Revision of Lecture Eight

- Baseband equivalent system and requirements of optimal transmit and receive filtering: (1) achieve zero ISI, and (2) maximise the receive SNR

- Three detection schemes:
  - Threshold detection receiver, matched filter receiver and correlation receiver
  - They are equivalent and all based on the principle of maximising the receive SNR
  - Detector is **important**, as every communication system has one
  - So you should know detector schematic diagrams and how they work

- Last component of modem receiver: demapper

- Note that maximising receive SNR is directly linked to minimum detection error, and in next two lectures we analyse performance of the system
  - Specifically, we analyse bit error ratio of the system in AWGN and fading channels

Performance in AWGN

- In AWGN channel, sampled quadrature (I or Q) component of receive signal is

$$r_k = \bar{r}_k + n_k = g_0 x_k + n_k$$

  - x_k: transmitted symbol at k, g_0: known CSI (for coherent system) and may be assumed $g_0 = 1$
- With optimal transmit & receive filtering as well as channel $G_c(f) = 1$,

$$G_{\mathrm{Rx}}(f) = G_{\mathrm{Tx}}(f) \quad \text{and} \quad G_{\mathrm{Tx}}(f) G_{\mathrm{Rx}}(f) \quad \text{is required Nyquist filter}$$

  - Note that

$$\int_{-\infty}^{\infty} |G_{\mathrm{Rx}}(f)|^2 df = 1 \qquad \int_{-\infty}^{\infty} |G_{\mathrm{Tx}}(f)|^2 \cdot |G_{\mathrm{Rx}}(f)|^2 df = 1$$

  - Thus, received noise sample n_k has power

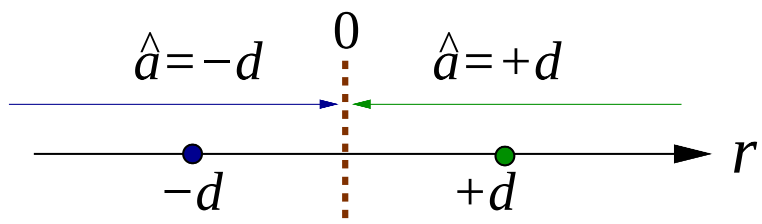
$$P_N = \sigma_n^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} |G_{\mathrm{Rx}}(f)|^2 df = \frac{N_0}{2}$$

  - Let $\bar{a^2}$ be average (I or Q) symbol power, received signal sample $\bar{r}_k$ has power

$$P_{\mathrm{Rx}} = \bar{a^2} \int_{-\infty}^{\infty} |G_{\mathrm{Tx}}(f)|^2 \cdot |G_{\mathrm{Rx}}(f)|^2 df = \bar{a^2}$$

- Detection error probability depends on noise probability density function, which is Gaussian with variance σ_n^2, and receiver output SNR$= P_{\mathrm{Rx}}/P_N$
  - Maximising receiver output SNR in AWGN leads to minimising detection error probability

BPSK Bit Error Rate



- BPSK transmitter: bit 0$\longrightarrow$ $a = +d$, bit 1$\longrightarrow$ $a = -d$
- As all detectors are equivalent, we assume threshold detector
  – Received sample is

$$r = a + n, \;\; a \in \{\pm d\} \;\; \text{and} \;\; n \in N(0, \sigma^2)$$

- As the **decision boundary** is $r = 0$, the **threshold** decision rule is

$$r > 0 \rightarrow \widehat{a} = d, \;\; r \leq 0 \rightarrow \widehat{a} = -d$$

- Using **Bayes** theorem, the error probability or BER is given by

$$\begin{aligned} P_e = P(\widehat{a} \neq a) \quad &= \quad P(a = d \cap \widehat{a} = -d) + P(a = -d \cap \widehat{a} = d) \\ &= \quad P(a = d)P(\widehat{a} = -d|a = d) + P(a = -d)P(\widehat{a} = d|a = -d) \end{aligned}$$

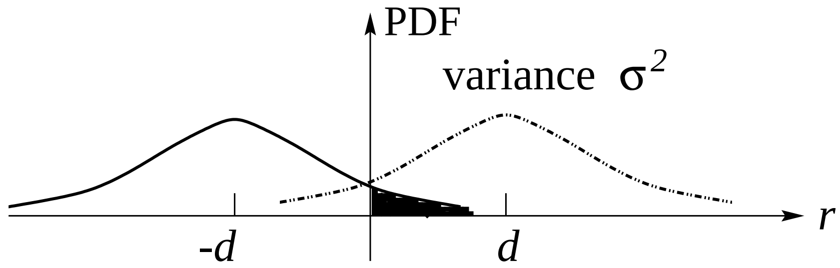
- As transmitted bit is equally likely to be 0 or 1, the two *a prior* probabilities are

$$P(a = d) = P(a = -d) = \frac{1}{2}$$

- Given $a = -d$, the decision $\widehat{a} = d$ means that $r = -d + n > 0$ or noise value $n > d$, and the **conditional probability** $P(\widehat{a} = d|a = -d)$ equals to

$$P(n > d) = \int_d^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \frac{1}{\sqrt{2\pi}} \int_{d/\sigma}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = Q\left(d/\sigma\right)$$

BPSK BER: Result

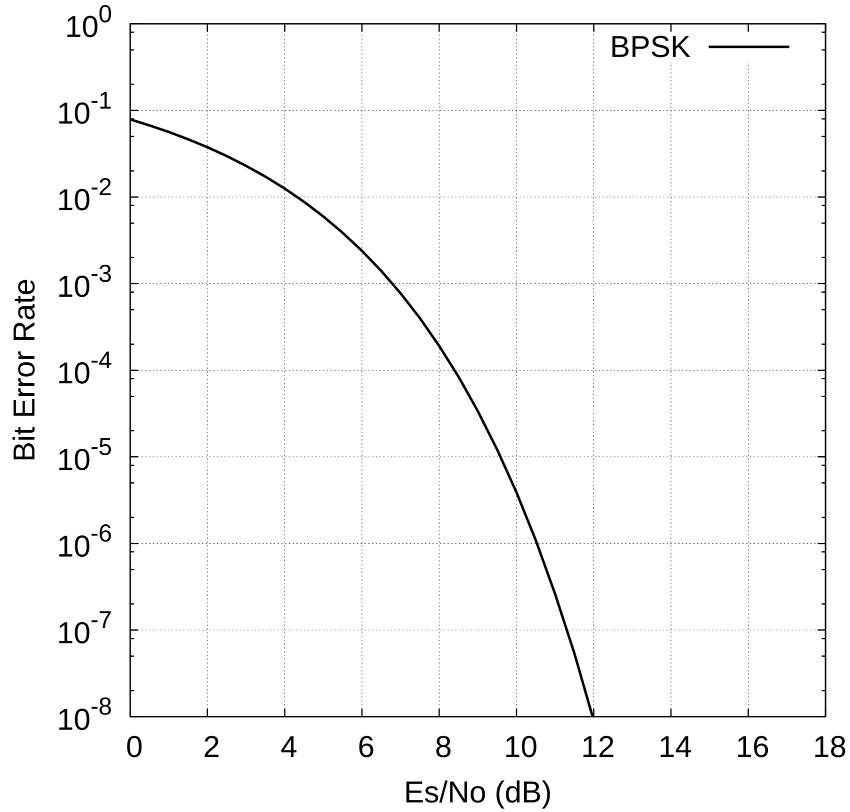


- Interpretation of conditional error probability or error Q-function $P(\widehat{a} = d | a = -d)$:
  - Gaussian tail area over threshold $r = 0$
- Similarly, the other conditional error probability

$$P(\widehat{x} = -d | x = d) = Q\left(d/\sigma\right)$$

- As average signal power $E_s = \frac{1}{2}(d^2 + d^2) = d^2$ and noise power $\frac{N_0}{2} = \sigma^2$, BPSK BER is

$$P_e = \frac{1}{2} Q\left(d/\sigma\right) + \frac{1}{2} Q\left(d/\sigma\right)$$
$$= Q\left(d/\sigma\right) = Q\left(\sqrt{\text{SNR}}\right)$$
$$= Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

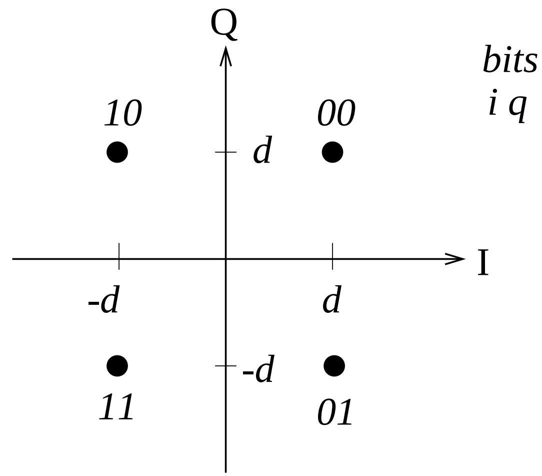


- Note the error Q-function rapidly decreases as the SNR increases
  - Maximising receive SNR leads to minimising error probability or bit error rate

4QAM Bit Error Rate

- 4QAM or QPSK: I and Q components are both BPSK, and average signal power is $E_s = 2d^2$, noise power $2\sigma^2 = N_0$
- Let $r = r_I + jr_Q$ be received signal sample, then decision rule is

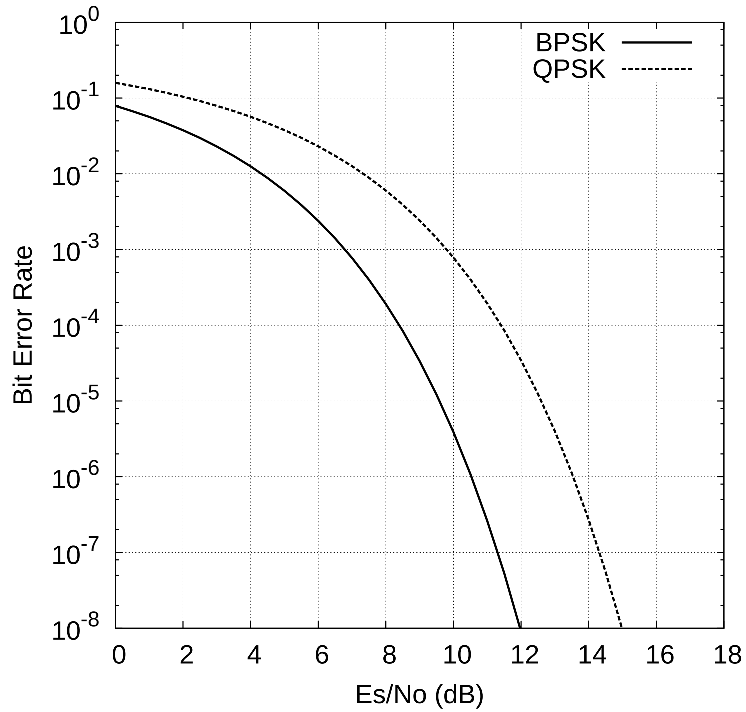
$$r_I, r_Q > 0 \rightarrow i, q = 0 \qquad r_I, r_Q \le 0 \rightarrow i, q = 1$$



- Applying BPSK result to both I and Q yields
  - $P_{e,I} = Q(d/\sigma)$ and $P_{e,Q} = Q(d/\sigma)$
- **Average error rate** for 4QAM is then

$$P_e = \frac{1}{2}(P_{e,I} + P_{e,Q}) = Q(d/\sigma) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

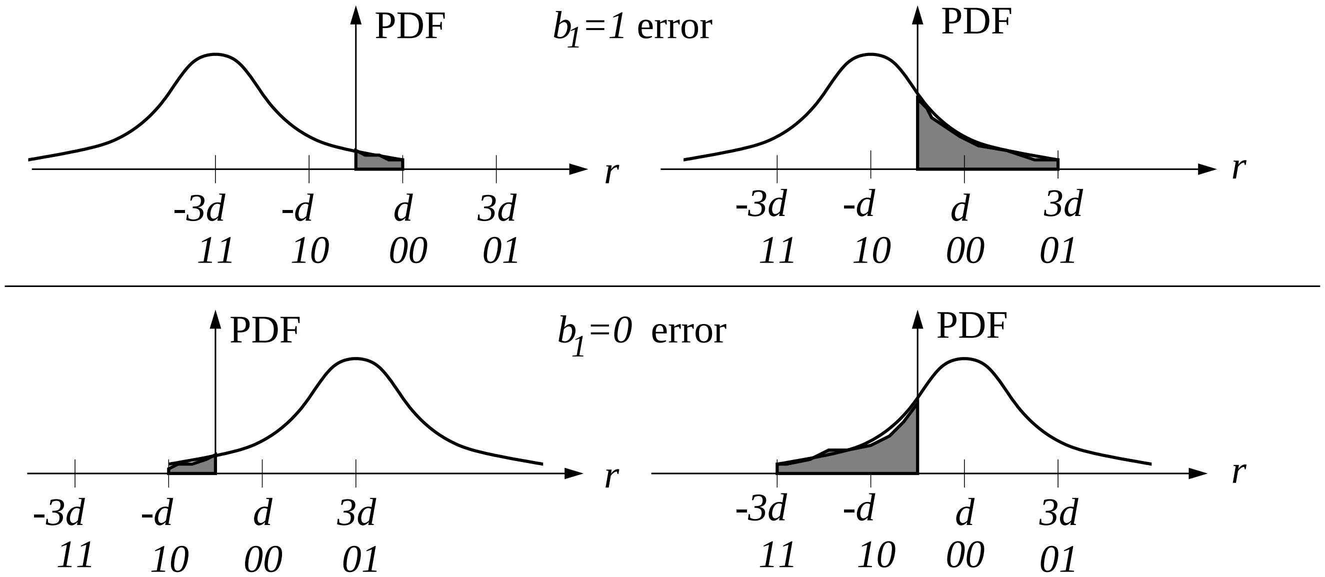
- In comparison to BPSK, who has $P_e = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$
  - For same bit rate R_b, 4QAM bandwidth is half
  - but requires higher signal power (factor of 2 or 3 dB) to achieve same level of BER
- When plot as functions of $\frac{E_b}{N_0}$, how two BER curves will look like? where E_b is average bit energy



4-ary Constellation BER

- 4-ary constellation: 2 bits per symbol and Gray coding $b_1 b_2 : 01, 00, 10, 11 \rightarrow 3d, d, -d, -3d$
  - Most significant bit b_1 and least significant bit b_2 have **different immunities to noise**, and are called **class-one** and **class-two** bits, respectively
  - Intrinsically, as though C1 and C2 bits are transmitted through two different **sub-channels**
- C1 bit decision rule: $r > 0 \rightarrow b_1 = 0, \quad r \leq 0 \rightarrow b_1 = 1$
- C1 bit BER:

$$\begin{aligned} P_{e,1} &= \frac{2}{4} Q\left(d/\sigma_n\right) \\ &+ \frac{2}{4} Q\left(3d/\sigma_n\right) \\ &= \frac{1}{2} Q\left(d/\sigma_n\right) \\ &+ \frac{1}{2} Q\left(3d/\sigma_n\right) \end{aligned}$$



- C1 bits b_1 are at a protection distance of d from the decision boundary ($r = 0$) for 50% of the time, and their protection distance is $3d$ for the other 50% of the time

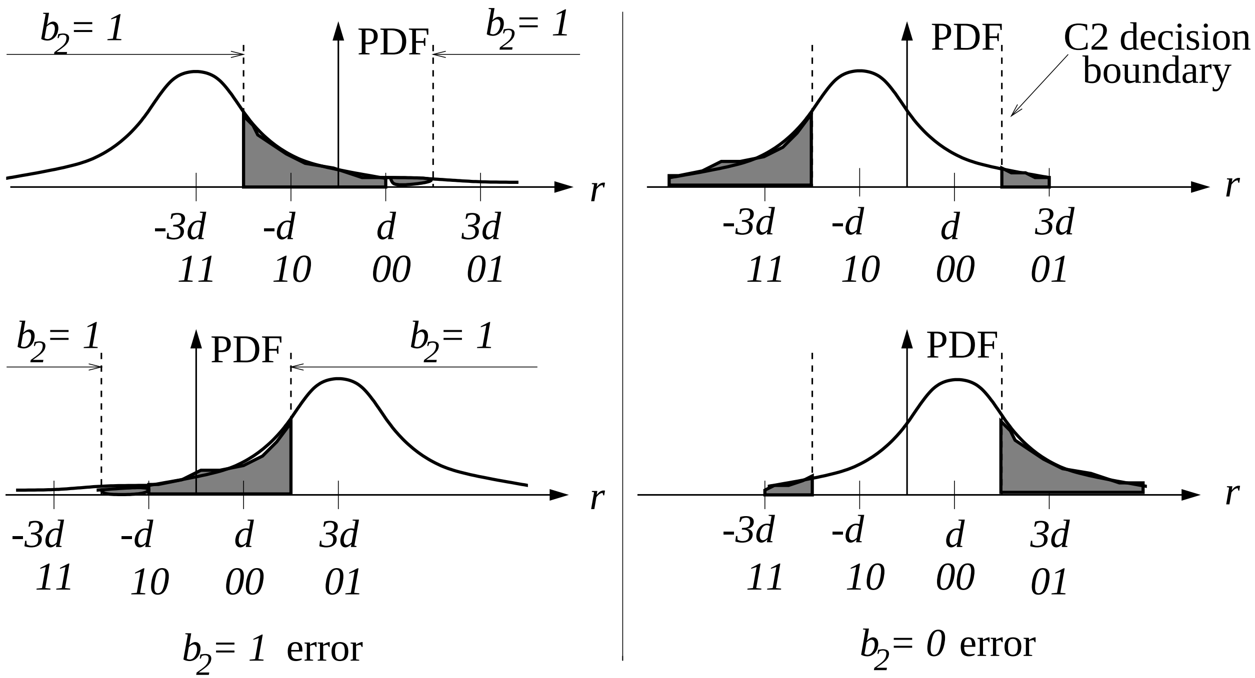
4-ary Constellation BER (continue)

- C2 bit decision rule: $r > 2d$ or $r \leq -2d \rightarrow b_2 = 1$, $-2d < r \leq 2d \rightarrow b_2 = 0$
  - For symbol $-3d$: when noise value $n > d$, $r > -2d$ and decision error occurs but when noise value $n > 5d$, $r > 2d$ and decision is correct again, thus the conditional error rate is $Q(d/\sigma_n) - Q(5d/\sigma_n)$
  - For symbol $-d$, it is $Q(d/\sigma_n) + Q(3d/\sigma_n)$
- C2 bit error rate

$$P_{e,2} = \frac{4}{4}Q(d/\sigma_n) + \frac{2}{4}Q(3d/\sigma_n) - \frac{2}{4}Q(5d/\sigma_n)$$

Note that

$$P_{e,2} \approx 2P_{e,1}$$



- **Average error rate** of 4-ary constellation:

$$P_e = \frac{1}{2}(P_{e,1} + P_{e,2}) = \frac{3}{4}Q\left(\frac{d}{\sigma_n}\right) + \frac{1}{2}Q\left(\frac{3d}{\sigma_n}\right) - \frac{1}{4}Q\left(\frac{5d}{\sigma_n}\right)$$

16QAM Bit Error Rate

- 16QAM: I and Q are both 4-ary
  - C1 bit decision:

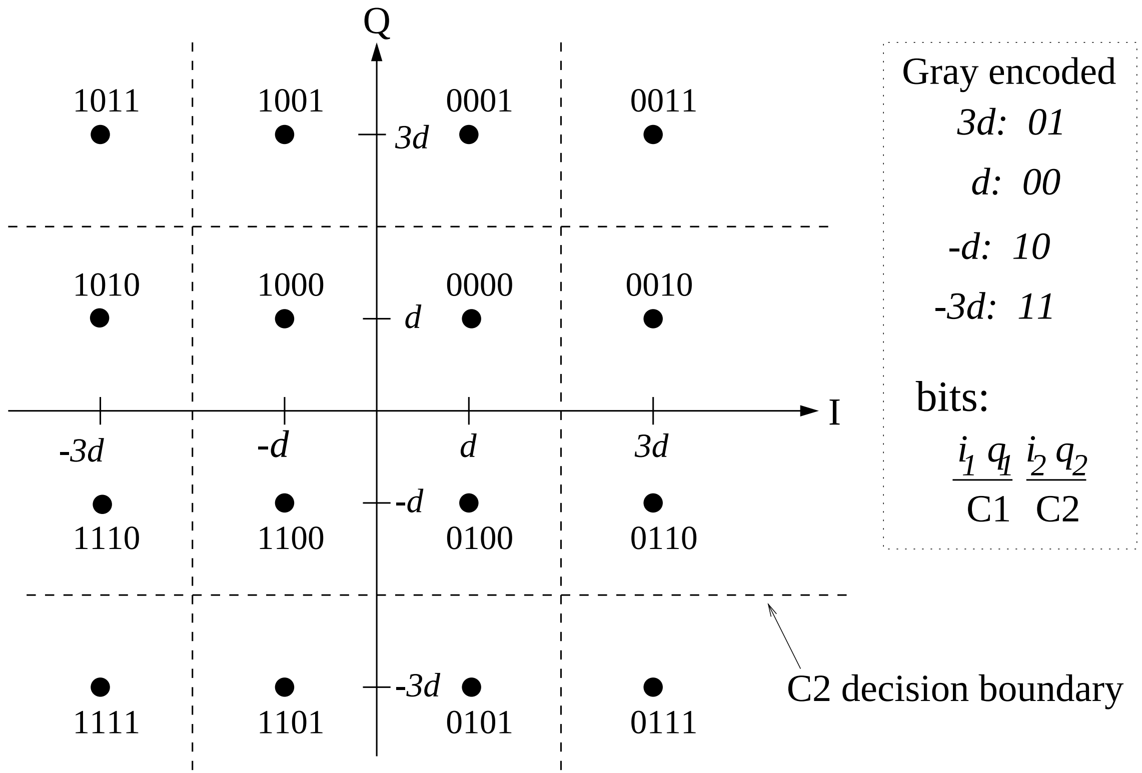
$$I, Q > 0 \rightarrow i_1, q_1 = 0$$

$$I, Q \leq 0 \rightarrow i_1, q_1 = 1$$

  - C2 bit decision:

$$I, Q > 2d \text{ or } I, Q \leq -2d \rightarrow i_2, q_2 = 1$$

$$-2d < I, Q \leq 2d \rightarrow i_2, q_2 = 0$$



- Signal power $E_s = 10d^2$ and noise power $\sigma_n^2 = \frac{N_0}{2} \rightarrow d/\sigma_n = \sqrt{E_s/5N_0}$, and 16QAM BER:

$$P_e = \frac{1}{2}(P_{e,I} + P_{e,Q}) = \frac{3}{4}Q\left(\frac{d}{\sigma_n}\right) + \frac{1}{2}Q\left(\frac{3d}{\sigma_n}\right) - \frac{1}{4}Q\left(\frac{5d}{\sigma_n}\right)$$

$$= \frac{3}{4}Q\left(\sqrt{E_s/5N_0}\right) + \frac{1}{2}Q\left(3\sqrt{E_s/5N_0}\right) - \frac{1}{4}Q\left(5\sqrt{E_s/5N_0}\right)$$

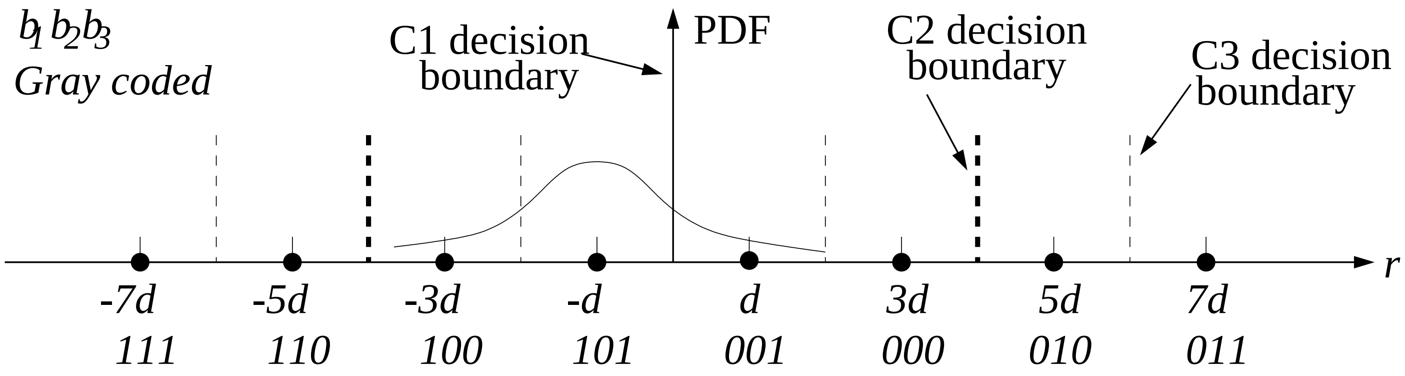
8-ary Constellation BER

- 8-ary: $b_1 b_2 b_3$ Gray coded. **3 classes** of bits, C1 b_1 has the highest immunity to noise, and C3 b_3 has the lowest, as though these three classes of bits were transmitted through 3 different **sub-channels**
  - One C1 decision boundary, two C2 decision boundaries, and four C3 decision boundaries

- C1 decision rule:

$$r > 0 \rightarrow b_1 = 0$$
$$r \le 0 \rightarrow b_1 = 1$$

Thus, conditional error rate for symbol $-7d$ is $Q\left(7d/\sigma_n\right)$ and so on



- C1 bit error rate:

$$P_{e,1} = \frac{1}{4}\left(Q\left(d/\sigma_n\right) + Q\left(3d/\sigma_n\right) + Q\left(5d/\sigma_n\right) + Q\left(7d/\sigma_n\right)\right)$$

- C2 decision: $r > 4d$ or $r \le -4d \rightarrow b_2 = 1, \quad -4d < r \le 4d \rightarrow b_2 = 0$

$$\begin{aligned} P_{e,2} \quad = \quad & \frac{1}{2}Q\left(d/\sigma_n\right) + \frac{1}{2}Q\left(3d/\sigma_n\right) + \frac{1}{4}Q\left(5d/\sigma_n\right) + \frac{1}{4}Q\left(7d/\sigma_n\right) \\ & -\frac{1}{4}Q\left(9d/\sigma_n\right) - \frac{1}{4}Q\left(11d/\sigma_n\right) \end{aligned}$$

8-ary Constellation BER (continue)

- C3 decision: $r > 6d$ or $r \le -6d$ or $-2d < r \le 2d \rightarrow b_3 = 1$

$$-6d < r \le -2d \text{ or } 2d < r \le 6d \rightarrow b_3 = 0$$

  – Hence, C3 error probability

$$\begin{aligned} P_{e,3} \quad = \quad & Q\left(d/\sigma_n\right) + \frac{3}{4}Q\left(3d/\sigma_n\right) - \frac{3}{4}Q\left(5d/\sigma_n\right) - \frac{1}{2}Q\left(7d/\sigma_n\right) \\ & + \frac{1}{2}Q\left(9d/\sigma_n\right) + \frac{1}{4}Q\left(11d/\sigma_n\right) - \frac{1}{4}Q\left(13d/\sigma_n\right) \end{aligned}$$

- Note $P_{e,3} \approx 2P_{e,2}$ and $P_{e,2} \approx 2P_{e,1}$
- Average error of 8-ary constellation is: $P_e = \frac{1}{3}(P_{e,1} + P_{e,2} + P_{e,3})$ or

$$P_e = \frac{7}{12}Q\left(d/\sigma_n\right) + \frac{1}{2}Q\left(3d/\sigma_n\right) - \frac{1}{12}Q\left(5d/\sigma_n\right) + \frac{1}{12}Q\left(9d/\sigma_n\right) - \frac{1}{12}Q\left(13d/\sigma_n\right)$$

Since the average symbol energy of 8-ary is $E_s = 21d^2$ and $\sigma_n^2 = N_0/2$,

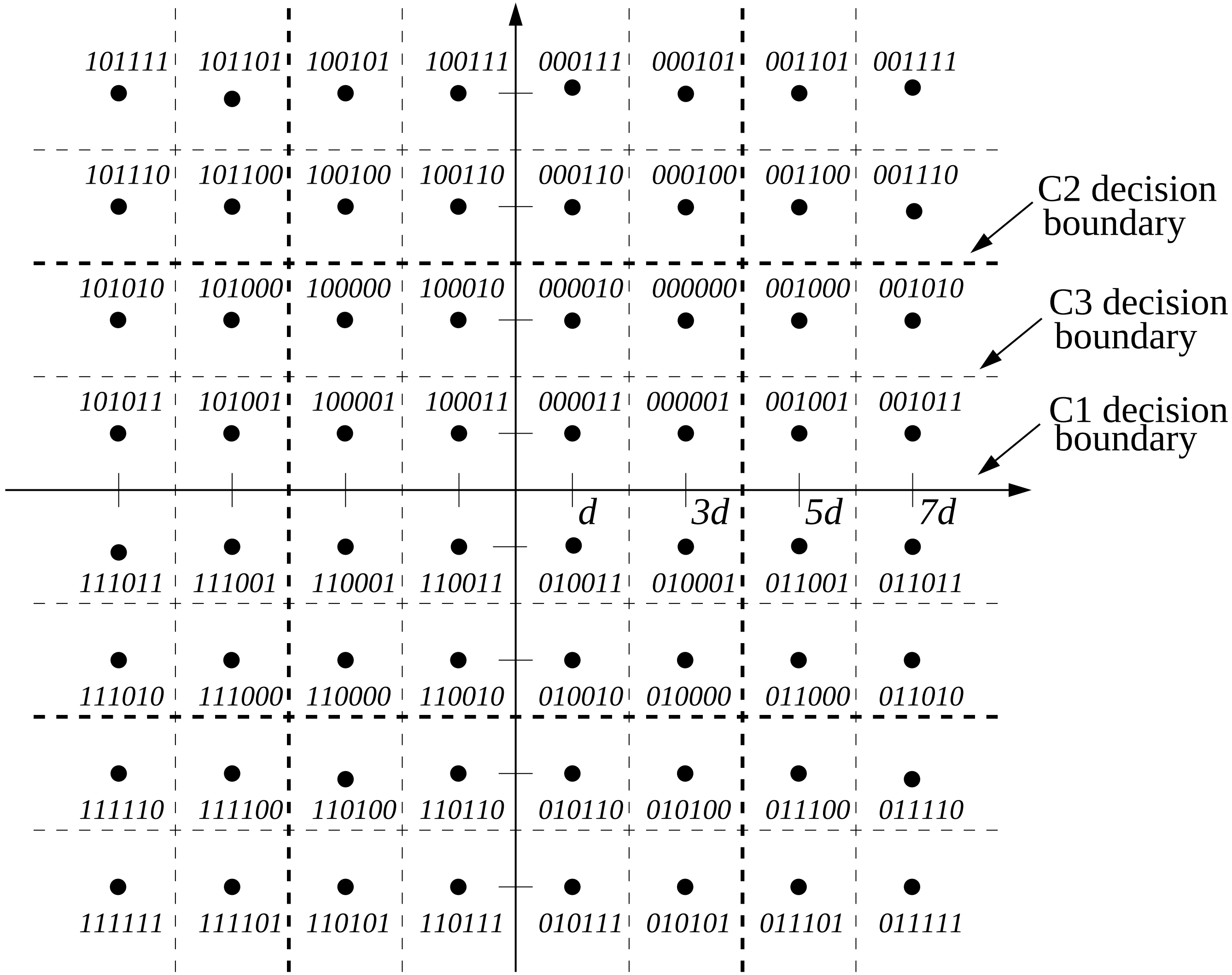
$$\begin{aligned} P_e = \tfrac{7}{12}Q\left(\sqrt{2E_s/21N_0}\right) + \tfrac{1}{2}Q\left(3\sqrt{2E_s/21N_0}\right) - \tfrac{1}{12}Q\left(5\sqrt{2E_s/21N_0}\right) \\ + \tfrac{1}{12}Q\left(9\sqrt{2E_s/21N_0}\right) - \tfrac{1}{12}Q\left(13\sqrt{2E_s/21N_0}\right) \end{aligned}$$

64QAM Bit Error Rate

- 64QAM: $i_1q_1i_2q_2i_3q_3$

  Three classes of bits

  I & Q are identical to 8-ary



64QAM BER (continue)

- C1 decision: $I, Q > 0 \rightarrow i_1, q_1 = 0,\ \ I, Q \leq 0 \rightarrow i_1, q_1 = 1$
- C2 decision: $I, Q > 4d$ or $I, Q \leq -4d \rightarrow i_2, q_2 = 1$

$$-4d < I, Q \leq 4d \rightarrow i_2, q_2 = 0$$

- C3 decision: $I, Q > 6d$ or $I, Q \leq -6d$ or $-2d < I, Q \leq 2d \rightarrow i_3, q_3 = 1$

$$-6d < I, Q \leq -2d \text{ or } 2d < I, Q \leq 6d \rightarrow i_3, q_3 = 0$$

- Average error of 64QAM is:

$$P_e = \frac{7}{12} Q\left(d/\sigma_n\right) + \frac{1}{2} Q\left(3d/\sigma_n\right) - \frac{1}{12} Q\left(5d/\sigma_n\right) + \frac{1}{12} Q\left(9d/\sigma_n\right) - \frac{1}{12} Q\left(13d/\sigma_n\right)$$

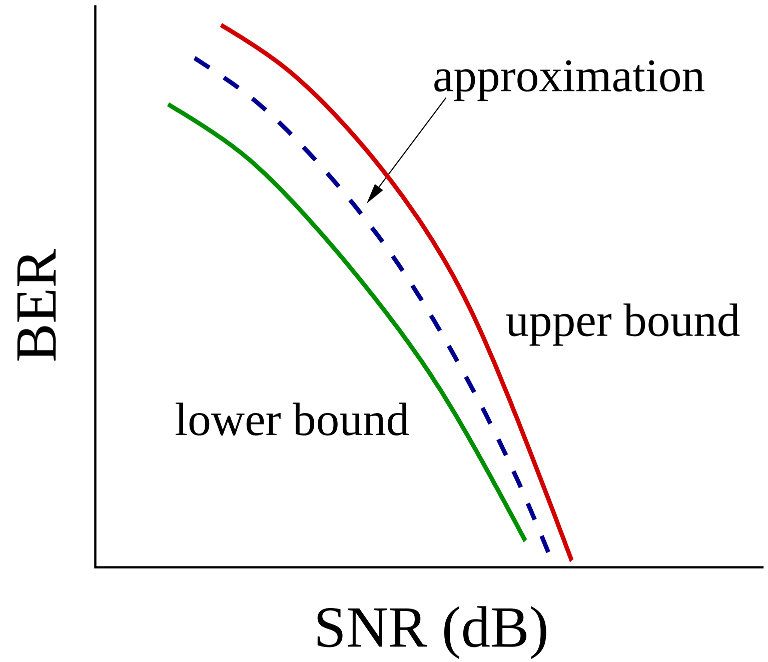
- Noting the average symbol energy of 64QAM is $E_s = 42d^2$,

$$P_e = \frac{7}{12} Q\left(\sqrt{E_s/21N_0}\right) + \frac{1}{2} Q\left(3\sqrt{E_s/21N_0}\right) - \frac{1}{12} Q\left(5\sqrt{E_s/21N_0}\right)$$

$$+ \frac{1}{12} Q\left(9\sqrt{E_s/21N_0}\right) - \frac{1}{12} Q\left(13\sqrt{E_s/21N_0}\right)$$

General Comments

- For generic high-order QAM, derive approximate bit error rate as $Q\left(\sqrt{\text{SNR}_{\text{equivalent}}}\right)$
  - In theory, one can define the upper bound which is larger than true BER
  - and the lower bound which is smaller than true BER
  - By making the two bounds very tight, ensure the approximation is very accurate
- In general, union bound is derived, which is very accurate approximation to represent the true bit error rate
  - In practice, union bound is often used to represent the BER of high-order QAM



- One may just simply use Monte Carlo simulation to evaluate the system's bit error rate
  - Sending large number of bits, and simply counting the bit errors: if total number of bits sent is N_b, and error counts is N_e

$$\text{BER} \approx \frac{N_e}{N_b}$$

  - How accurate this approximation? You must ensure at least a few hundreds of error counts!
- Systems like fibre networks, BER$< 10^{-9} \sim 10^{-12}$, may have to build prototype to evaluate, or
  - Importance sampling simulation may be adopted to evaluate these systems

Summary

- Implication of optimal Tx and Rx filter design: the area under $|G_{\mathrm{Rx}}(f)|^2$ is unity
- Decision theory: error probability, Bayes theorem, a priori probability, conditional probability, Q-function
- 4QAM (I & Q are 2-ary or BPSK): decision rule, BER derivation
- 16QAM (I & Q are 4-ary): C1 and C2 bits, two virtual sub-channels and different noise immunity, decision rules, BER derivation
- 64QAM (I & Q are 8-ary): C1, C2 and C3 bits, three virtual sub-channels and different noise immunity, decision rule, BER derivation

For 256QAM or higher, simplified approximation rather than exact derivation is used for BER calculation