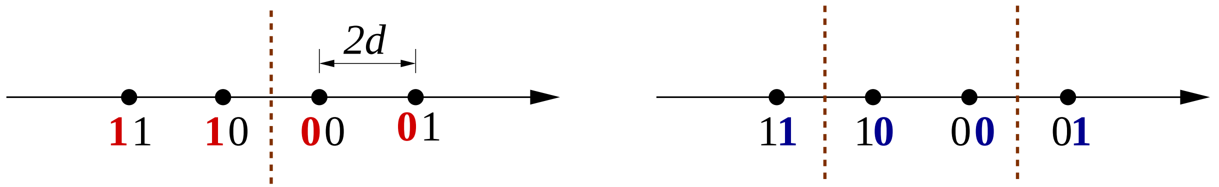
Revision of Lecture Nine

- AWGN channel: decision variable $r_k = \bar{r}_k + n_k = g_0 s_k + n_k$, where channel g_0 is known
  - AWGN n_k with PSD $\frac{N_0}{2}$, average symbol energy E_s, average channel SNR $\widetilde{\Lambda} = \frac{E_s}{N_0}$
- Bit error ratio performance of BPSK and QPSK
  - BPSK $P_e = Q\left(\sqrt{\frac{2E_s}{N_0}}\right) = Q\left(\sqrt{2\widetilde{\Lambda}}\right)$, QPSK $P_e = Q\left(\sqrt{\frac{E_s}{N_0}}\right) = Q\left(\sqrt{\widetilde{\Lambda}}\right)$
- Bit error ratio performance of 16QAM
  - 4-ary modulation: two classes of bits, and $P_{e,2} \approx 2P_{e,1}$



  - 16QAM $P_e = \frac{3}{4}Q\left(\sqrt{\frac{E_s}{5N_0}}\right) + \frac{1}{2}Q\left(\sqrt{\frac{9E_s}{5N_0}}\right) - \frac{1}{4}Q\left(\sqrt{\frac{25E_s}{5N_0}}\right) \approx \frac{3}{4}Q\left(\sqrt{\frac{\widetilde{\Lambda}}{5}}\right)$
- Bit error ratio performance of 64QAM
  - 8-ary modulation: three classes of bits, and $P_{e,3} \approx 2P_{e,2}$ and $P_{e,2} \approx 2P_{e,1}$
  - 64QAM $P_e = \frac{7}{12}Q\left(\sqrt{E_s/21N_0}\right) + \frac{1}{2}Q\left(3\sqrt{E_s/21N_0}\right) - \frac{1}{12}Q\left(5\sqrt{E_s/21N_0}\right) + \frac{1}{12}Q\left(9\sqrt{E_s/21N_0}\right) - \frac{1}{12}Q\left(13\sqrt{E_s/21N_0}\right) \approx \frac{7}{12}Q\left(\sqrt{\frac{\widetilde{\Lambda}}{21}}\right)$
- This lecture we consider fading channel performance

Flat Rayleigh Fading Channels

- A narrowband channel is represented by $c(t) = \alpha(t) \cdot e^{j\phi(t)}$. Assume that fading is sufficiently slow, $c(t)$ and $\phi(t)$ are symbol invariant $\rightarrow$ during one symbol period

$$c = \alpha \cdot e^{j\phi}$$

  – We consider uncorrelated Rayleigh fading, i.e. Doppler spread$\rightarrow \infty$

- The Rayleigh fading envelope α has a probability density function

$$p_\alpha(\alpha) = \frac{\alpha}{\alpha_0^2} \exp\left(-\frac{\alpha^2}{2\alpha_0^2}\right), \quad \alpha \geq 0$$

  and the channel phase ϕ is uniformly distributed in $[-\pi, \ \pi]$ with PDF

$$p_\phi(\phi) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \phi \leq \pi \\ 0, & \text{otherwise} \end{cases}$$

- Note that Re$[c]$ and Im$[c]$ are i.i.d. Gaussian with variance α_0^2, and α has mean $E[\alpha] = \bar{\alpha} = \sqrt{\frac{\pi}{2}}\alpha_0$, 2nd moment $E[\alpha^2] = 2\alpha_0^2$ and variance $\frac{4-\pi}{2}\alpha_0^2$

Flat Fading Performance

- Given the transmitted baseband signal $m(t)$ with average energy E_s, the received signal is

$$r(t) = \alpha \cdot e^{j\phi} \cdot m(t) + n(t)$$

  where $n(t)$ is AWGN with PSD $\frac{N_0}{2}$
- Define the instantaneous channel SNR

$$\lambda = \alpha^2 \frac{E_s}{N_0}$$

  Note that $\lambda > 0$ is a chi-square distribution with PDF

$$p_\lambda(\lambda) = \frac{1}{\Lambda} e^{-\lambda/\Lambda}$$

- The **average channel SNR** Λ is defined as

$$\Lambda = E[\lambda] = \bar{\lambda} = 2\alpha_0^2 \frac{E_s}{N_0}$$

- Let $P(\lambda)$ be the instantaneous error probability, the **average error probability** is then defined as

$$P_e = \int_0^\infty P(\lambda) p_\lambda(\lambda) d\lambda = \frac{1}{\Lambda} \int_0^\infty P(\lambda) e^{-\lambda/\Lambda} d\lambda$$

4QAM Flat Fading Performance

- **Fading does not change the decision boundaries** $I, Q = 0$, but $I, Q = d$ becomes $I, Q = \alpha d$
- Using non-fading result $P_e = Q(d/\sqrt{N_0/2}) \rightarrow$ the **instantaneous error probability**

$$P_e(\lambda) = Q\left(\frac{\alpha d}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\lambda}\right)$$

- Note $\lambda \geq 0$ and $\Lambda = E[\lambda] = 2\alpha_0^2 E_s/N_0$, the **average error probability** is

$$P_e = \frac{1}{\Lambda}\int_0^\infty Q\left(\sqrt{\lambda}\right) e^{-\lambda/\Lambda} d\lambda$$

- Note that $\lambda = \alpha^2 E_s/N_0$ and $E_s = 2d^2$, the **close-form solution** for P_e is:

$$P_e = \int_0^\infty Q\left(\frac{\alpha d}{\sqrt{N_0/2}}\right) p_\alpha(\alpha) d\alpha = \int_0^\infty Q\left(\frac{\alpha d}{\sqrt{N_0/2}}\right) \frac{\alpha}{\alpha_0^2} \exp\left(-\frac{\alpha^2}{2\alpha_0^2}\right) d\alpha$$

$$= \frac{1}{2}\left(1 - \sqrt{\frac{2\alpha_0^2 \frac{d^2}{N_0}}{1 + 2\alpha_0^2 \frac{d^2}{N_0}}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{\Lambda}{2 + \Lambda}}\right)$$

4QAM Flat Fading Performance (Derivation)

- Note

integration formula $$\int_0^\infty 2Q\left(\sqrt{2}\beta x\right) e^{-\mu x^2} x\, dx = \frac{1}{2\mu}\left(1 - \frac{\beta}{\sqrt{\mu + \beta^2}}\right)$$

average error probability $$P_e = \int_0^\infty Q\left(\frac{\alpha d}{\sqrt{N_0/2}}\right) \frac{\alpha}{\alpha_0^2} \exp\left(-\frac{\alpha^2}{2\alpha_0^2}\right) d\alpha$$

$$= \int_0^\infty 2Q\left(\sqrt{2}\frac{d\alpha_0}{\sqrt{N_0/2}} \frac{\alpha}{\sqrt{2}\alpha_0}\right) \exp\left(-\frac{\alpha^2}{2\alpha_0^2}\right) \frac{\alpha}{\sqrt{2}\alpha_0} d\left(\frac{\alpha}{\sqrt{2}\alpha_0}\right)$$

- Letting $\mu = 1$, $\beta = \frac{d\,\alpha_0}{\sqrt{N_0/2}}$ and $x = \frac{\alpha}{\sqrt{2}\alpha_0}$ in the above **integration formula** leads to

$$P_e = \frac{1}{2}\left(1 - \frac{\frac{d\,\alpha_0}{\sqrt{N_0/2}}}{\sqrt{1 + \left(\frac{d\,\alpha_0}{\sqrt{N_0/2}}\right)^2}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{2\alpha_0^2 \frac{d^2}{N_0}}{1 + 2\alpha_0^2 \frac{d^2}{N_0}}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{\Lambda}{2 + \Lambda}}\right)$$

where $\Lambda = 2\alpha_0^2 \frac{E_s}{N_0}$, $E_s = 2d^2 \Rightarrow \Lambda = \frac{4\alpha_0^2 d^2}{N_0}$

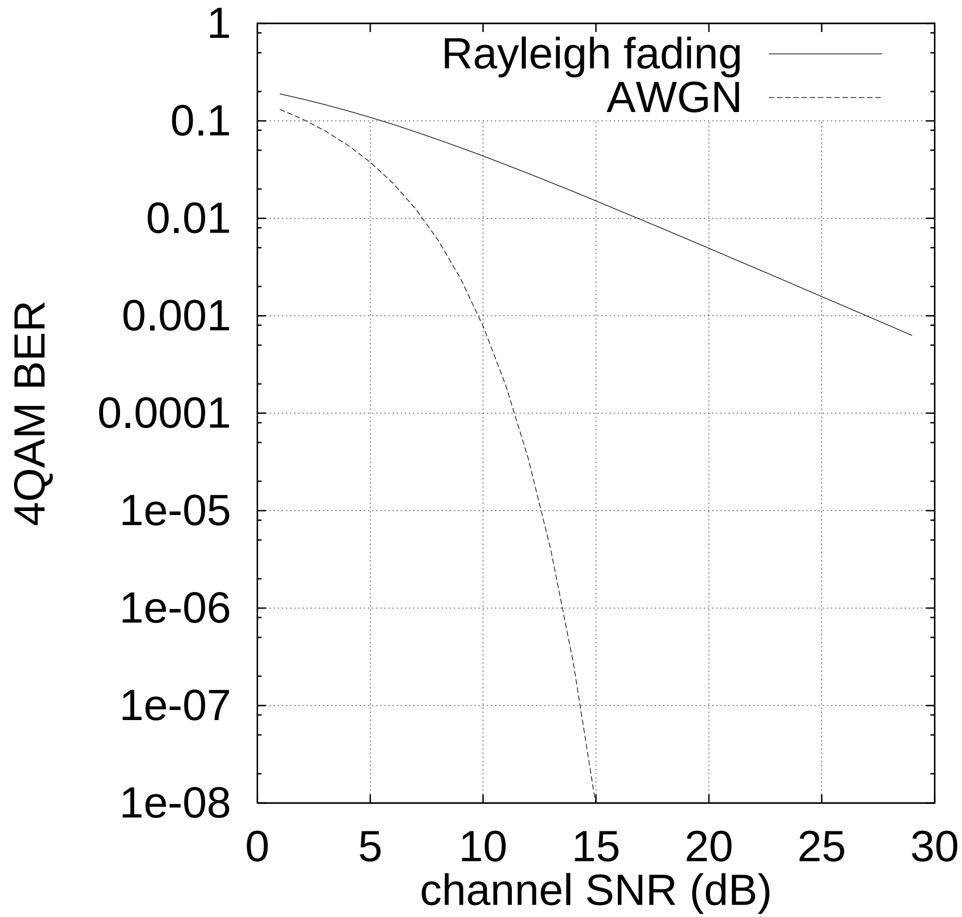
4QAM Fading / Non-Fading BER Comparison

- In fading, average SNR $\Lambda = 2\alpha_0^2 \frac{E_s}{N_0}$
  - Average BER

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\Lambda}{2+\Lambda}}\right)$$

- Compare with AWGN, average SNR is defined as $\widetilde{\Lambda} = E_s/N_0$
  - Average error probability

$$P_e = Q\left(\sqrt{\widetilde{\Lambda}}\right)$$



- Clearly, fading is "a big **killer**" limiting communication system's performance
  - Many mobile communication technologies developed in past three decades are "**counter-fading measures**"

16QAM Flat Fading Performance: C1 BER

- Motivate by success of applying non-fading 4QAM to fading 4QAM: try the same to 16QAM?
- For C1 bits, the decision boundaries are not changed:

$$I, Q > 0 \rightarrow i_1, q_1 = 0, \;\; I, Q \leq 0 \rightarrow i_1, q_1 = 1$$

But $\pm d$, $\pm 3d$ become $\pm \alpha d$, $\pm 3\alpha d$. Also $E_s = 10d^2$ and $\Lambda = 20\alpha_0^2 d^2/N_0$

- Thus, the instantaneous C1 error probability is

$$P_{e,1}(\lambda) = \frac{1}{2}Q\left(\frac{\alpha d}{\sigma_n}\right) + \frac{1}{2}Q\left(\frac{3\alpha d}{\sigma_n}\right) = \frac{1}{2}Q\left(\sqrt{\frac{\lambda}{5}}\right) + \frac{1}{2}Q\left(3\sqrt{\frac{\lambda}{5}}\right)$$

- The average C1 error probability is therefore (**success** again!)

$$P_{e,1} = \frac{1}{\Lambda}\int_0^{\infty} P_{e,1}(\lambda) e^{-\lambda/\Lambda} d\lambda = \frac{1}{4}\left(1 - \sqrt{\frac{2\alpha_0^2 \frac{d^2}{N_0}}{1 + 2\alpha_0^2 \frac{d^2}{N_0}}}\right) + \frac{1}{4}\left(1 - \sqrt{\frac{18\alpha_0^2 \frac{d^2}{N_0}}{1 + 18\alpha_0^2 \frac{d^2}{N_0}}}\right)$$

$$= \frac{1}{4}\left(1 - \sqrt{\frac{\Lambda}{10 + \Lambda}}\right) + \frac{1}{4}\left(1 - \sqrt{\frac{9\Lambda}{10 + 9\Lambda}}\right)$$

16QAM Fading Performance: C2 BER

- For C2 bits, the decision boundaries are changed to:

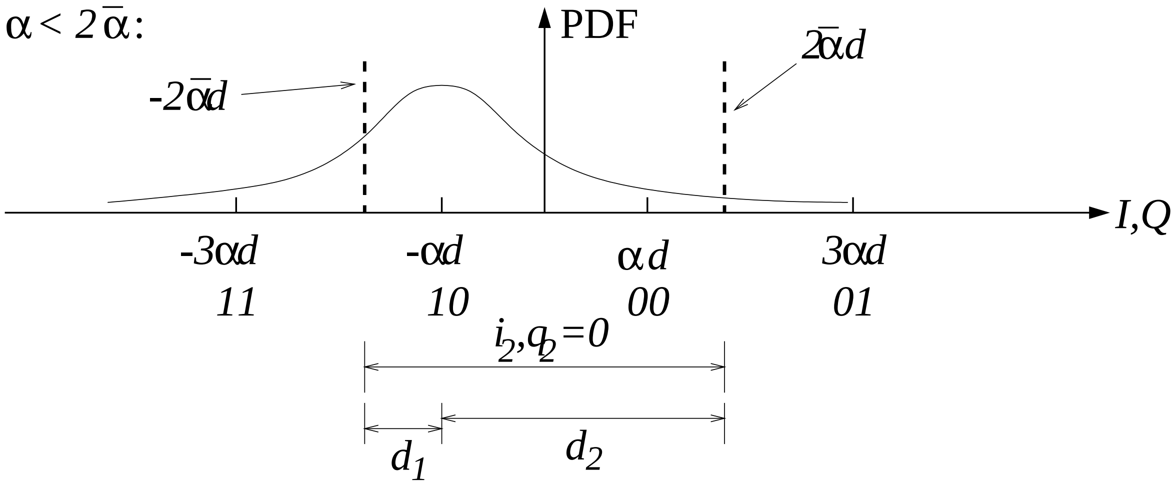
$$I, Q > 2\bar{\alpha}d \text{ or } I, Q \leq -2\bar{\alpha}d \rightarrow i_2, q_2 = 1$$

$$-2\bar{\alpha}d < I, Q \leq 2\bar{\alpha}d \rightarrow i_2, q_2 = 0$$

Note that the average value of α, i.e. $\bar{\alpha}$, has to be used for decision threshold

- Two cases of $i_2, q_2 = 0$ error need consideration

1. $i_2, q_2 = 0$ error in case of $\alpha < 2\bar{\alpha}$
   - Instantaneous symbols $-\alpha d$ and αd are within region defined by two decision boundaries
   - Error occurs when noise makes received signal outside the region and instantaneous C2 bit $= 0$ error probability for $\alpha < 2\bar{\alpha}$ is

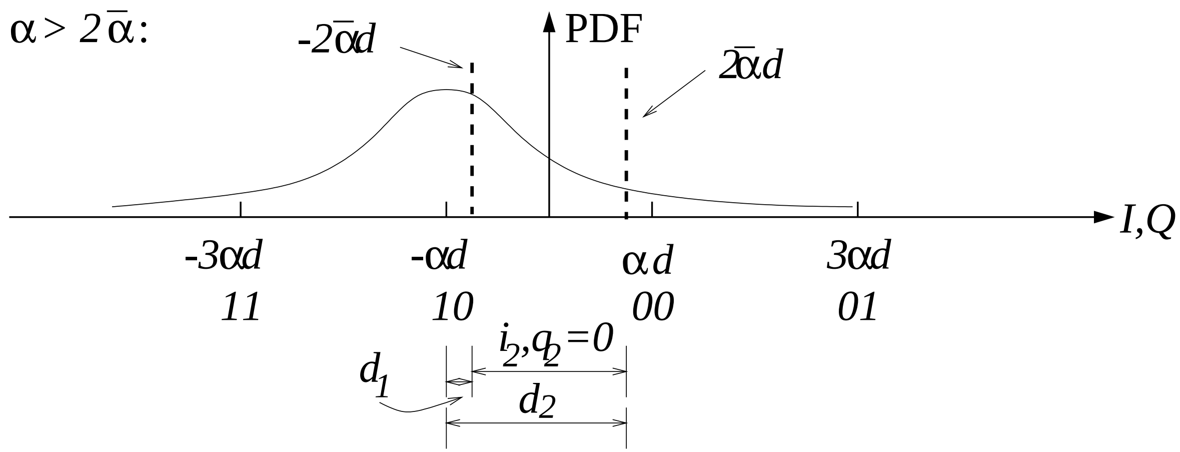


$$P_{2,0,<2}(\alpha) = Q\left(d_1/\sqrt{N_0/2}\right) + Q\left(d_2/\sqrt{N_0/2}\right) = Q\left(\frac{(2\bar{\alpha}-\alpha)d}{\sqrt{N_0/2}}\right) + Q\left(\frac{(2\bar{\alpha}+\alpha)d}{\sqrt{N_0/2}}\right)$$

16QAM: C2 BER (continue)

2. $i_2, q_2 = 0$ error in the case of $\alpha > 2\bar{\alpha}$: Instantaneous symbols $-\alpha d$ and αd are outside region defined by two decision boundaries

   - Correct decision occurs only when noise moves received signal inside the region
   - Thus instantaneous C2 bit $= 0$ error probability for $\alpha > 2\bar{\alpha}$ is



$$P_{2,0,>2}(\alpha) =$$

$$1 - Q\left(d_1/\sqrt{N_0/2}\right) + Q\left(d_2/\sqrt{N_0/2}\right) = 1 - Q\left(\frac{(-2\bar{\alpha} + \alpha)d}{\sqrt{N_0/2}}\right) + Q\left(\frac{(2\bar{\alpha} + \alpha)d}{\sqrt{N_0/2}}\right)$$

- Note $\alpha > 0$, the average error for $i_2, q_2 = 0$ is therefore

$$P_{2,0} = \int_0^{2\bar{\alpha}} P_{2,0,<2}(\alpha) p_\alpha(\alpha) d\alpha + \int_{2\bar{\alpha}}^{\infty} P_{2,0,>2}(\alpha) p_\alpha(\alpha) d\alpha$$

  - There exists closed form solution for this integration but it is very complicated
  - So far not too bad

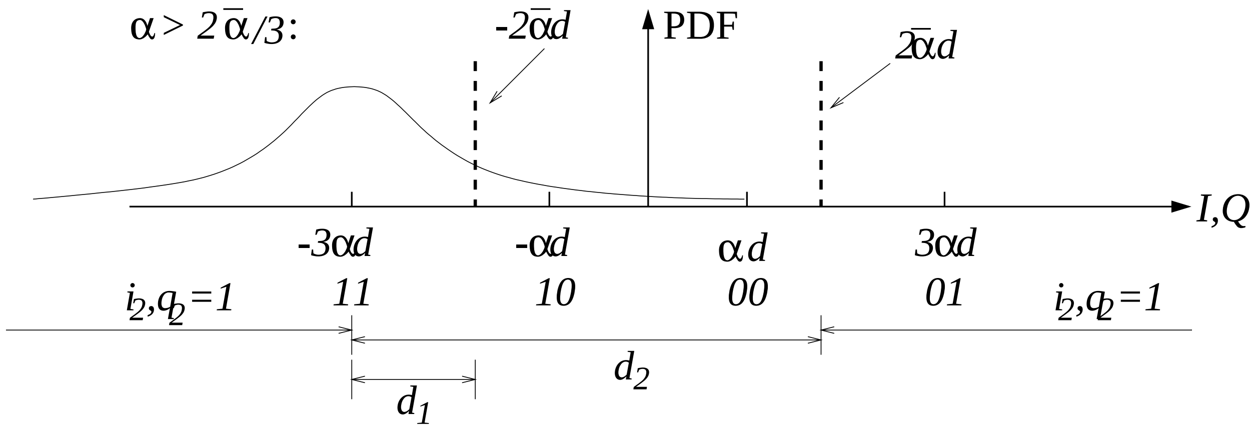
16QAM: C2 BER (continue)

- Two cases of $i_2, q_2 = 1$ error need consideration

1. $i_2, q_2 = 1$ error in case of $\alpha > 2\bar{\alpha}/3$

   – Instantaneous symbols $-3\alpha d$ and $3\alpha d$ are within correct regions corresponding to respective decision boundaries

   – Error occurs when noise makes received signal outside the corresponding region



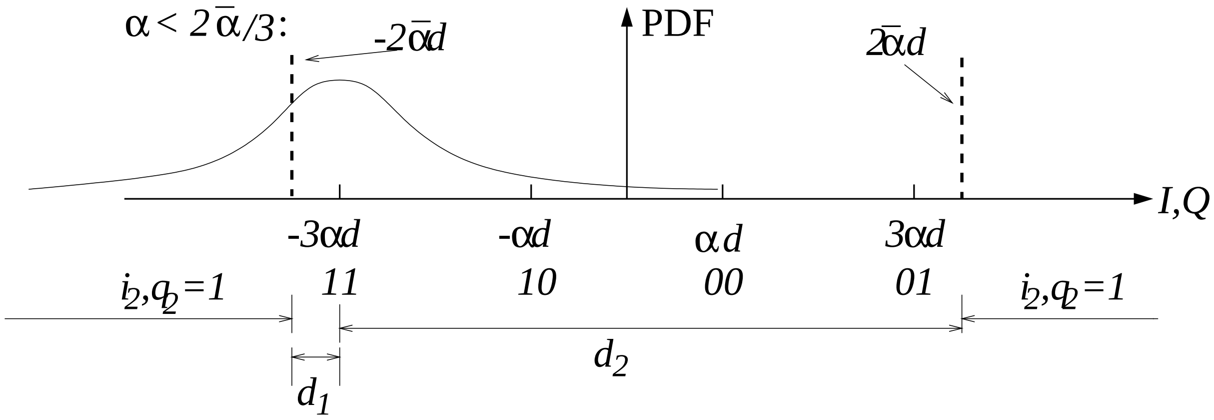
   – Thus instantaneous C2 bit $= 1$ error probability for $\alpha > 2\bar{\alpha}/3$ is

$$P_{2,1,>2/3}(\alpha) = Q\left(d_1/\sqrt{N_0/2}\right) - Q\left(d_2/\sqrt{N_0/2}\right)$$
$$= Q\left(\frac{(3\alpha - 2\bar{\alpha})d}{\sqrt{N_0/2}}\right) - Q\left(\frac{(3\alpha + 2\bar{\alpha})d}{\sqrt{N_0/2}}\right)$$

   – Oh, not, nightmare now

16QAM: C2 BER (continue)

2. $i_2, q_2 = 1$ error in case of $\alpha < 2\bar{\alpha}/3$



- Instantaneous symbols $-3\alpha d$ and $3\alpha d$ are outside correct regions defined by respective decision boundaries
- Correct decision occurs only when noise moves received signal to the correct region
- The instantaneous C2 bit $= 1$ error probability for $\alpha < 2\bar{\alpha}/3$ is

$$P_{2,1,<2/3}(\alpha) = 1 - Q\left(d_1/\sqrt{N_0/2}\right) - Q\left(d_2/\sqrt{N_0/2}\right)$$

$$= 1 - Q\left(\frac{(2\bar{\alpha} - 3\alpha)d}{\sqrt{N_0/2}}\right) - Q\left(\frac{(2\bar{\alpha} + 3\alpha)d}{\sqrt{N_0/2}}\right)$$

- The average error for $i_2, q_2 = 1$ is therefore

$$P_{2,1} = \int_0^{2\bar{\alpha}/3} P_{2,1,<2/3}(\alpha)p_\alpha(\alpha)d\alpha + \int_{2\bar{\alpha}/3}^{\infty} P_{2,1,>2/3}(\alpha)p_\alpha(\alpha)d\alpha$$

- Really nightmare, too much for me

16QAM Fading /Non-Fading BER Comparison

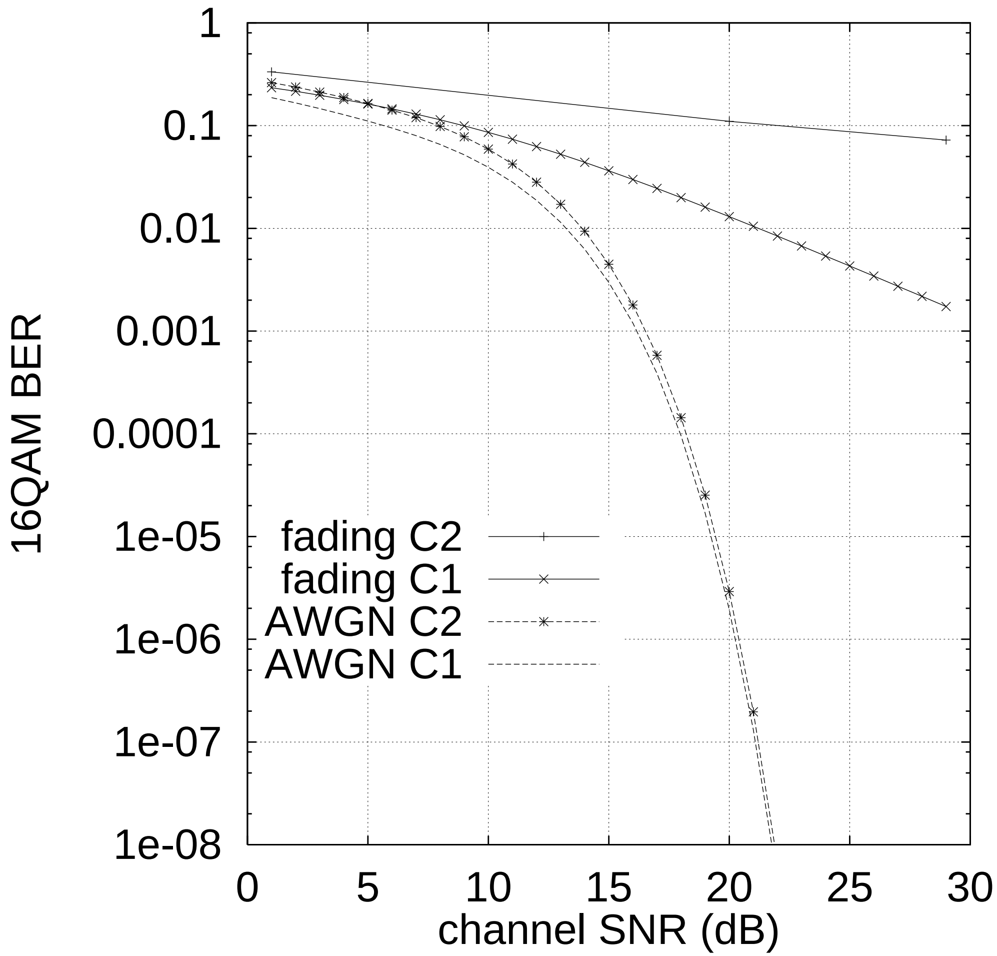
- OK, just to continue, we known
  - average C2 error probability is

$$P_{e,2} = \frac{1}{2}(P_{2,0} + P_{2,1})$$

  - Therefore average error probability for 16QAM is

$$P_e = \frac{1}{2}(P_{e,1} + P_{e,2})$$

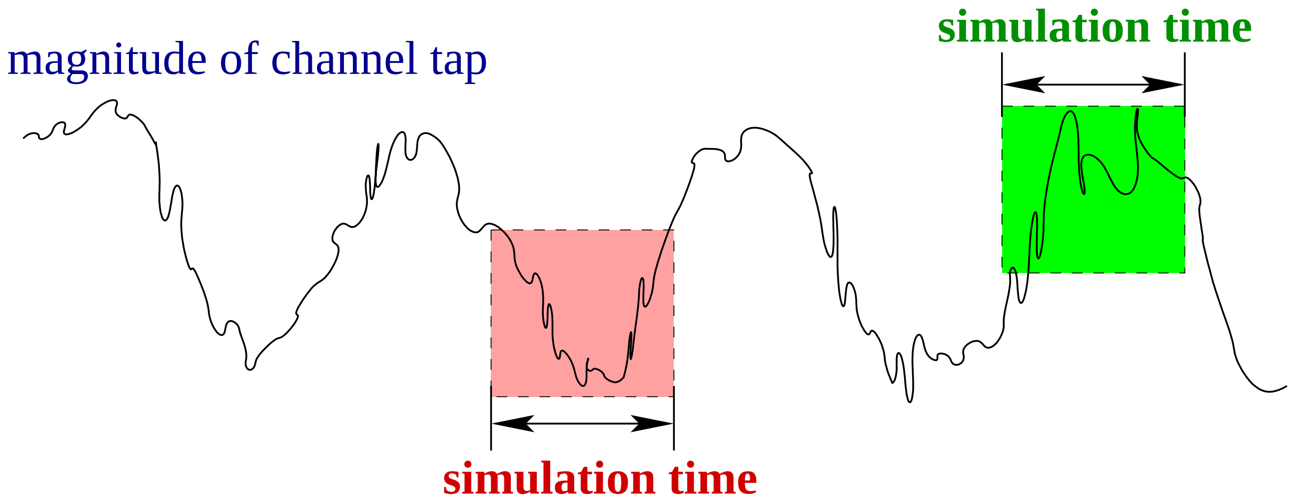
- Although no closed-form solution for $P_{e,2}$ and P_e, we can still make some sense out of them
  - 16QAM fading / non-fading BER:
  - Fading degrades BER performance seriously ⇒ counter fading measures



- Alternatively, Monte Carlo simulation is often used to evaluate fading BER
  - Recall slide 48 for flat Rayleigh fading channel simulation

Fading Channel Simulation Considerations

- For **uncorrelated** fading , Re$\{c\}$ and Im$\{c\}$ are i.i.d. Gaussians with zero mean and unit variance
  - Channel is simulated by generating such a sequence of $\{c_k\}$
- For **correlated** fading, $\{c_k\}$ can be generated by flat Rayleigh fading channel simulator of slide 48
- For AWGN channel BER simulation, at least a few hundred of error counts should be obtained
  - For fading channel BER simulation, this is **insufficient**



  - If you simulate at pink simulation zone, channel is in a deep fade, you get too bad BER
  - If you simulate at green simulation zone, channel gain is high, you get too good BER
- Basically, all magnitude possibilities of channel tap c should be simulated or seen, such a sequence $\{c_k\}$ can be very very long, and simulation is really **time-consuming**
  - Roughly tens of “cycles” in channel fading envelope should be simulated to get accurate average fading BER $\longrightarrow$ for slow fading channel, this is **very very long**

Summary

- Narrowband Rayleigh fading channel:
  - Fading envelope and phase PDFs, instantaneous and average channel SNRs, instantaneous and average error probabilities
- Fading channel BER performance analysis
  - For 4QAM, by applying non-fading BER result, closed-form BER solution of fading channel is obtained
  - Apply same approach to 16QAM is less fruitful, but nevertheless useful insight can be obtained
  - Fading degrades BER performance seriously $\Rightarrow$ counter fading measures are necessary
- Practical considerations in Monte Carlo simulation of fading channel performance
  - Basically, all magnitude possibilities of channel tap c should be simulated or seen
  - This corresponds to roughly tens of "cycles" in channel fading envelope