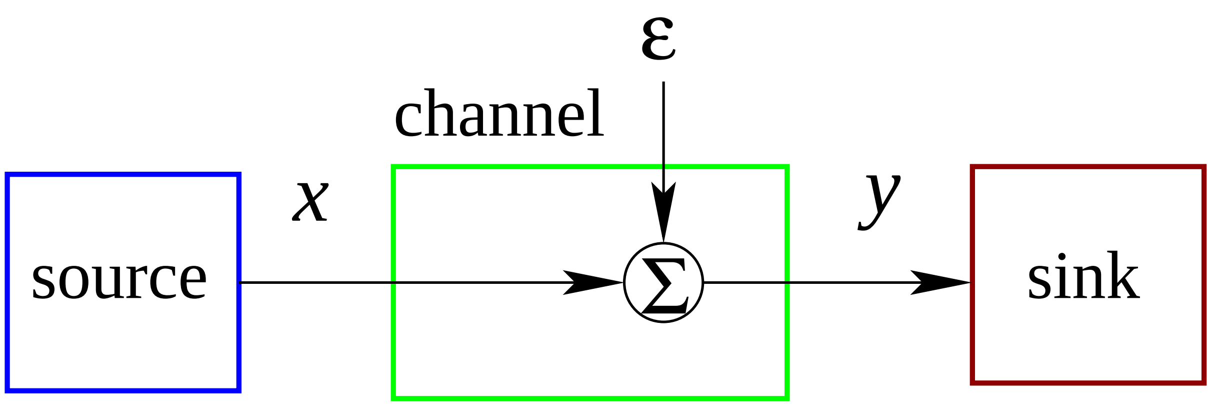
- Gaussian PDF with zero mean and variance σ_x^2:

$$p(x) = \frac{1}{\sqrt{2\pi}\sigma_x} e^{-(x^2/2\sigma_x^2)}$$

- The maximum entropy can be shown to be

$$H_{\max}(x) = \log_2 \sqrt{2\pi e}\sigma_x = \frac{1}{2} \log_2 2\pi e \sigma_x^2$$

Gaussian Channel



- Signal with Gaussian PDF attains maximum entropy, thus we consider **Gaussian channel**
- Channel output y is linked to channel input x by

$$y = x + \varepsilon$$

  - Channel AWGN ε is independent of channel input x, having **noise power** $N_P = \sigma_\varepsilon^2$
  - Assume Gaussian channel, i.e. channel input x has a Gaussian PDF, having **signal power** $S_P = \sigma_x^2$
  - Basic linear system theory: Gaussian signal operated by linear operator remains Gaussian
- Channel output signal y is also Gaussian, i.e. having a Gaussian PDF with power σ_y^2

$$\sigma_y^2 = S_P + N_P = \sigma_x^2 + \sigma_\varepsilon^2$$

Capacity for Continuous Channels

- Thus channel output y has a Gaussian PDF

$$p(y) = \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\left(y^2/2\sigma_y^2\right)}$$

- with power $\sigma_y^2 = S_P + N_P$, and entropy of y attains maximum value

$$H_{\max}(y) = \frac{1}{2}\log_2 2\pi e(S_P + N_P)$$

- Since $I(x,y) = H(y) - H(y|x)$, and $H(y|x) = H(\varepsilon)$ with ε being AWGN

$$H(y|x) = \frac{1}{2}\log_2 2\pi e N_P$$

- Therefore, the average mutual information

$$I(x,y) = \frac{1}{2}\log_2\left(1 + \frac{S_P}{N_P}\right) \quad \text{[bits/symbol]}$$

Shannon-Hartley Law

- With a sampling rate of $f_s = 2 \cdot B$, the Gaussian channel capacity is given by

$$C = f_s \cdot I(x, y) = B \cdot \log_2 \left(1 + \frac{S_P}{N_P}\right) \qquad \text{(bits/second)}$$

  where B is the signal bandwidth

  - For digital communications, signal bandwidth B (Hz) is **channel bandwidth**
  - Sampling rate f_s is the **symbol rate** (symbols/second)
  - Channel noise power is $N_P = N_0 \cdot B$, where N_0 is the power spectral density of the channel AWGN

- Two basic resources of communication: bandwidth and signal power

  - Increasing the SNR $\frac{S_P}{N_P}$ increases the channel capacity
  - Increasing the channel bandwidth B increases the channel capacity

- Gaussian channel capacity is often a good approximation for practical digital communication channels

Bandwidth and SNR Trade off

- From the definition of channel capacity, we can trade the channel bandwidth B for the SNR or signal power S_P, and vice versa
  - Depending on whether B or S_P is more precious, we can increase one and reduce the other, and yet maintain the same channel capacity
  - A noiseless analogue channel ($S_P/N_P = \infty$) has an infinite capacity
- C increases as B increases, but it does not go to infinity as $B \to \infty$; rather C approaches an upper limit

$$C = B \log_2 \left(1 + \frac{S_P}{N_0 B}\right) = \frac{S_P}{N_0} \log_2 \left(1 + \frac{S_P}{N_0 B}\right)^{N_0 B / S_P}$$

Recall that

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

We have

$$C_\infty = \lim_{B \to \infty} C = \frac{S_P}{N_0} \log_2 e = 1.44 \frac{S_P}{N_0}$$

Bandwidth and SNR Trade off – Example

- **Q**: A channel has an SNR of 15. If the channel bandwidth is reduced by half, determine the increase in the signal power required to maintain the same channel capacity

- **A**:

$$B \cdot \log_2 \left(1 + \frac{S_P}{N_0 B}\right) = B^{'} \cdot \log_2 \left(1 + \frac{S^{'}_P}{N_0 B^{'}}\right)$$

$$4 \cdot B = \frac{B}{2} \cdot \log_2 \left(1 + \frac{(S^{'}_P / S_P) \cdot S_P}{N_0 B / 2}\right)$$

$$8 = \log_2 \left(1 + 30 \frac{S^{'}_P}{S_P}\right)$$

$$256 = 1 + 30 \frac{S^{'}_P}{S_P} \quad \longrightarrow \quad S^{'}_P = 8.5 S_P$$

Summary

- Channel capacity, a fundamental physical quantity, defines maximum rate that information can be transfer across channel error-free
  - It is based on concept of maximum achievable mutual information between channel input and output, either defined as [bits/symbol] or [bits/s]
  - Channel capacity for discrete channels, e.g. channel capacity for BSC
  - Shannon theorem

- Channel capacity for continuous channels
  - Continuous-valued signal attains maximum entropy, if its PDF is Gaussian
  - Gaussian channel capacity: Shannon-Hartley law

$$C = B \cdot \log_2 \left(1 + \frac{S_P}{N_P}\right) \qquad \text{(bits/second)}$$

    where B is channel bandwidth; $\text{SNR} = \frac{S_P}{N_P}$
  - Bandwidth and signal power trade off

- Shannon's information theory provides underlying principles for communication and information processing systems