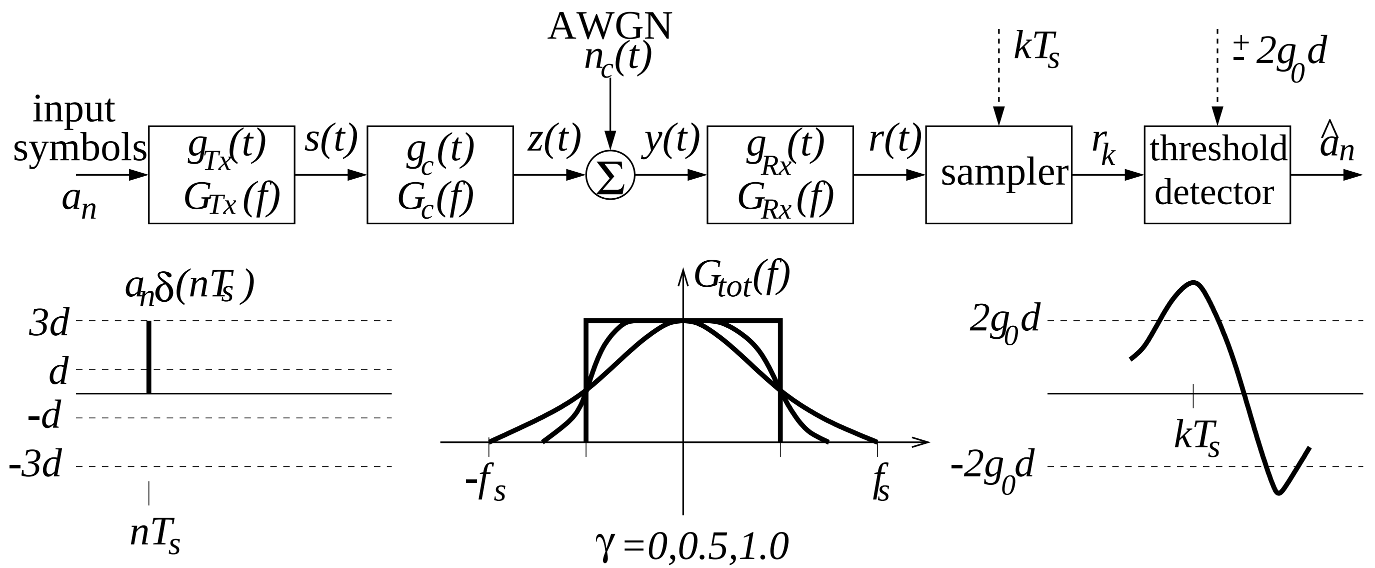
Revision of Modem so far

- Previous three lectures have covered most components of Modem, including
  - Two key **performance measures**: bandwidth efficiency and power efficiency, and pulse shaping
  - Several digital modulation schemes, with emphasis on PSK and QAM, mapping bits to symbols, and constellation **design considerations**
  - Carrier recovery (at least for BPSK, later we will generalise to QAM), allow receiver to remove carrier to get back to **baseband** signal
  - Clock recover (at least for binary signalling, later we will generalise to multilevel signalling), allow receiver to detect transmitted symbols and map back to bits

- This lecture, we exam baseband equivalent system and detector
  - In Digital Coding and Transmission, we learn Tx & Rx pulse shaping filter pair are designed for
  - We will see the connection with **optimal detection**

Baseband Equivalent System

- For M-QAM, I and Q branches are identical to a one-dimensional $\sqrt{M}$-ary system
- **Baseband equivalent** I or Q system (with 16QAM example) is:



- Transmitted signal, received signal and sampled received signal are, respectively,

$$s(t) = \sum_n a_n g_{\text{Tx}}(t - nT_s), \qquad r(t) = \sum_n a_n g_{\text{tot}}(t - nT_s) + n(t)$$

$$r_k = \sum_n a_n g_{n-k} + n_k = g_0 a_k + \sum_{\substack{n \\ n \neq k}} a_n g_{n-k} + n_k$$

  - Note $g_{\text{tot}}(t) = g_{\text{Rx}}(t) \star g_{\text{c}}(t) \star g_{\text{Tx}}(t)$ and $\{g_n\}$ are symbol-spaced samples of $g_{\text{tot}}(t)$

Optimal Tx & Rx Filter Design

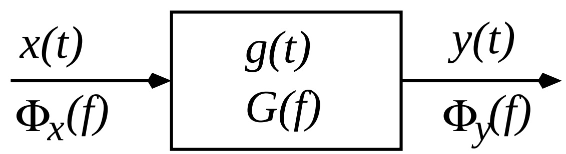
- **Optimal design**: Tx & Rx filters are identical to square root of required Nyquist raised cosine filter

  1. **Achieve zero ISI**:

     $$\sum_{\substack{n \\ n\neq k}} a_n g_{n-k} = 0 \Longrightarrow G_{\text{tot}}(f) = G_{\text{Rx}}(f)G_c(f)G_{\text{Tx}}(f) \text{ is required Nyquist filter}$$

     Ideal channel $G_c(f) = 1$, the combined $G_{\text{Rx}}(f)G_{\text{Tx}}(f)$ is required Nyquist filter
  2. **Maximise the received signal to noise ratio** $\Longrightarrow$ $g_{\text{Rx}}(t)$ matches to $g_{\text{Tx}}(t)$: $G_{\text{Rx}}(f) = G_{\text{Tx}}(f)$ identical to square root of required Nyquist raised cosine filter

- Linear system theory revisit
  - Informative in transfer domain
  - Recall power is area under PSD



Linear system

  - PSD of system output $y(t)$ is $\Phi_y(f) = |G(f)|^2\Phi_x(f)$, and power of $y(t)$ is thus

$$P_y = \int_{-\infty}^{\infty} \Phi_y(f)df = \int_{-\infty}^{\infty} |G(f)|^2\Phi_x(f)df$$

Maximising SNR

- The channel noise has a PSD $N(f) = \frac{N_0}{2}$ and it passes through $G_{\rm Rx}(f)$, thus the **noise power** at receiver is:

$$P_N = \int_{-\infty}^{\infty} |G_{\rm Rx}(f)|^2 N(f) df = \frac{N_0}{2} \int_{-\infty}^{\infty} |G_{\rm Rx}(f)|^2 df$$

- **Power of the transmitted signal** $s(t)$ is:

$$P_{\rm Tx} = \bar{a^2} \int_{-\infty}^{\infty} |G_{\rm Tx}(f)|^2 df = \bar{a^2} \int_{-\infty}^{\infty} \left| \frac{G_{\rm tot}(f)}{G_{\rm Rx}(f)} \right|^2 df$$

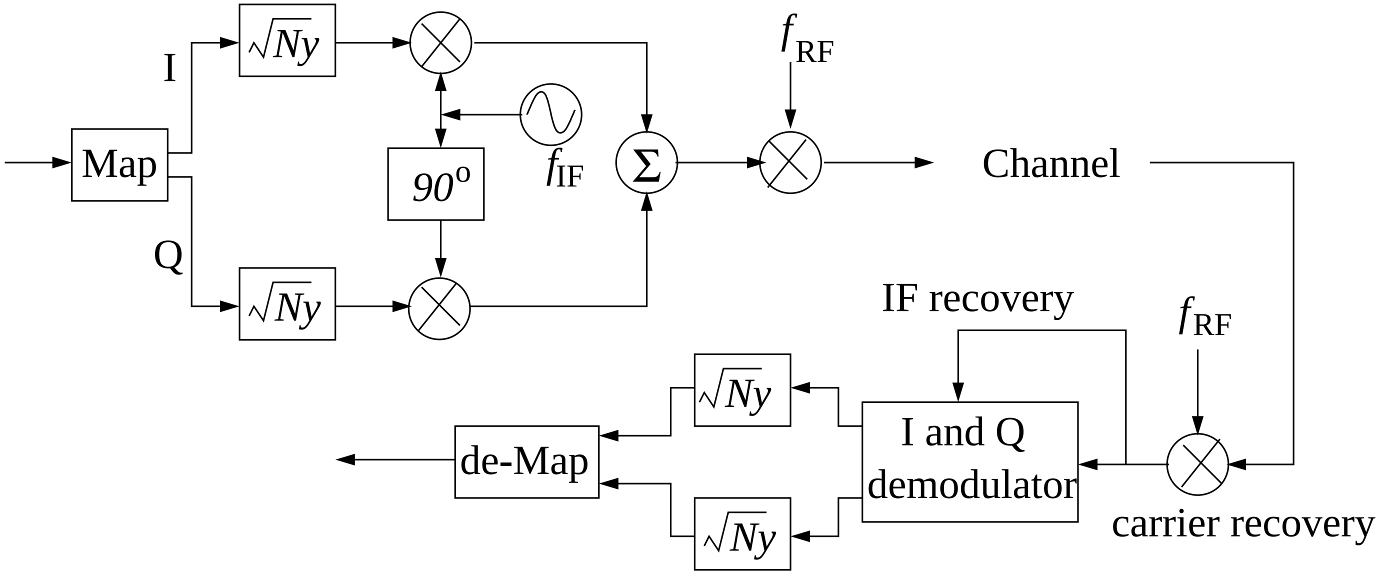
where $\bar{a^2}$ is average symbol power and we assume $G_c(f) = 1$

- **Maximising the received SNR** is equivalent to minimising P_N under the constraint of a constant $P_{\rm Tx}$. Standard optimisation result yields:
  - $g_{\rm Rx}(t)$ matches to $g_{\rm Tx}(t)$, or

$$G_{\rm Tx}(f) = G_{\rm Rx}(f) \text{ is square root of required Nyquist filter}$$

QAM Modem

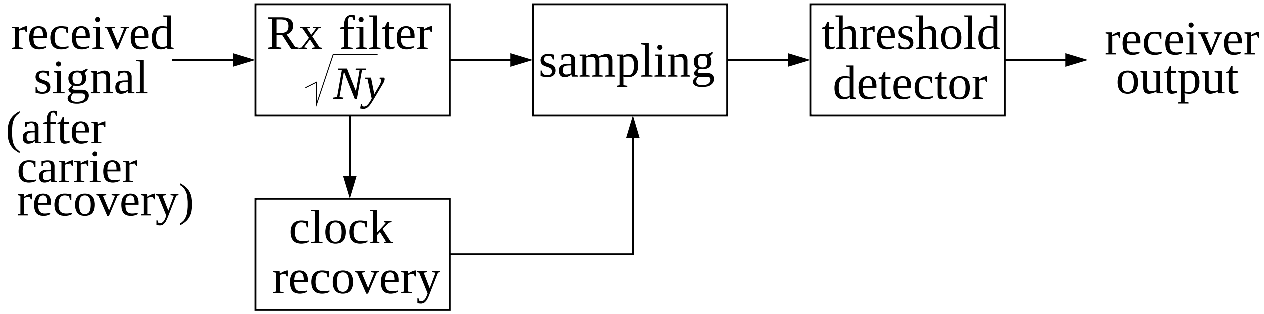
- QAM Modem of slide 92:
- We have discussed all operations at transmitter
- Receiver operations are:
  - Received signal after carrier recovery is demodulated into I/Q baseband signals



  - Clock recovery is needed for each baseband signal to obtain timing information and a detector recovers the transmitted I & Q symbols
  - Finally, a de-mapping converts symbol stream into bit stream
- We will discuss carrier and clock recovery for QAM later
- Here we first concentrate on **detection** part
  - There are many equivalent detection schemes, and three common ones are discussed
  - They are equivalent, all based on principle of maximising received signal to noise ratio
  - Maximising received SNR, in ideal AWGN, is equivalent to minimise detection error

Threshold Detection

- Receiver using optimal **threshold detection**
  - Optimal receive filter: square root of raised cosine filter identical to transmit filter
  - This as discussed previously maximises the receive SNR

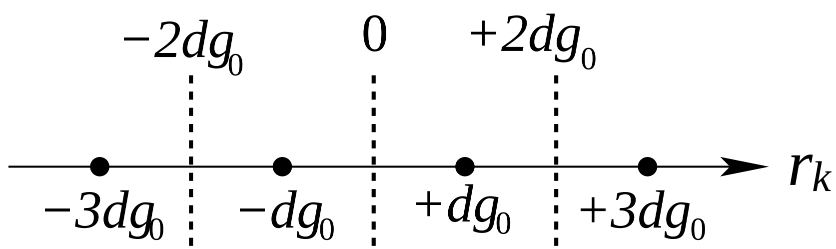


- For 16QAM example, I and Q have a 4-ary constellation, and let symbols at levels $3d$, d, $-d$ and $-3d$ be denoted as a_{+3}, a_{+1}, a_{-1} and a_{-3}
  - i.e. transmitted I or Q symbols $x_k \in \{a_{-3}, a_{-1}, a_{+1}, a_{+3}\}$
- Received signal sample

$$r_k = g_0 x_k + \varepsilon_k$$

  - g_0: channel state information (CSI), ε_k channel AWGN sample
- The thresholds at detector are set to $+2g_0d$, 0 and $-2g_0d$, and the decision is made according to where the sample r_k lies:

$$\widehat{x}_k = \begin{cases} a_{+3}, & \text{if } \ r_k > +2g_0 d \\ a_{+1}, & \text{if } \ 0 < r_k \le +2g_0 d \\ a_{-1}, & \text{if } \ -2g_0 d < r_k \le 0 \\ a_{-3}, & \text{if } \ r_k \le -2g_0 d \end{cases}$$

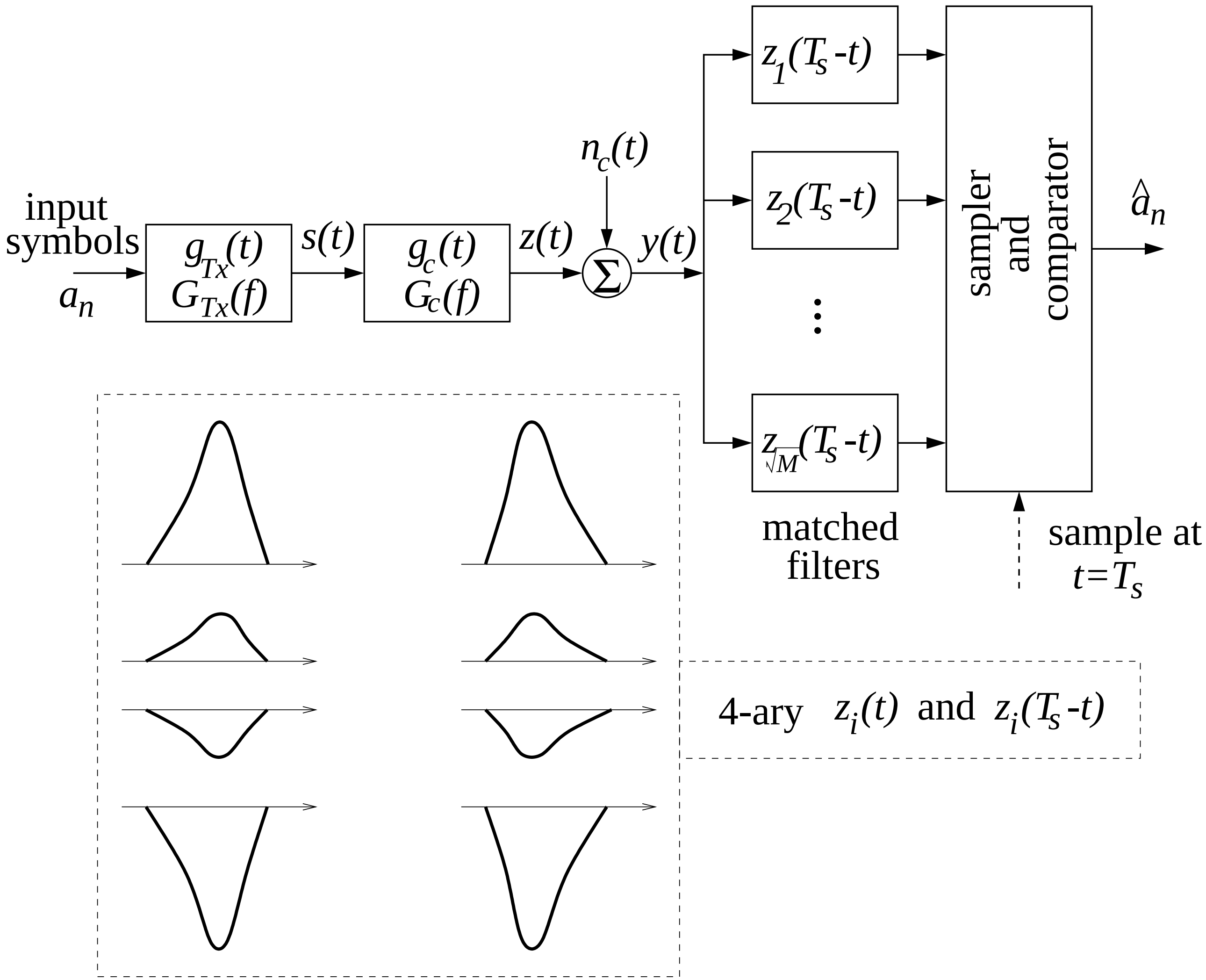


Matched Filter Detection

- Receiver using optimal **matched filter detection**

- Receiver consists of a bank of filters, each **matched** to one of the received waveforms
  - $z_i(T_s - t)$ is matched to $z_i(t)$

- Using the same principle of maximising received SNR
  - implemented as maximising the SNR at the output of matched filter for a given received signal waveform



Matched Filter Derivation

- Input to the receive filter is:

$$y(t) = \sum_n a_n h(t - nT_s) + n_c(t) = z(t) + n_c(t)$$

  - where $h(t) = g_{\rm Tx}(t) \star g_c(t)$. Let matched filter output be $r(t) = \bar{r}(t) + n(t)$

$$\bar{r}(t) = \mathcal{F}^{-1}[G_{\rm Rx}(f)Z(f)] = \int_{-\infty}^{\infty} G_{\rm Rx}(f)Z(f)e^{j2\pi ft}df$$

  - where $Z(f) = \mathcal{F}[z(t)]$. Note that $r_k = \bar{r}_k + n_k$ and the noise n_k has a variance

$$\sigma_n^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |G_{\rm Rx}(f)|^2 df$$

- The aim is to maximise the SNR at sampling instant $t = T_s$:

$$\text{SNR}_{T_s} = \frac{\bar{r}_k^2}{\sigma_n^2} = \frac{\left|\int_{-\infty}^{\infty} G_{\rm Rx}(f)Z(f)e^{j2\pi fT_s}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty}|G_{\rm Rx}(f)|^2 df}$$

- Using Schwartz's inequality leads to

$$\left|\int_{-\infty}^{\infty} G_{\rm Rx}(f)Z(f)e^{j2\pi fT_s}df\right|^2 \le \int_{-\infty}^{\infty}|G_{\rm Rx}(f)|^2 df \cdot \int_{-\infty}^{\infty}|Z(f)|^2 df$$

  - with equality holds if

$$G_{\rm Rx}(f) = cZ^*(f)e^{-j2\pi fT_s}$$

    where * denotes the complex conjugate

Matched Filter Derivation (continue)

- With this optimal $G_{\text{Rx}}(f)$, SNR_{T_s} is maximised

$$\text{SNR}_{T_s}^{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |Z(f)|^2 df$$

  – The optimal receive filter in time domain is then given by

$$g_{\text{Rx}}(t) = \mathcal{F}^{-1}[cZ^*(f)e^{-j2\pi f T_s}] = \begin{cases} c \cdot z(T_s - t), & 0 \le t \le T_s \\ 0, & \text{otherwise} \end{cases}$$

- Summary of **matched filter structure** in slide 106
  1. The received $z(t)$ takes waveforms $z_i(t)$, $1 \le i \le \sqrt{M}$, each corresponding to a symbol point. If $G_c(f) = 1$, $z_i(t) = s_i(t)$, i.e. waveforms of transmitted signal $s(t)$

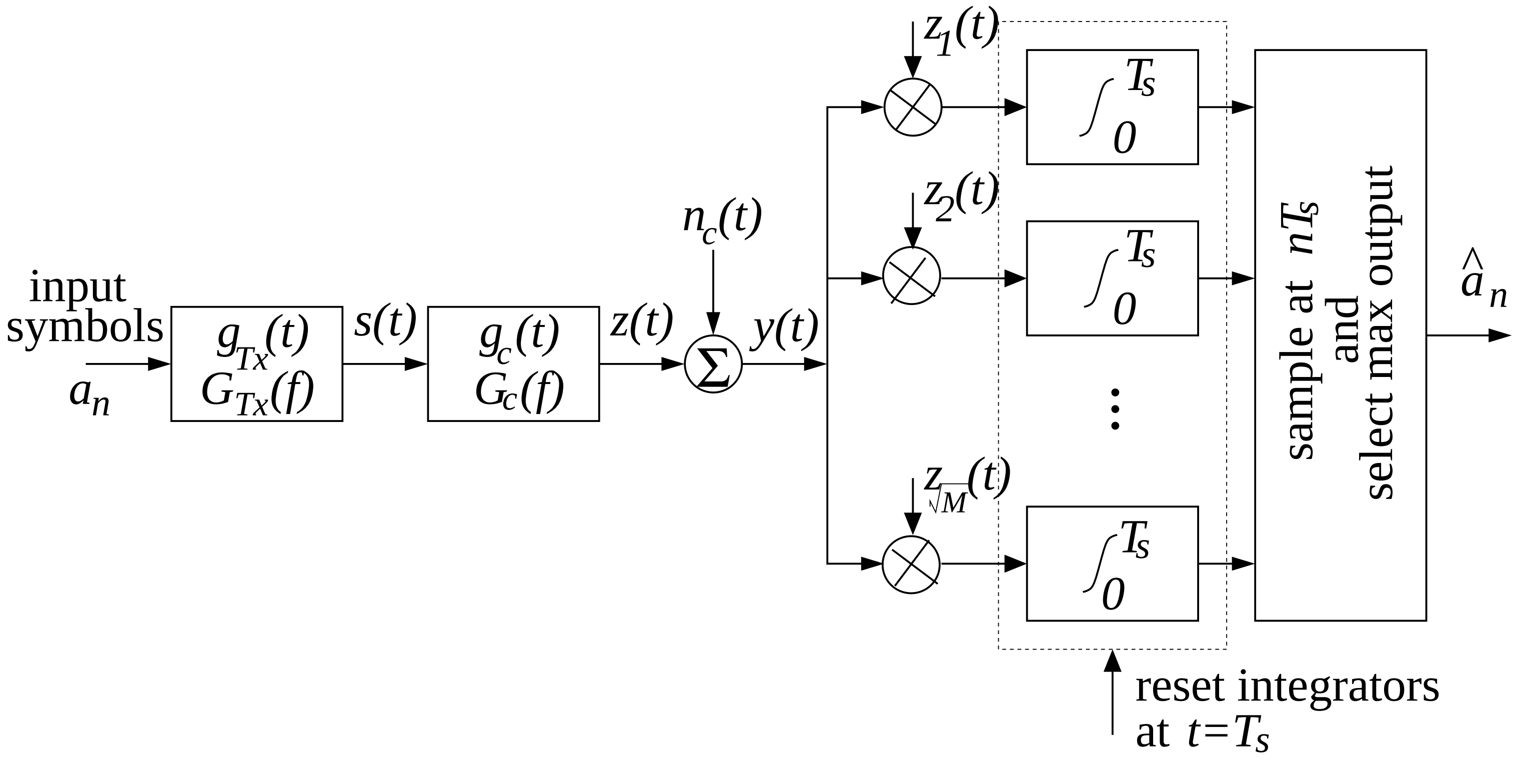
     As $s(t)$ has a shape of square root of Nyquist raised cosine pulse, Rx filter $g_{\text{Rx}}(t)$ has a similar shape. This is the same requirements of optimal Tx & Rx filtering
  2. The matched filter receiver consists of a bank of filters $z_i(T_s - t)$, $1 \le i \le \sqrt{M}$, each of which is **matched** to one of the received waveforms $z_i(t)$, $1 \le i \le \sqrt{M}$

     If the jth symbol point is transmitted, the waveform of $z(t)$ is $z_j(t)$. The output of matched filter $z_j(T_s - t)$ will be the largest, and all other matched filter outputs will be small

     The comparator can easily infer which symbol point is transmitted
- Matched filter detector is a commonly used receiver structure. In practice, $z_i(t)$ are unknown (channel is unknown), so $s_i(T_s - t)$ are used

Correlation Receiver

- **Correlation receiver**: an alternative implementation of matched filter receiver



- It first multiplies the received signal with the prototype signals $z_i(t)$, $1 \leq i \leq \sqrt{M}$, integrates and dumps them at kT_s

- This is followed by a decision circuit to choose the largest output

Correlation Receiver Derivation

- Integrator (receiver) output is convolution of received signal with receiver filter

$$r(t) = y(t) \star g_{\mathrm{Rx}}(t) = \int_0^t y(\tau) g_{\mathrm{Rx}}(t-\tau) d\tau$$

- Choosing $c = 1$ in the optimal Rx filter $g_{\mathrm{Rx}}(t-\tau) = c \cdot z(T_s - (t-\tau))$ leads to

$$r(t) = \int_0^t y(\tau) z(T_s - t + \tau) d\tau$$

- Integrate and dump at every $t = T_s$:

$$r(t = T_s) = r_k = \int_0^{T_s} y(\tau) z(\tau) d\tau$$

- In practice, received prototypes $z_i(t)$, $1 \le i \le \sqrt{M}$, are unknown, and transmitted prototypes $s_i(t)$ are used

Receiver Demapper

- For $M = 2^n$-ary constellation $\mathcal{X} = \{\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_M\}$, **mapper** at transmitter maps every n bits to a symbol
$$\{b_0, b_1, \cdots, b_{n-1}\} \rightarrow x \in \mathcal{X}$$
  - Transmitted bit sequence comes from channel coding encoder
- At receiver with **optimal** detection structure discussed previously, received signal sample is
$$y = g_0 x + \varepsilon$$
  - g_0: CSI, ε: channel AWGN sample with power N_0
  - **Detector** estimates transmitted symbol based on received signal sample
$$y \rightarrow \widehat{x} \in \mathcal{X}$$
- **Demapper** then maps this symbol estimate into corresponding estimated bits
$$\widehat{x} \rightarrow \{\widehat{b}_0, \widehat{b}_1, \cdots, \widehat{b}_{n-1}\}$$
  - This is called **hard** demapper, as it produces "hard" estimates of transmitted bits
- Estimated hard bits are passed to channel coding decoder
  - This type of channel coding decoder is called hard-input decoder, as it accepts "hard" bit input
- There are soft channel coding decoders accepting "soft" bit input
  - **Soft** demapper produces "soft" estimates or log likelihood ratios (LLRs) of $\{b_0, b_1, \cdots, b_{n-1}\}$

Soft Demapper

- Look at ith bit b_i, and divide $\mathcal{X}$ into two subsets with $b_i = 0$ and $b_i = 1$, respectively

$$\mathcal{X} = \mathcal{X}_i^{(0)} \bigcup \mathcal{X}_i^{(1)}$$

- Optimal **log-MAP** demapper calculates LLR of b_i

$$L_i = \log \frac{P(b_i = 0|y)}{P(b_i = 1|y)} = \log \frac{\sum_{x \in \mathcal{X}_i^{(0)}} p(y|x)}{\sum_{x \in \mathcal{X}_i^{(1)}} p(y|x)}$$

  – As PDF of ε is Gaussian

$$p(y|x) = \frac{1}{\pi N_0} \exp\left(-|y - g_0 x|^2 / N_0\right)$$

- **Max-log-MAP** approximation is near optimal
  – With max-sum approximation

$$\sum_j z_j \approx \max_j z_j, \ \forall z_j \geq 0$$

  – LLR L_i of b_i can be calculated according to

$$L_i \approx \log \frac{\max_{x \in \mathcal{X}_i^{(0)}} p(y|x)}{\max_{x \in \mathcal{X}_i^{(1)}} p(y|x)} = -\frac{1}{N_0} \left(\min_{x \in \mathcal{X}_i^{(0)}} |y - g_0 x|^2 - \min_{x \in \mathcal{X}_i^{(1)}} |y - g_0 x|^2 \right)$$

Summary

- Baseband equivalent system, and optimal transmit and receive filtering:

  $G_{\text{Tx}}(f) = G_{\text{Rx}}(f)$ and combined $G_{\text{Tx}}(f)G_{\text{Rx}}(f)$ is required Nyquist filter

  1. Achieve zero ISI, and
  2. maximise the receive SNR

- Receiver detector structure
  - Threshold detection receiver, matched filter receiver and correlation receiver
  - They are equivalent and all based on the principle of maximising the receive SNR
  - Note that maximum receive SNR is directly linked to minimum detection error

- Receiver demapper
  - Hard demapper and soft demapper