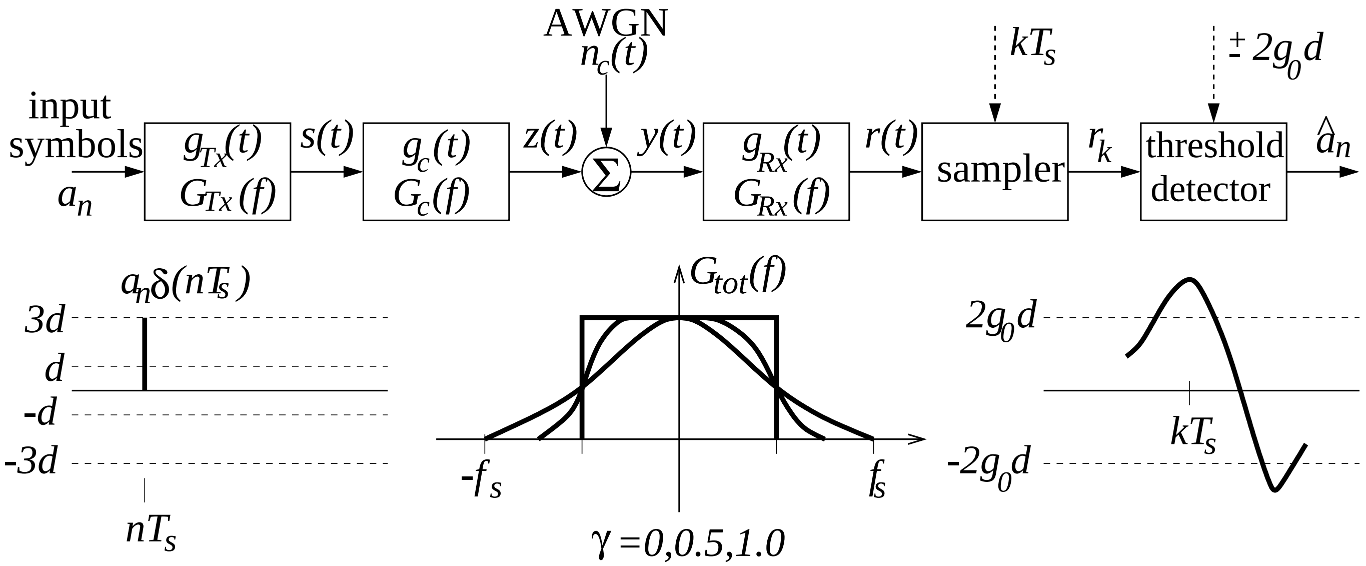
Revision of Modem so far

- Previous three lectures have covered most of Modem, e.g.
  - Several digital modulation schemes, with emphasis on PSK and QAM
  - Mapping bits to symbols, type of constellations, and constellation **design considerations**
  - Carrier recovery (at least for BPSK, later we will generalise to QAM), this allows to remove carrier to get back to **baseband** signal
  - Time recover (at least for binary signalling, later we will generalise to multilevel signalling), this allows receiver **detector** to detect transmitted symbols and maps them back to bits
- This lecture, we exam baseband equivalent system and detector
  - Can any one answer what Tx & Rx filtering pair are designed for?
  - We will see the connection with **optimal detection**

Baseband Equivalent System

- For M-QAM, I and Q branches are identical to one-dimensional $\sqrt{M}$-ary system
- **Baseband equivalent** I or Q system (with 16QAM example) is:



- Transmitted signal, received signal and sampled received signal are, respectively,

$$s(t) = \sum_n a_n g_{\mathrm{Tx}}(t - nT_s), \qquad r(t) = \sum_n a_n g_{\mathrm{tot}}(t - nT_s) + n(t)$$

$$r_k = \sum_n a_n g_{n-k} + n_k = g_0 a_k + \sum_{\substack{n \\ n \neq k}} a_n g_{n-k} + n_k$$

Optimal Tx & Rx Filter Design

- **Optimal design**: Tx & Rx filters are identical to square root of Nyquist raised cosine filter

  Note $g_{\text{tot}}(t) = g_{\text{Rx}}(t) \star g_{\text{c}}(t) \star g_{\text{Tx}}(t)$ and $\{g_n\}$ are symbol-spaced samples of $g_{\text{tot}}(t)$

1. **Achieve zero ISI**:

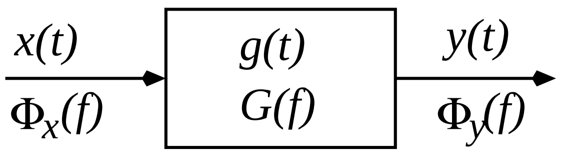
$$\sum_{\substack{n \\ n \neq k}} a_n g_{n-k} = 0 \Longrightarrow G_{\text{tot}}(f) = G_{\text{Rx}}(f) G_c(f) G_{\text{Tx}}(f) \text{is a Nyquist filter}$$

   If $G_c(f) = 1$, the combined $G_{\text{Rx}}(f) G_{\text{Tx}}(f)$ is a Nyquist filter

2. **Maximise the received signal to noise ratio** $\Longrightarrow G_{\text{Rx}}(f) = G_{\text{Tx}}(f)$

- Linear system theory revisit

Recall power is the area under PSD



Linear system

PSD of system output $y(t)$ is $\Phi_y(f) = |G(f)|^2 \Phi_x(f)$, and power of $y(t)$ is thus

$$P_y = \int_{-\infty}^{\infty} \Phi_y(f) df = \int_{-\infty}^{\infty} |G(f)|^2 \Phi_x(f) df$$

Maximising SNR (continue)

- The channel noise has a PSD $N(f) = \frac{N_0}{2}$ and it passes through $G_{\text{Rx}}(f)$, thus the **noise power** at receiver is:

$$P_N = \int_{-\infty}^{\infty} |G_{\text{Rx}}(f)|^2 N(f) df = \frac{N_0}{2} \int_{-\infty}^{\infty} |G_{\text{Rx}}(f)|^2 df$$

- **Power of the transmitted signal** $s(t)$ is:

$$P_{\text{Tx}} = \bar{a^2} \int_{-\infty}^{\infty} |G_{\text{Tx}}(f)|^2 df = \bar{a^2} \int_{-\infty}^{\infty} \left| \frac{G_{\text{tot}}(f)}{G_{\text{Rx}}(f)} \right|^2 df$$

assuming $G_c(f) = 1$, where $\bar{a^2}$ is the average symbol power

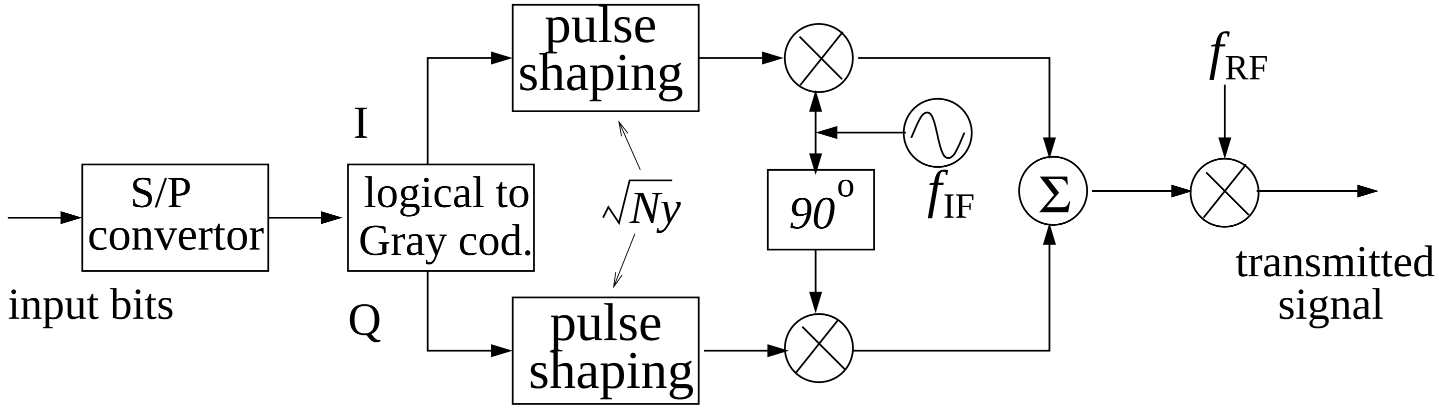
- **Maximising the received SNR** is equivalent to minimising P_N under the constraint of a constant P_{Tx}. Standard optimisation result yields:

$$G_{\text{Tx}}(f) = G_{\text{Rx}}(f)$$

QAM Modem

- Re-plot transmitter of QAM Modem from slide 77

  We have discussed all operations in it



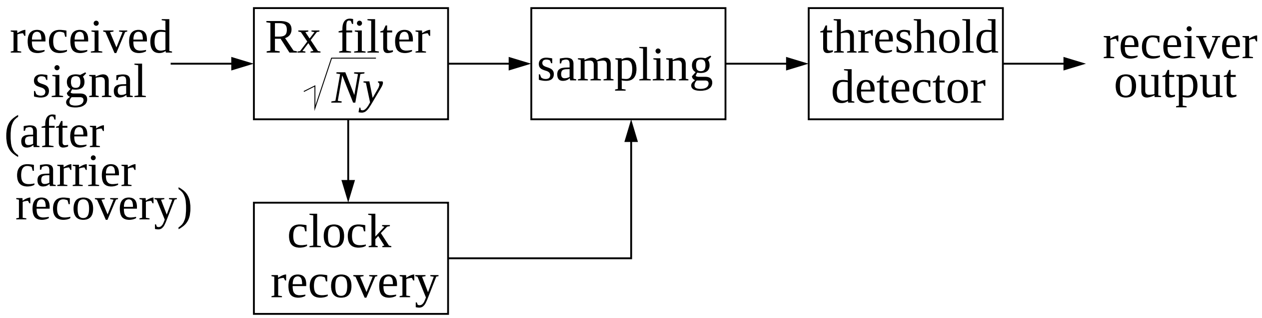
- Now recall QAM receiver in slide 77
  - Received signal after carrier recovery is demodulated into I and Q baseband signals
  - Clock recovery is needed for each baseband signal to obtain timing information and a detector recovers the transmitted I & Q symbols
  - Finally, a de-mapping converts symbol stream into bit stream

- We will discuss carrier and clock recovery for QAM later

- Here we first concentrate on **detection** part
  - There are many equivalent detection schemes, and three common ones are discussed
  - They are equivalent, all based on principle of maximising received signal to noise ratio
  - Maximising received SNR, in ideal AWGN, is equivalent to minimise detection error

Threshold Detection

- Receiver using **threshold detection**
  - Optimal receive filter, square root of raised cosine filter identical to transmit filter
  - This as discussed previously maximises the SNR



- For 16QAM example, I and Q have a 4-ary constellation, and let symbols at levels $3d$, d, $-d$ and $-3d$ be denoted as a_{+3}, a_{+1}, a_{-1} and a_{-3}

  The thresholds at detector are set to $+2g_0d$, 0 and $-2g_0d$, and the decision is made according to where the sample r_k lies:

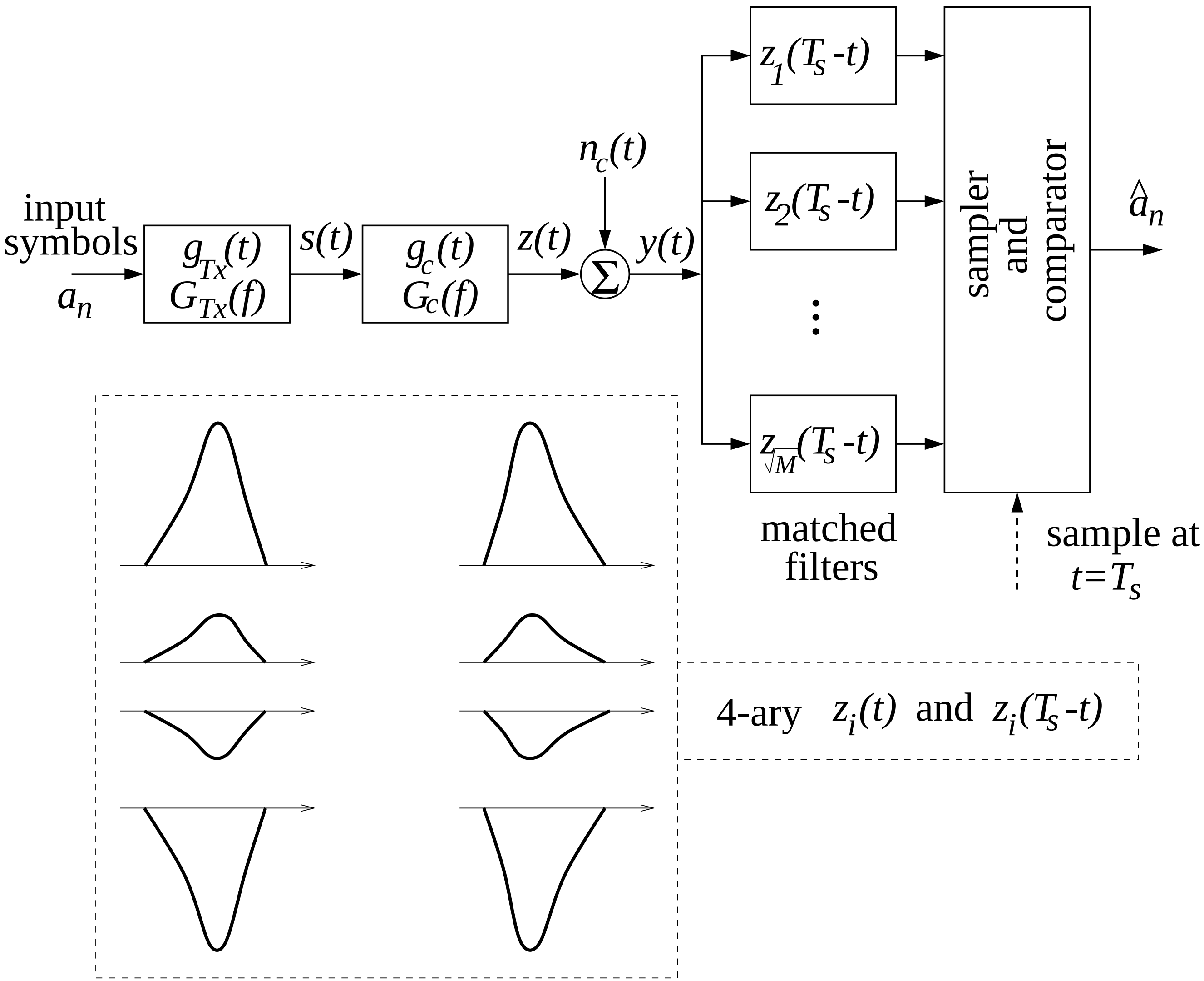
$$\hat{a}_k = \begin{cases} a_{+3}, & \text{if } r_k > +2g_0d \\ a_{+1}, & \text{if } 0 < r_k \leq +2g_0d \\ a_{-1}, & \text{if } -2g_0d < r_k \leq 0 \\ a_{-3}, & \text{if } r_k \leq -2g_0d \end{cases}$$

Matched Filter Detection

- Receiver using **matched filter detection**
  - Receiver consists of a bank of filters, each **matched** to one of the received waveforms

    $z_i(T_s - t)$ is matched to $z_i(t)$

  - Using the same principle of maximising received SNR, implemented as maximising the SNR at the output of matched filter for a given received signal waveform



Matched Filter Derivation

- Input to the receive filter is:

$$y(t) = \sum_n a_n h(t - nT_s) + n_c(t) = z(t) + n_c(t)$$

where $h(t) = g_{\text{Tx}}(t) \star g_c(t)$. Let matched filter output be $r(t) = \bar{r}(t) + n(t)$

$$\bar{r}(t) = \mathcal{F}^{-1}[G_{\text{Rx}}(f)Z(f)] = \int_{-\infty}^{\infty} G_{\text{Rx}}(f)Z(f)e^{j2\pi f t}df$$

where $Z(f) = \mathcal{F}[z(t)]$. Note that $r_k = \bar{r}_k + n_k$ and the noise n_k has a variance

$$\sigma_n^2 = \frac{N_0}{2}\int_{-\infty}^{\infty} |G_{\text{Rx}}(f)|^2 df$$

- The aim is to maximise the SNR at sampling instant $t = T_s$:

$$\text{SNR}_{T_s} = \frac{\bar{r}_k^2}{\sigma_n^2} = \frac{\left|\int_{-\infty}^{\infty} G_{\text{Rx}}(f)Z(f)e^{j2\pi f T_s}df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |G_{\text{Rx}}(f)|^2 df}$$

Matched Filter Derivation (continue)

- Using Schwartz's inequality leads to

$$\left| \int_{-\infty}^{\infty} G_{\mathrm{Rx}}(f) Z(f) e^{j2\pi f T_s} df \right|^2 \le \int_{-\infty}^{\infty} |G_{\mathrm{Rx}}(f)|^2 df \cdot \int_{-\infty}^{\infty} |Z(f)|^2 df$$

  with equality holds if

$$G_{\mathrm{Rx}}(f) = c Z^*(f) e^{-j2\pi f T_s}$$

  where * denotes the complex conjugate

- With this optimal $G_{\mathrm{Rx}}(f)$, SNR_{T_s} is maximised

$$\mathsf{SNR}_{T_s}^{\max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |Z(f)|^2 df$$

- The optimal receive filter in time domain is then given by

$$g_{\mathrm{Rx}}(t) = \mathcal{F}^{-1}[c Z^*(f) e^{-j2\pi f T_s}] = \begin{cases} c \cdot z(T_s - t), & 0 \le t \le T_s \\ 0, & \text{otherwise} \end{cases}$$

Matched Filter Summary

Recall matched filter structure in slide 91

- The received $z(t)$ takes waveforms $z_i(t)$, $1 \le i \le \sqrt{M}$, each corresponding to a symbol point. If $G_c(f) = 1$, $z_i(t) = s_i(t)$, waveforms of transmitted signal $s(t)$

  As $s(t)$ has a shape of square root of Nyquist raised cosine pulse, Rx filter $g_{\text{Rx}}(t)$ has a similar shape. This is the same requirements of optimal Tx & Rx filtering

- The matched filter receiver consists of a bank of filters $z_i(T_s - t)$, $1 \le i \le \sqrt{M}$, each of which is **matched** to one of the received waveforms $z_i(t)$, $1 \le i \le \sqrt{M}$

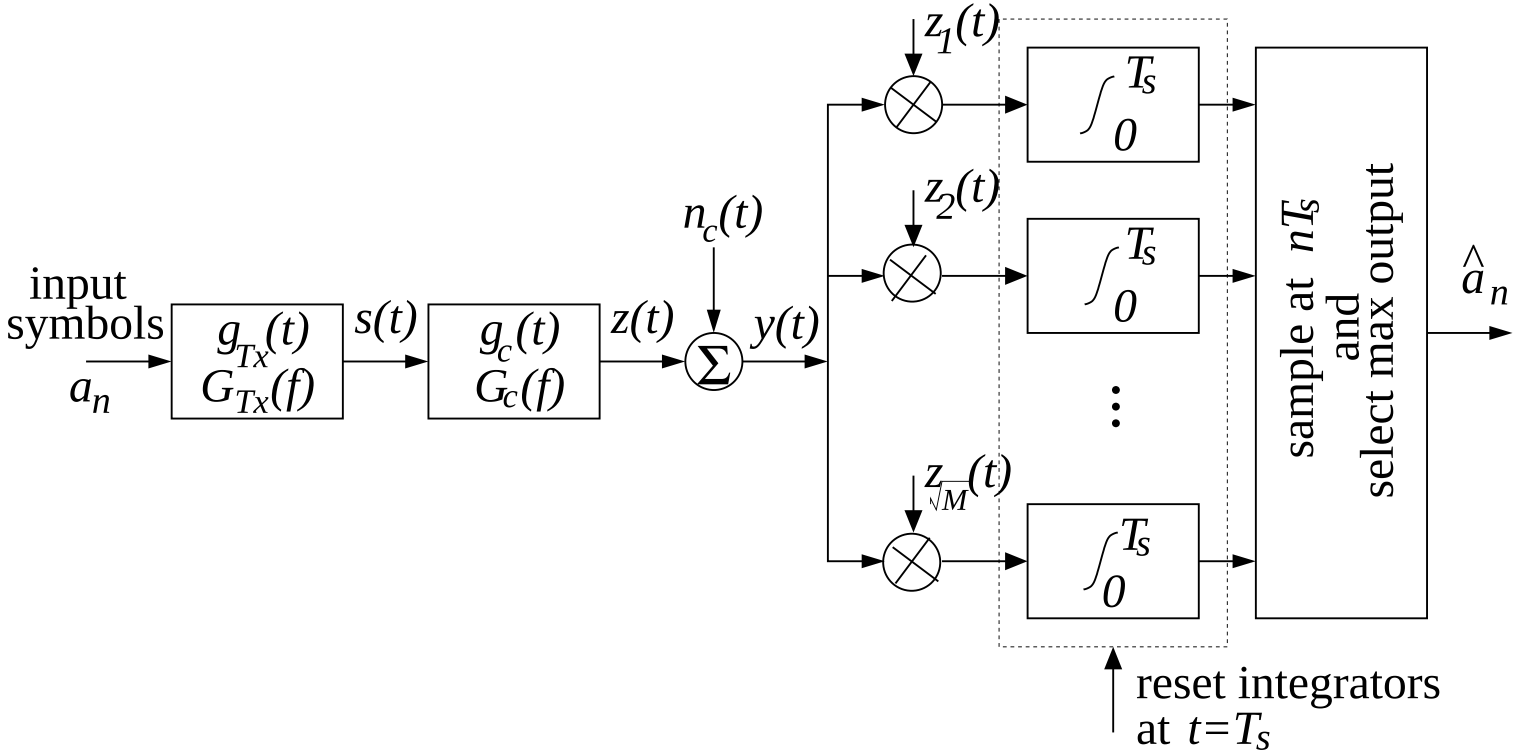
  If the jth symbol point is transmitted, the waveform of $z(t)$ is $z_j(t)$. The output of the matched filter $z_j(T_s - t)$ will be the largest, and all the other matched filter outputs will be small

  The comparator can easily infer which symbol point is transmitted

- Matched filter detector is a commonly used receiver structure. In practice, $z_i(t)$ are unknown (channel is unknown), so $s_i(T_s - t)$ are used

Correlation Receiver

- **Correlation receiver**: an alternative implementation of matched filter receiver



- It first multiplies the received signal with the prototype signals $z_i(t)$, $1 \leq i \leq \sqrt{M}$, integrates and dumps them at kT_s

  This is followed by a decision circuit to choose the largest output

Correlation Receiver Derivation

- Integrator (receiver) output is convolution of received signal with receiver filter

$$r(t) = y(t) \star g_{\mathrm{Rx}}(t) = \int_0^t y(\tau) g_{\mathrm{Rx}}(t-\tau) d\tau$$

- Choosing $c = 1$ in the optimal Rx filter $g_{\mathrm{Rx}}(t-\tau) = c \cdot z(T_s - (t-\tau))$ leads to

$$r(t) = \int_0^t y(\tau) z(T_s - t + \tau) d\tau$$

- Integrate and dump at every $t = T_s$:

$$r(t = T_s) = r_k = \int_0^{T_s} y(\tau) z(\tau) d\tau$$

- In practice, received prototypes $z_i(t)$, $1 \le i \le \sqrt{M}$, are unknown, and transmitted prototypes $s_i(t)$ are used

Summary

- Baseband equivalent system and optimal Tx & Rx filtering:

  (1) achieve zero ISI, and (2) maximise the receive SNR, that is,

$$G_{\text{Tx}}(f) = G_{\text{Rx}}(f) \text{ and combined } G_{\text{Tx}}(f)G_{\text{Rx}}(f) \text{ is a Nyquist filter}$$

- QAM transmitter revisit

- QAM receivers: detector structure

  Threshold detection receiver, matched filter receiver and correlation receiver

  Their schematic diagrams, and how they works

  They are equivalent and all based on the principle of maximising the receive SNR

  Note that maximum Rx SNR is directly linked to minimum detection error