Revision of Lecture 5

- We have discussed all basic components of MODEM, including equalisation
  - Dispersive channel caused by insufficient channel bandwidth and/or multipath distorting
  - Effect of dispersive channel: cause ISI
  - Equalisation aims to make the combined channel and equaliser again a Nyquist system
- Dispersive channel modelling
- Equaliser design and adaptive equaliser

MODEM components

pulse shaping Tx/Rx filter pair

modulator/demodulator

bits $\overset{map}{\leftrightarrow}$ symbols

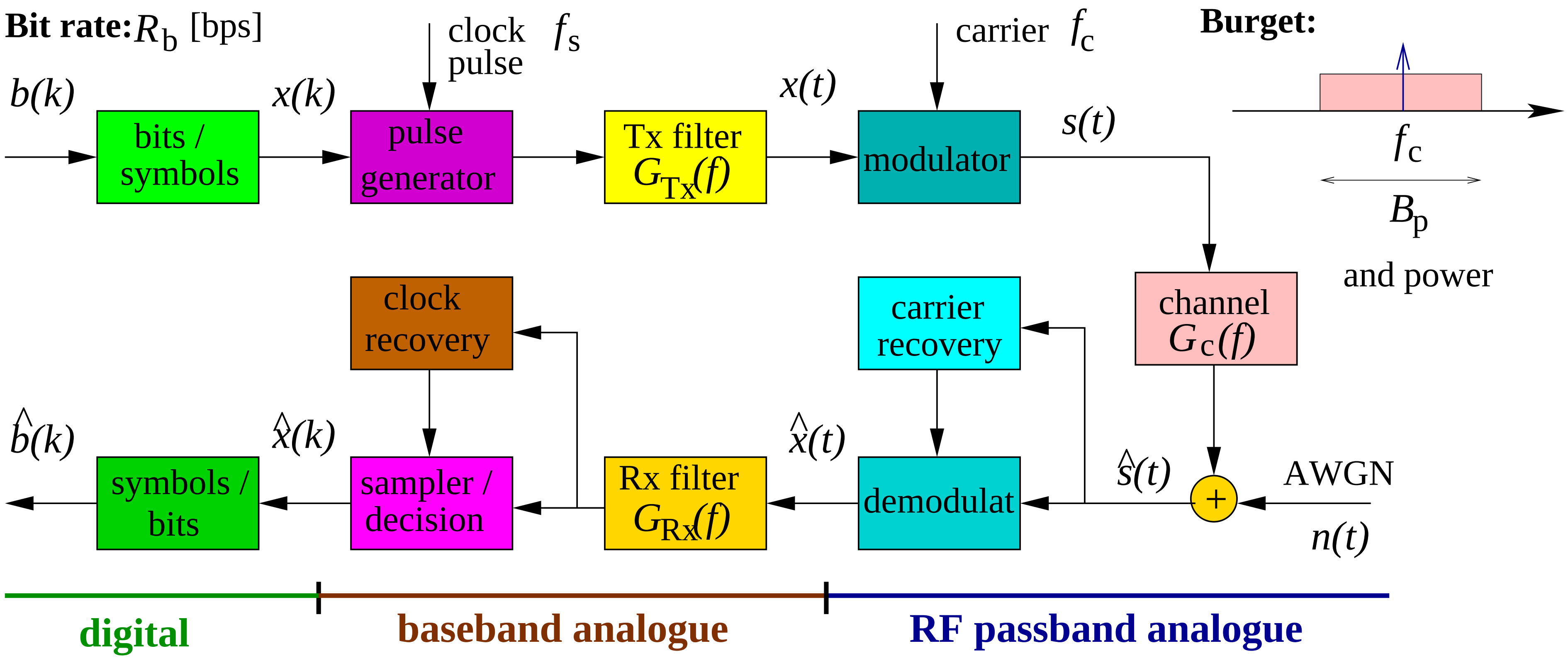
equalisation (distorting channel)

bit error rate and other issues

This lecture: how to evaluate performance of a digital communication system, i.e. bit error rate, and non-coherent system design

MODEM Overview

- Schematic of **MODEM** (modulation and demodulation) with **basic components**:



- This is for the **ideal** AWGN channel. If the channel is **dispersive**, then **equaliser** is required at receiver, as discussed in the previous lecture

Motivations

- MODEM aims at transferring information at required rate reliably over channel, subject to given bandwidth and power resource constraint
- Reliability is quantified by error probability of the system, namely, bit error rate
  - For speech, bit error rate of MODEM often requires to be below 1 in 100
  - For data, bit error rate of MODEM often requires to be below 1 in 1000
- In MODEM design, specifically pulse shaping design, we learn that receive filter should be matched to transmit filter
  - In this way, the receive signal to noise ratio (SNR) is maximized
  - This maximised the receive SNR is equivalent to minimise bit error rate
  - We will see how this is so by discussing how to evaluation bit error rate
- In MODEM design, specifically carrier recovery design, we learn that accurate carrier phase recovery is vital for coherent demodulation
  - Coherent system has better performance but carrier recovery circuit operated at very high RF frequency is difficult and expense
  - We will discuss so-call non-coherent system without carrier recovery

Error Probability (BPSK): Introduction

- Recall **pulse shaping** purposes: i) achieve zero ISI, and ii) maximise receive signal to noise ratio (SNR)
  - Why maximise receive SNR? $\longrightarrow$ best **detection accuracy**
- BPSK transmitter: bit 0: $x = +d$, bit 1: $x = -d$
- BPSK receiver: the received sample is

$$y = x + n, \;\; x \in \{\pm d\} \;\; \text{and} \;\; n \in \mathcal{N}(0, \sigma^2)$$

  - **Decision boundary** is $y = 0$, and **decision rule** is

$$y > 0 \rightarrow \hat{x} = d \text{ or bit 0}, \;\; y \leq 0 \rightarrow \hat{x} = -d \text{ or bit 1}$$

- Average signal power is $E_s = \frac{1}{2}(d^2 + d^2) = d^2$, and receive SNR is

$$\text{SNR} = \frac{E_s}{\sigma^2} = \frac{2d^2}{N_o}$$

  - AWGN power spectral density is $N_o/2$, hence noise power $\sigma^2 = N_o/2$

Error Probability (BPSK): Derivation

- Using **Bayes** theorem, the error probability or BER is given by

$$\begin{aligned} P_e = & P(\hat{x} \neq x) = P(x = d \cap \hat{x} = -d) + P(x = -d \cap \hat{x} = d) \\ = & P(x = d) P(\hat{x} = -d | x = d) + P(x = -d) P(\hat{x} = d | x = -d) \end{aligned}$$

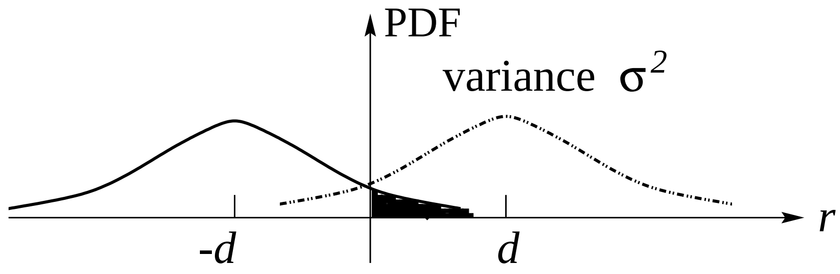
- As transmitted bit is equally likely to be 0 or 1, the two *a prior* probabilities are

$$P(x = d) = P(x = -d) = \frac{1}{2}$$

- Given $x = -d$, the decision $\hat{x} = d$ means that $y = -d + n > 0$ or noise value $n > d$, and the **conditional probability** $P(\hat{x} = d | x = -d)$ is equal to:

$$P(n > d) = \int_d^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{r^2}{2\sigma^2}\right) dr = \frac{1}{\sqrt{2\pi}} \int_{d/\sigma}^{\infty} \exp\left(-\frac{v^2}{2}\right) dv = Q\left(d/\sigma\right)$$

  – Interpretation of **conditional error probability** $P(\hat{x} = d | x = -d)$: Gaussian tail area over threshold $y = 0$

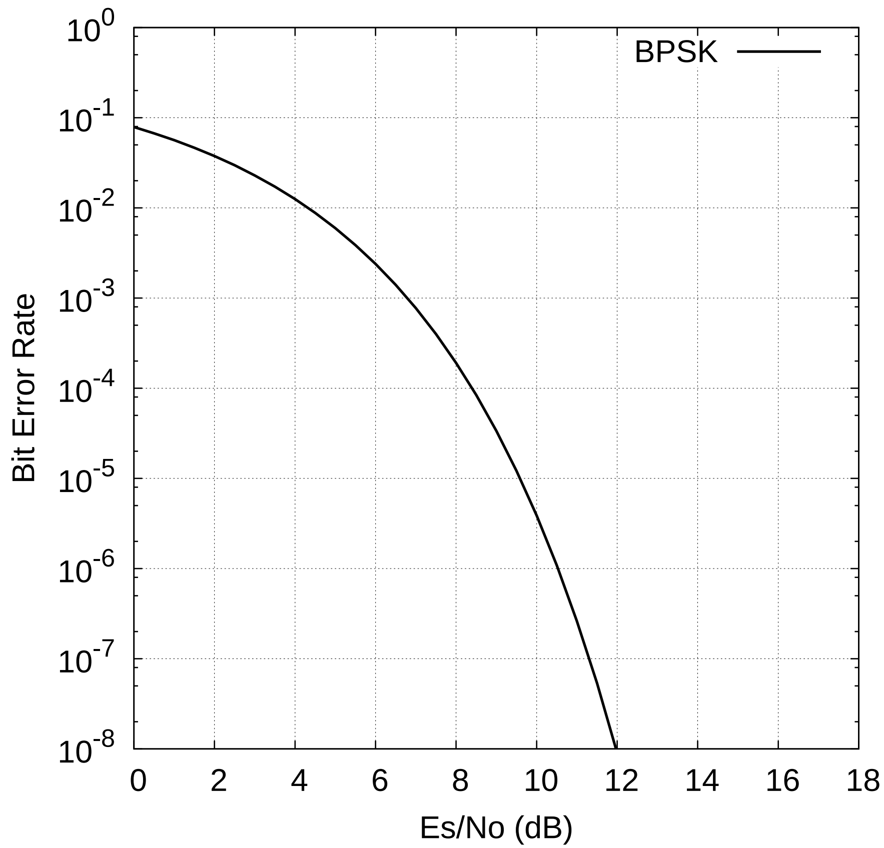


Error Probability (BPSK): Results

- Similarly, the other conditional error probability $P(\hat{x} = -d | x = d) = Q\left(d/\sigma\right)$

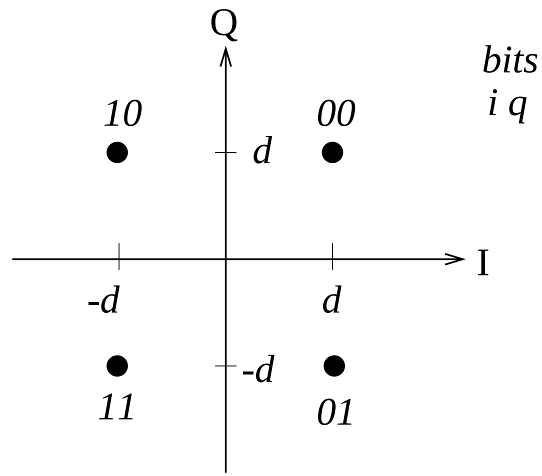
- Note signal power $E_s = \frac{1}{2}(d^2 + d^2) = d^2$ and noise power $\frac{N_0}{2} = \sigma^2$, BER is:

$$\begin{aligned} P_e =& \frac{1}{2} Q\left(d/\sigma\right) + \frac{1}{2} Q\left(d/\sigma\right) \\ =& Q\left(d/\sigma\right) = Q\left(\sqrt{\mathsf{SNR}}\right) \\ =& Q\left(\sqrt{\frac{2E_s}{N_0}}\right) \end{aligned}$$



- Maximize receive SNR $\rightarrow$ minimise error probability or bit error rate

Error Probability (QPSK)



- QPSK: I and Q components are both BPSK, and average signal power is $E_s = 2d^2$, noise power $2\sigma^2 = N_o$
- Let $y = y_I + \mathrm{j}y_Q$ be received signal sample, then decision rule is

$$y_I, y_Q > 0 \rightarrow i, q = 0 \quad y_I, Y_Q \leq 0 \rightarrow i, q = 1$$

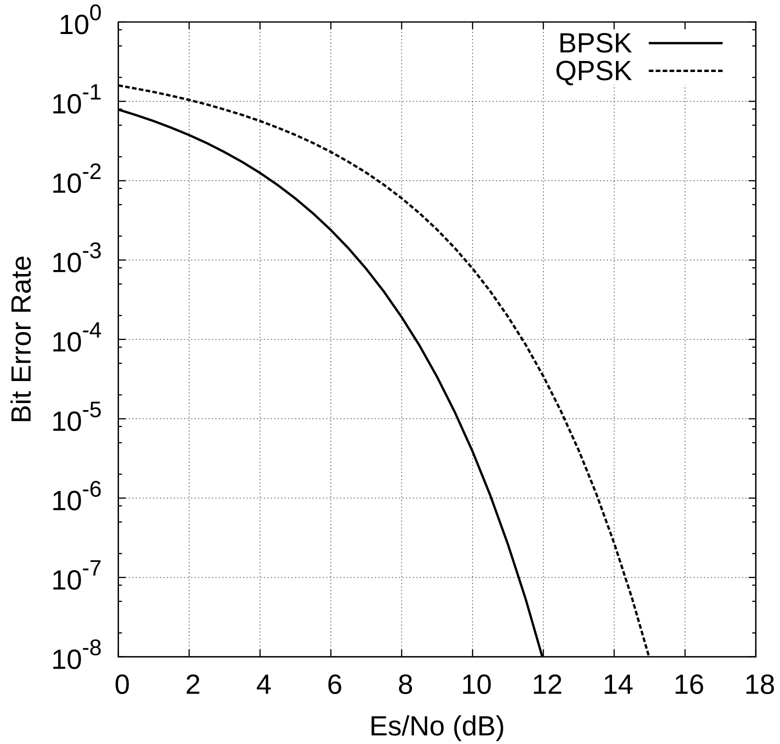
- Applying BPSK result to both I and Q yields
  - $P_{e,I} = Q(d/\sigma)$ and $P_{e,Q} = Q(d/\sigma)$
- **Average error rate** for QPSK is then

$$P_e = \frac{1}{2}(P_{e,I} + P_{e,Q}) = Q(d/\sigma) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

- In comparison, average error rate for BPSK

$$P_e = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

- For same bit rate R_b, QPSK bandwidth is half of BPSK, but requires higher signal power (factor of 2 or 3 dB) to achieve same level of BER



Coherent Receiver

(a) Carrier recovery for demodulation

- Received signal $\hat{s}(t) = A\cos\left(\omega_c t + \varphi\right) + n(t)$
- Receiver local carrier $\cos\left(\omega_c t + \tilde{\varphi}\right)$
- Carrier recovery (e.g. phase lock loop) circuit locks $\tilde{\varphi}$ to φ:

$$\Delta\varphi = \varphi - \tilde{\varphi} \to 0 \quad \text{i.e.} \quad \tilde{\varphi} \to \varphi$$

- Demodulation leads to recovered baseband signal

$$y(t) = x(t+\tau) + n(t)$$

where $X(t)$ is transmitted baseband signal

(b) Timing recovery for sampling

- Align receiver clock with transmitter clock, so that sampling $\Rightarrow$ no ISI

$$y_k = x_k + n_k$$

where $\{x_k\}$ are transmitted symbols, and $\{n_k\}$ are noise samples

Non-coherent Receiver

(a) No carrier recovery for demodulation

- Received signal $\hat{s}(t) = A\cos\left(\omega_c t + \varphi\right) + n(t)$
- Receiver local carrier $\cos\left(\omega_c t + \tilde{\varphi}\right)$
- No carrier recovery,

$$\phi = \Delta\varphi = \varphi - \tilde{\varphi} \neq 0 \quad \text{i.e.} \quad \tilde{\varphi} \neq \varphi$$

- Demodulation leads to recovered baseband signal

$$y(t) = x(t+\tau)e^{\mathrm{j}\phi} + n(t)$$

(b) Timing recovery for sampling

- Align receiver clock with transmitter clock, sampling results in

$$y_k = x_k e^{\mathrm{j}\phi} + n_k$$

- There is a random **unknown** channel state information $e^{\mathrm{j}\phi}$
- Could not recover transmitted symbols $\{x_k\}$ properly from $\{y_k\}$!

Differential Detection

(a) Differential encoding at transmitter for transmission

– Symbols $\{x_k\} \Rightarrow \{c_k\}$ for transmission by differential encoding

$$c_k = \begin{cases} 1, & k = 0 \\ x_k \cdot c_{k-1}, & k \geq 1 \end{cases}$$

– $\{c_k\}$ are transmitted, not symbols $\{x_k\}$, and as $c_k \cdot c_{k-1}^* = x_k \cdot \left(c_{k-1} \cdot c_{k-1}^*\right)$,

$$x_k = \frac{c_k \cdot c_{k-1}^*}{|c_{k-1}|^2} \tag{1}$$

(b) Non-coherent detection

– Receiver samples

$$y_k = c_k \cdot |h| \cdot e^{\mathrm{j}\phi} + n_k$$

$|h|$: magnitude of combined channel tap, $\phi \neq 0$: unknown phase

– Differential decoding leads to recovered symbols

$$\hat{x}_k = \frac{y_k \cdot y_{k-1}^*}{|y_{k-1}|^2} \tag{2}$$

Differential Detection (derivation)

- Note

$$\begin{aligned} y_k \cdot y_{k-1}^* =& \left(c_k \cdot |h| \cdot e^{\mathrm{j}\phi} + n_k\right) \cdot \left(c_{k-1}^* \cdot |h| \cdot e^{-\mathrm{j}\phi} + n_{k-1}^*\right) \\ =& c_k \cdot c_{k-1}^* \cdot |h|^2 \cdot e^{\mathrm{j}(\phi-\phi)} + n_k \cdot n_{k-1}^* + c_k \cdot |h| \cdot e^{\mathrm{j}\phi} \cdot n_{k-1}^* + n_k \cdot c_{k-1}^* \cdot |h| \cdot e^{-\mathrm{j}\phi} \\ |y_{k-1}|^2 =& c_{k-1} \cdot c_{k-1}^* \cdot |h|^2 + n_{k-1} \cdot n_{k-1}^* + c_{k-1} \cdot |h| \cdot e^{\mathrm{j}\phi} \cdot n_{k-1}^* + n_{k-1} \cdot c_{k-1}^* \cdot |h| \cdot e^{-\mathrm{j}\phi} \end{aligned}$$

  - When noise n_k is very small, $n_k \cdot n_{k-1}^*$ and $n_{k-1} \cdot n_{k-1}^*$ are even smaller, and we have

$$y_k \cdot y_{k-1}^* \approx c_k \cdot c_{k-1}^* \cdot |h|^2 \quad \text{and} \quad |y_{k-1}|^2 \approx |c_{k-1}|^2 \cdot |h|^2$$

- Thus,

$$\hat{x}_k = \frac{y_k \cdot y_{k-1}^*}{|y_{k-1}|^2} \approx \frac{c_k \cdot c_{k-1}^*}{|c_{k-1}|^2} + \bar{n}_k = x_k + \bar{n}_k$$

  - Unknown ϕ has been removed, but power of enhanced noise $\bar{n}_k$ is **larger** than that of n_k
- Comparison of coherent system and non-coherent system
  - Coherent detection require expensive and complex carrier recovery circuit, but has better bit error rate of detection

$$\hat{x}_k = x_k + n_k$$

  - Non-coherent detection does not require expensive and complex carrier recovery circuit, but has poorer bit error rate of detection (power of $\bar{n}_k$ larger than that of n_k)

$$\hat{x}_k = x_k + \bar{n}_k$$

Differential PSK

(a) For differential **phase shift keying**, $|c_{k-1}|^2 = \text{con}$ and $x_k = \frac{c_k \cdot c_{k-1}^*}{\text{con}}$

- $x_k \leftarrow$ phase of $c_k \cdot c_{k-1}^*$
- $\hat{x}_k \leftarrow$ phase of $y_k \cdot y_{k-1}^*$

(b) At receiver, **differential decoding** (2) becomes

$$\hat{x}_k = \frac{y_k \cdot y_{k-1}^*}{|c_{k-1}|^2} = \frac{y_k \cdot y_{k-1}^*}{\text{con}} \tag{3}$$

- For convenience, assuming $|h|^2 = 1$ (or $|h|^2$ is known), then

$$\frac{y_k \cdot y_{k-1}^*}{\text{con}} = \frac{c_k \cdot c_{k-1}^*}{\text{con}} + \frac{n_k \cdot n_{k-1}^*}{\text{con}} + \frac{c_k}{\text{con}} \cdot e^{j\phi} \cdot n_{k-1}^* + n_k \cdot e^{-j\phi} \cdot \frac{c_{k-1}^*}{\text{con}}$$

- Noting magnitudes of $\frac{c_k}{\text{con}}$ and $\frac{c_{k-1}^*}{\text{con}}$ are 1, $\frac{n_k \cdot n_{k-1}^*}{\text{con}}$ is much smaller than the last two terms, while $e^{j\phi} \cdot n_{k-1}^*$ and $n_k \cdot e^{-j\phi}$ have the same variance as n_k,

$$\hat{x}_k \approx x_k + 2n_k \tag{4}$$

- Compared with coherent detection of $\hat{x}_k \approx x_k + n_k$
  - Non-coherent detection **doubles** noise or its SNR is 3 dB worse off
- In general, differential systems do not need to acquire channel state information
  - This important advantage makes differential systems widely used in practice

Differential BPSK

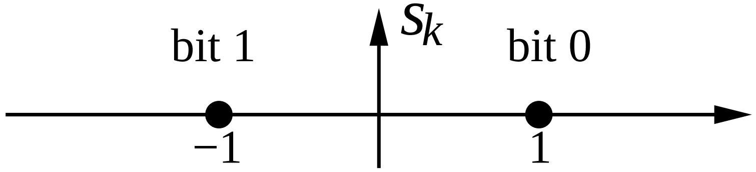
(a) For differential **binary phase shift keying**, at transmitter

– Bit sequence $\{b_k\}$ with $b_k \in \{0, 1\}$ are encoded by

$$d_k = \overline{b_k \oplus d_{k-1}}$$

Initial condition d_0 is given and known to both transmitter and receiver, e.g. $d_0 = 1$

– Encoded bit sequence $\{d_k\}$ is then BPSK modulated into BPSK symbol sequence $\{s_k\}$



– Transmitted signal

$$x(t) = \sqrt{\frac{2E_s}{T_s}} \cos\left(2\pi f_c t + \frac{1}{2}(1 - s_k)\pi\right), \;\; kT_s \le t < (k+1)T_s$$

where f_c is carrier frequency, E_s symbol energy, and T_s symbol duration

(b) For **coherent demodulator**, carrier recovery acquires carrier phase of incoming signal $y(t)$

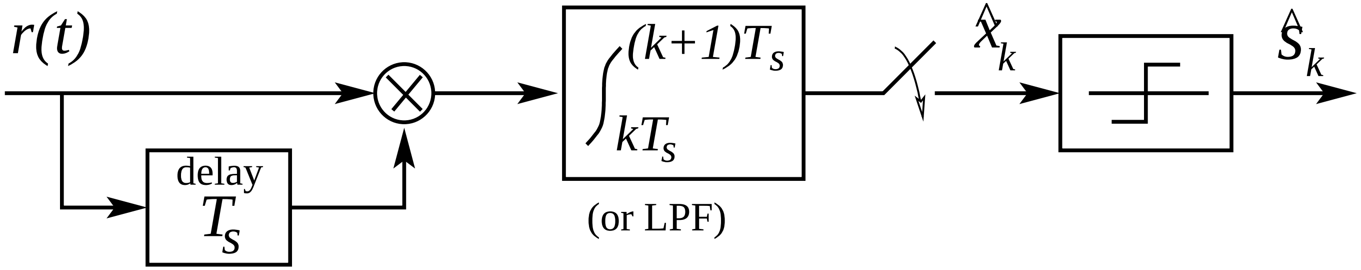
– $y(t)$ is coherently demodulated into baseband signal $r(t)$

– After timing recovery, sampled baseband received signal $\{r_k\}$ used for detection

$$\{r_k\} \rightarrow \{\hat{s}_k\} \rightarrow \{\hat{d}_k\} \rightarrow \{\hat{b}_k\} \text{ with } \hat{b}_k = \overline{\hat{d}_k \oplus \hat{d}_{k-1}}$$

DBPSK (Continue)

(c) For **noncoherent demodulator**, no local carrier is needed



- Let $A^2 = \frac{2E_s}{T_s}$ and $\frac{1}{2}(1 - s_k)\pi = \varphi_k$, then received RF signal (minus noise)

$$r(t) = A\cos\left(2\pi f_c t + \varphi_k + \vartheta\right), \ kT_s \le t < (k+1)T_s$$

- Noting $\cos\alpha\cos\beta = \frac{1}{2}\cos(\alpha+\beta) + \frac{1}{2}\cos(\alpha-\beta)$, input to integrator

$$A^2\cos\left(4\pi f_c t + 2\vartheta + \varphi_k + \varphi_{k-1}\right) + A^2\cos\left(\varphi_k - \varphi_{k-1}\right)$$

- First term average over one period is zero, and $\cos\left(\varphi_k - \varphi_{k-1}\right) = 1$ if $s_k = s_{k-1}$; $\cos\left(\varphi_k - \varphi_{k-1}\right) = -1$ if $s_k = -s_{k-1}$
- Hence, after timing recovery, from sampled signal $\{\hat{x}_k\}$, we know:

$$\hat{x}_k = \begin{cases} E_s, & s_k = s_{k-1} \\ -E_s, & s_k = -s_{k-1} \end{cases}$$

- Detection is achieved by $\{\hat{x}_k\} \rightarrow \{\hat{s}_k\} \rightarrow \{\hat{d}_k\} \rightarrow \{\hat{b}_k\}$

Summary

- Digital communication system performance, bit error rate, evaluation
  - BPSK – $Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$, E_s: average symbol energy, $N_0/2$: two-side noise power spectral density
  - QPSK – $Q\left(\sqrt{\frac{E_s}{N_0}}\right)$, E_s: average symbol energy; $N_0/2$: two-side noise power spectral density
  - Pulse shaping: receive filter is matched to transmit filter $\rightarrow$ maximise receive signal to noise ratio (SNR) $\rightarrow$ minimise bit error rate
  - Channel capacity: trade-off between bandwidth and power, e.g. consider BPSK and QPSK, can you see?

- Non-coherent systems: do not need expensive and complex carrier recovery, but have poorer performance, compared with coherent systems
  - Differential encoding at transmitter
  - Differential decoding/detection at receiver
  - differential phase shift keying systems