Revision of Lecture Eight

- Baseband equivalent system and requirements of optimal transmit and receive filtering: (1) achieve zero ISI, and (2) maximise the receive SNR

- Three detection schemes:

  – Threshold detection receiver, matched filter receiver and correlation receiver
  – They are equivalent and all based on the principle of maximising the receive SNR
  – Detector is **important**, as every communication system has one
  – So you should know detector schemetic diagrams and how they work

- Note that maximising receive SNR is directly linked to minimum detection error, and in next two lectures we analyse performance of the system

  Specifically, we analyse bit error ratio of the system in AWGN and fading channels

Performance in AWGN

- In AWGN, channel is ideal and sampled quadrature (I or Q) component is $r_k = \bar{r}_k + n_k$
- With optimal Tx & Rx filtering as well as channel $G_c(f) = 1$,

$$G_{\text{Rx}}(f) = G_{\text{Tx}}(f) \quad \text{and} \quad G_{\text{Tx}}(f)G_{\text{Rx}}(f) \quad \text{is a Nyquist filter}$$

  – Note that

$$\int_{-\infty}^{\infty} |G_{\text{Rx}}(f)|^2 df = 1 \qquad \int_{-\infty}^{\infty} |G_{\text{Tx}}(f)|^2 \cdot |G_{\text{Rx}}(f)|^2 df = 1$$

  – Thus, received noise sample n_k has power

$$P_N = \sigma_n^2 = \frac{N_0}{2} \int_{-\infty}^{\infty} |G_{\text{Rx}}(f)|^2 df = \frac{N_0}{2}$$

  – Let $\bar{a^2}$ be average (I or Q) symbol power, received signal sample $\bar{r}_k$ has power

$$P_{\text{Rx}} = \bar{a^2} \int_{-\infty}^{\infty} |G_{\text{Tx}}(f)|^2 \cdot |G_{\text{Rx}}(f)|^2 df = \bar{a^2}$$

- Detection error probability depends on noise probability density function, which is Gaussian with variance σ_n^2, and receiver output SNR$= P_{\text{Rx}}/P_N$
  – Maximising receiver output SNR in AWGN leads to minimising detection error probability

BPSK Bit Error Rate

- BPSK or 2-ary (bit 0: $a = +d$, bit 1: $a = -d$): the received sample is

$$r = a + n, \ \ a \in \{\pm d\} \ \ \text{and} \ \ n \in N(0, \sigma^2)$$

- As the decision boundary is $r = 0$, the threshold decision rule is

$$r > 0 \rightarrow \hat{a} = d, \ r \leq 0 \rightarrow \hat{a} = -d$$

- Using Bayes theorem, the error probability or BER is given by

$$P_e = P(\hat{a} \neq a) = P(a = d \cap \hat{a} = -d) + P(a = -d \cap \hat{a} = d)$$

$$= P(a = d)P(\hat{a} = -d|a = d) + P(a = -d)P(\hat{a} = d|a = -d)$$

- As transmitted bit is equally likely to be 0 or 1, the two $a \ prior$ probabilities are

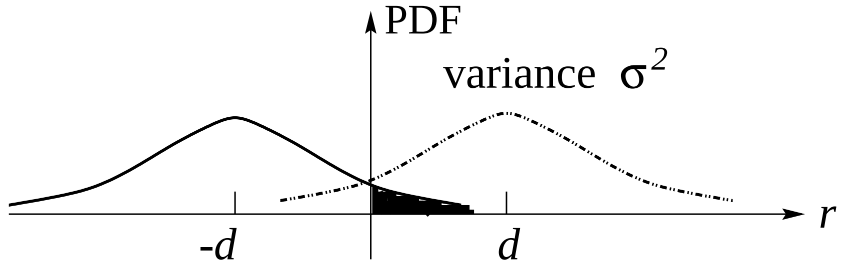
$$P(a = d) = P(a = -d) = \frac{1}{2}$$

BPSK BER (continue)

- Given $a = -d$, the decision $\hat{a} = d$ means that $r = -d + n > 0$ or noise value $n > d$, and the conditional probability $P(\hat{a} = d | a = -d) =$

$$P(n > d) = \int_d^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \frac{1}{\sqrt{2\pi}} \int_{d/\sigma}^{\infty} \exp\left(-\frac{y^2}{2}\right) dy = Q\left(d/\sigma\right)$$

  - Interpretation of conditional error probability $P(\hat{a} = d | a = -d)$: Gaussian tail area over threshold $r = 0$

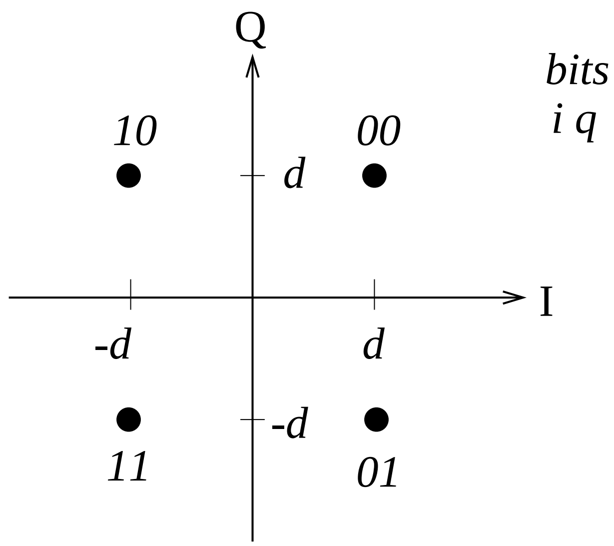


- Similarly, the other conditional error probability $P(\hat{a} = -d | a = d) = Q\left(d/\sigma\right)$

- Note signal power $E_s = \frac{1}{2}(d^2 + d^2) = d^2$ and noise power $\frac{N_0}{2} = \sigma^2$, BER is

$$P_e = \frac{1}{2} Q\left(d/\sigma\right) + \frac{1}{2} Q\left(d/\sigma\right) = Q\left(d/\sigma\right) = Q\left(\sqrt{\frac{2E_s}{N_0}}\right)$$

4QAM Bit Error Rate



- 4QAM: I and Q components are both BPSK, and average signal power is $E_s = 2d^2$
- Let I or Q denote respective received signal sample, then decision rule is

$$I, Q > 0 \rightarrow i, q = 0 \quad I, Q \leq 0 \rightarrow i, q = 1$$

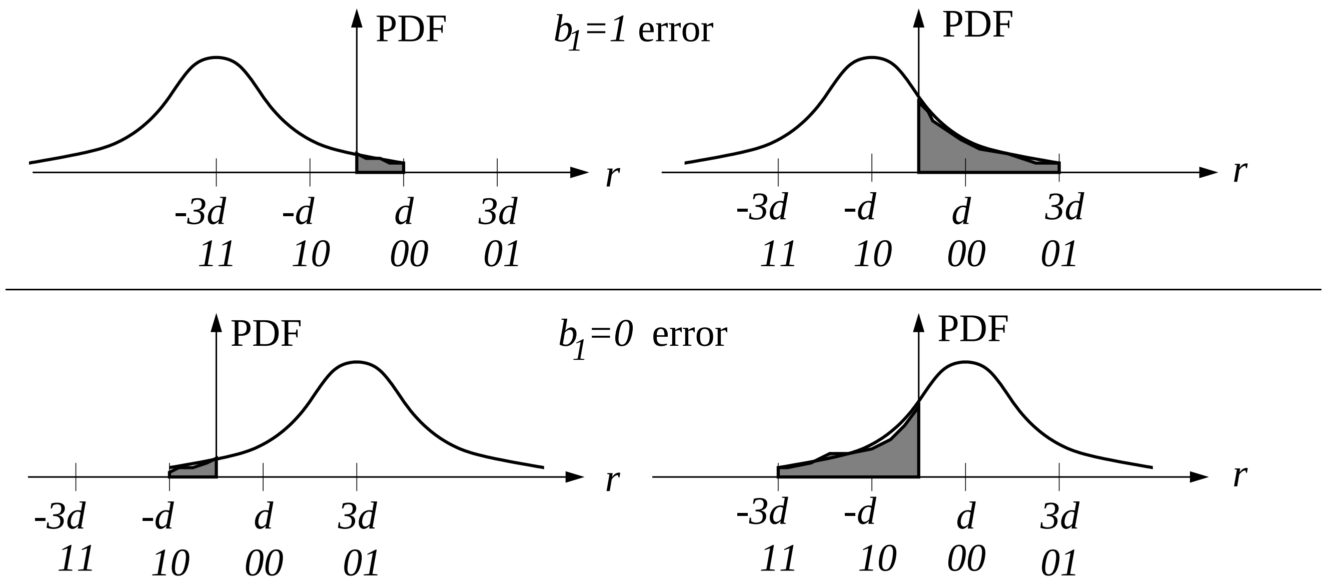
- Applying BPSK result to both I and Q yields
  - i bit error rate: $P_{e,I} = Q\left(d/\sigma_n\right)$
  - q bit error rate: $P_{e,Q} = Q\left(d/\sigma_n\right)$
- **Average error rate** for 4QAM is then

$$P_e = \frac{1}{2}\left(P_{e,I} + P_{e,Q}\right) = Q\left(d/\sigma_n\right) = Q\left(\sqrt{\frac{E_s}{N_0}}\right)$$

4-ary Constellation BER

- 4-ary constellation: 2 bits per symbol and Gray coding $b_1 b_2 : 01, 00, 10, 11 \rightarrow 3d, d, -d, -3d$
  - Most significant bit b_1 and least significant bit b_2 have **different immunities to noise**, and are called **class-one** and **class-two** bits, respectively
  - Intrinsically, as though C1 and C2 bits are transmitted through two different **sub-channels**
- C1 bit decision rule: $r > 0 \rightarrow b_1 = 0, \quad r \leq 0 \rightarrow b_1 = 1$
- C1 bit BER:

$$\begin{aligned} P_{e,1} &= \frac{2}{4} Q\left(d/\sigma_n\right) \\ &+ \frac{2}{4} Q\left(3d/\sigma_n\right) \\ &= \frac{1}{2} Q\left(d/\sigma_n\right) \\ &+ \frac{1}{2} Q\left(3d/\sigma_n\right) \end{aligned}$$



- C1 bits b_1 are at a protection distance of d from the decision boundary ($r = 0$) for 50% of the time, and their protection distance is $3d$ for the other 50% of the time

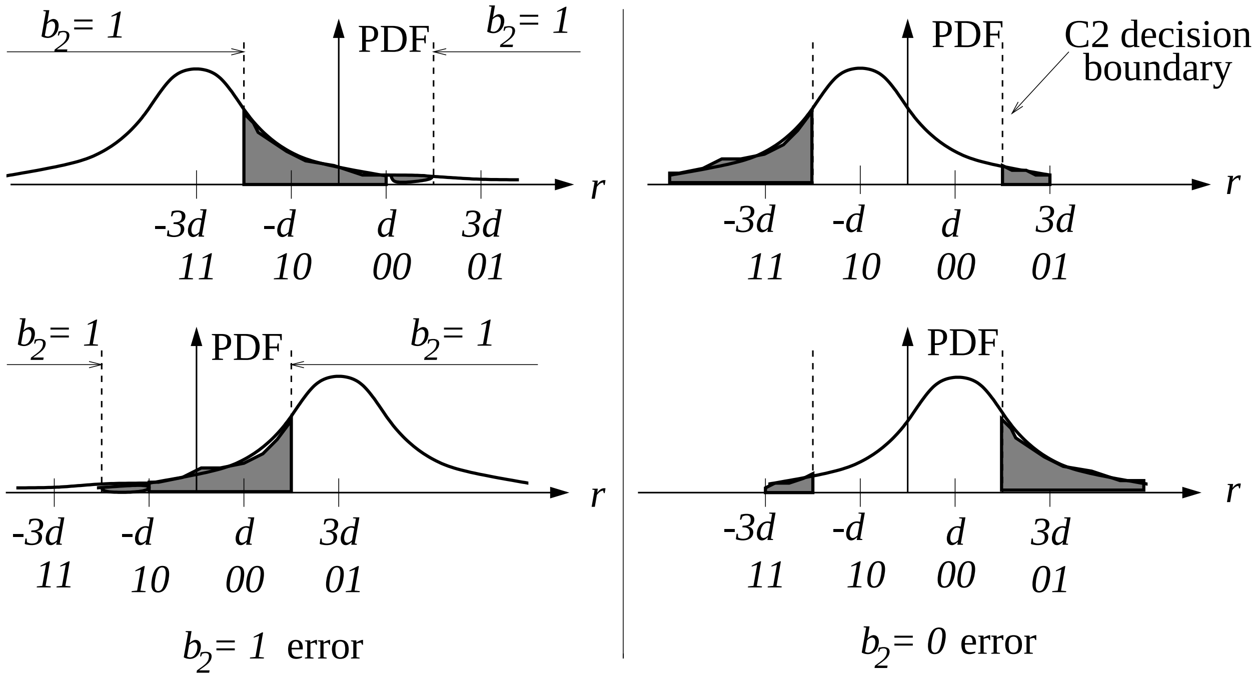
4-ary Constellation BER (continue)

- C2 bit decision rule: $r > 2d$ or $r \leq -2d \rightarrow b_2 = 1$, $-2d < r \leq 2d \rightarrow b_2 = 0$
  - For symbol $-3d$: when noise value $n > d$, $r > -2d$ and decision error occurs but when noise value $n > 5d$, $r > 2d$ and decision is correct again, thus the conditional error rate is $Q\left(d/\sigma_n\right) - Q\left(5d/\sigma_n\right)$
  - For symbol $-d$, it is $Q\left(d/\sigma_n\right) + Q\left(3d/\sigma_n\right)$
- C2 bit error rate

$$P_{e,2} = \frac{4}{4}Q\left(d/\sigma_n\right) + \frac{2}{4}Q\left(3d/\sigma_n\right) - \frac{2}{4}Q\left(5d/\sigma_n\right)$$

Note that

$$P_{e,2} \approx 2P_{e,1}$$



- **Average error rate** of 4-ary constellation:

$$P_e = \frac{1}{2}(P_{e,1} + P_{e,2}) = \frac{3}{4}Q\left(\frac{d}{\sigma_n}\right) + \frac{1}{2}Q\left(\frac{3d}{\sigma_n}\right) - \frac{1}{4}Q\left(\frac{5d}{\sigma_n}\right)$$

16QAM Bit Error Rate

- 16QAM: I and Q are both 4-ary
  - C1 bit decision:

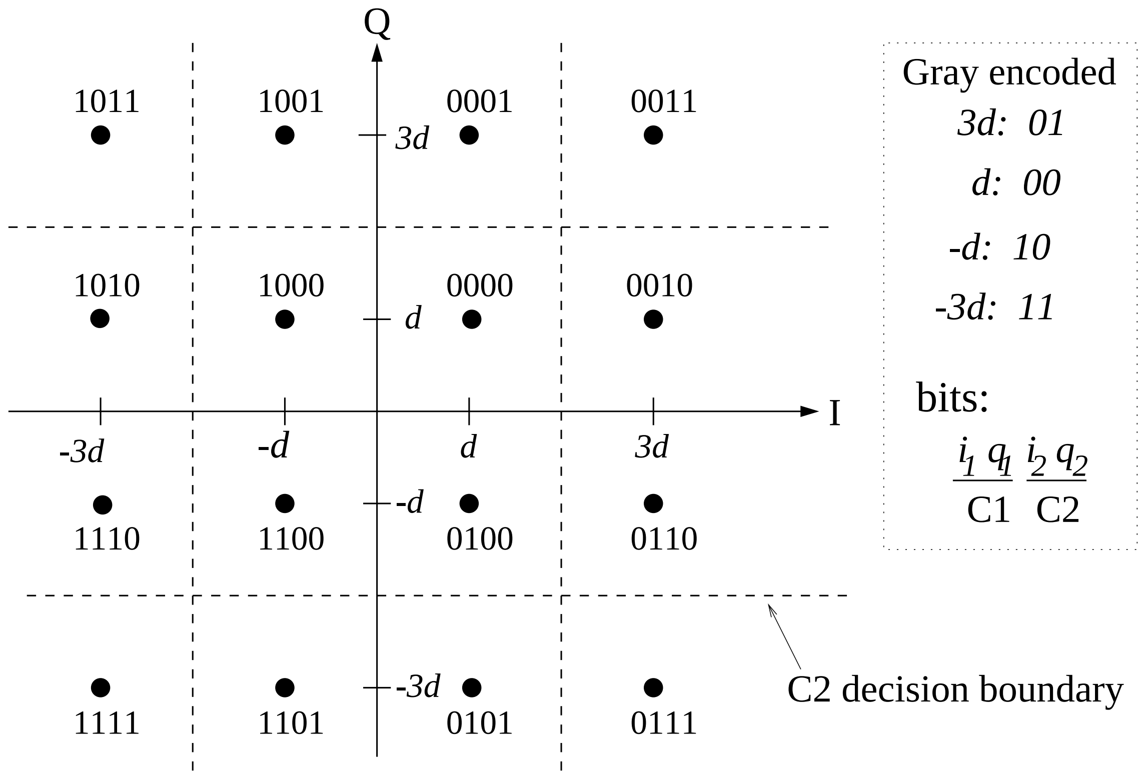
$$I, Q > 0 \rightarrow i_1, q_1 = 0$$

$$I, Q \leq 0 \rightarrow i_1, q_1 = 1$$

  - C2 bit decision:

$$I, Q > 2d \text{ or } I, Q \leq -2d \rightarrow i_2, q_2 = 1$$

$$-2d < I, Q \leq 2d \rightarrow i_2, q_2 = 0$$



- Signal power $E_s = 10d^2$ and noise power $\sigma_n^2 = \frac{N_0}{2} \rightarrow d/\sigma_n = \sqrt{E_s/5N_0}$, and 16QAM BER:

$$P_e = \frac{1}{2}(P_{e,I} + P_{e,Q}) = \frac{3}{4}Q\left(\frac{d}{\sigma_n}\right) + \frac{1}{2}Q\left(\frac{3d}{\sigma_n}\right) - \frac{1}{4}Q\left(\frac{5d}{\sigma_n}\right)$$

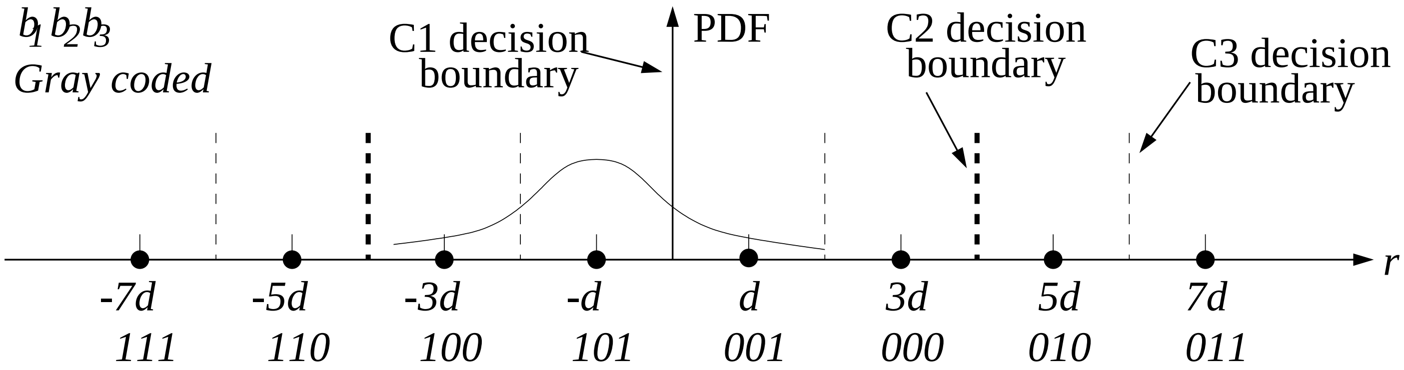
$$= \frac{3}{4}Q\left(\sqrt{E_s/5N_0}\right) + \frac{1}{2}Q\left(3\sqrt{E_s/5N_0}\right) - \frac{1}{4}Q\left(5\sqrt{E_s/5N_0}\right)$$

8-ary Constellation BER

- 8-ary: $b_1b_2b_3$ Gray coded. **3 classes** of bits, C1 b_1 has the highest immunity to noise, and C3 b_3 has the lowest, as though these three classes of bits were transmitted through 3 different **sub-channels**
  - One C1 decision boundary, two C2 decision boundaries, and four C3 decision boundaries
- C1 decision rule:

$$r > 0 \rightarrow b_1 = 0$$
$$r \leq 0 \rightarrow b_1 = 1$$

Thus, conditional error rate for symbol $-7d$ is $Q\left(7d/\sigma_n\right)$ and so on



- C1 bit error rate:

$$P_{e,1} = \frac{1}{4}\left(Q\left(d/\sigma_n\right) + Q\left(3d/\sigma_n\right) + Q\left(5d/\sigma_n\right) + Q\left(7d/\sigma_n\right)\right)$$

- C2 decision: $r > 4d$ or $r \leq -4d \rightarrow b_2 = 1, \quad -4d < r \leq 4d \rightarrow b_2 = 0$

$$P_{e,2} = \tfrac{1}{2}Q\left(d/\sigma_n\right) + \tfrac{1}{2}Q\left(3d/\sigma_n\right) + \tfrac{1}{4}Q\left(5d/\sigma_n\right) + \tfrac{1}{4}Q\left(7d/\sigma_n\right) - \tfrac{1}{4}Q\left(9d/\sigma_n\right) - \tfrac{1}{4}Q\left(11d/\sigma_n\right)$$

8-ary Constellation BER (continue)

- C3 decision: $r > 6d$ or $r \le -6d$ or $-2d < r \le 2d \rightarrow b_3 = 1$

$$-6d < r \le -2d \text{ or } 2d < r \le 6d \rightarrow b_3 = 0$$

$$P_{e,3} = Q\left(d/\sigma_n\right) + \frac{3}{4}Q\left(3d/\sigma_n\right) - \frac{3}{4}Q\left(5d/\sigma_n\right) - \frac{1}{2}Q\left(7d/\sigma_n\right)$$
$$+\frac{1}{2}Q\left(9d/\sigma_n\right) + \frac{1}{4}Q\left(11d/\sigma_n\right) - \frac{1}{4}Q\left(13d/\sigma_n\right)$$

- Note $P_{e,3} \approx 2P_{e,2}$ and $P_{e,2} \approx 2P_{e,1}$
- Average error of 8-ary constellation is: $P_e = \frac{1}{3}(P_{e,1} + P_{e,2} + P_{e,3})$ or

$$P_e = \frac{7}{12}Q\left(d/\sigma_n\right) + \frac{1}{2}Q\left(3d/\sigma_n\right) - \frac{1}{12}Q\left(5d/\sigma_n\right) + \frac{1}{12}Q\left(9d/\sigma_n\right) - \frac{1}{12}Q\left(13d/\sigma_n\right)$$

Since the average symbol energy of 8-ary is $E_s = 21d^2$ and $\sigma_n^2 = N_0/2$,

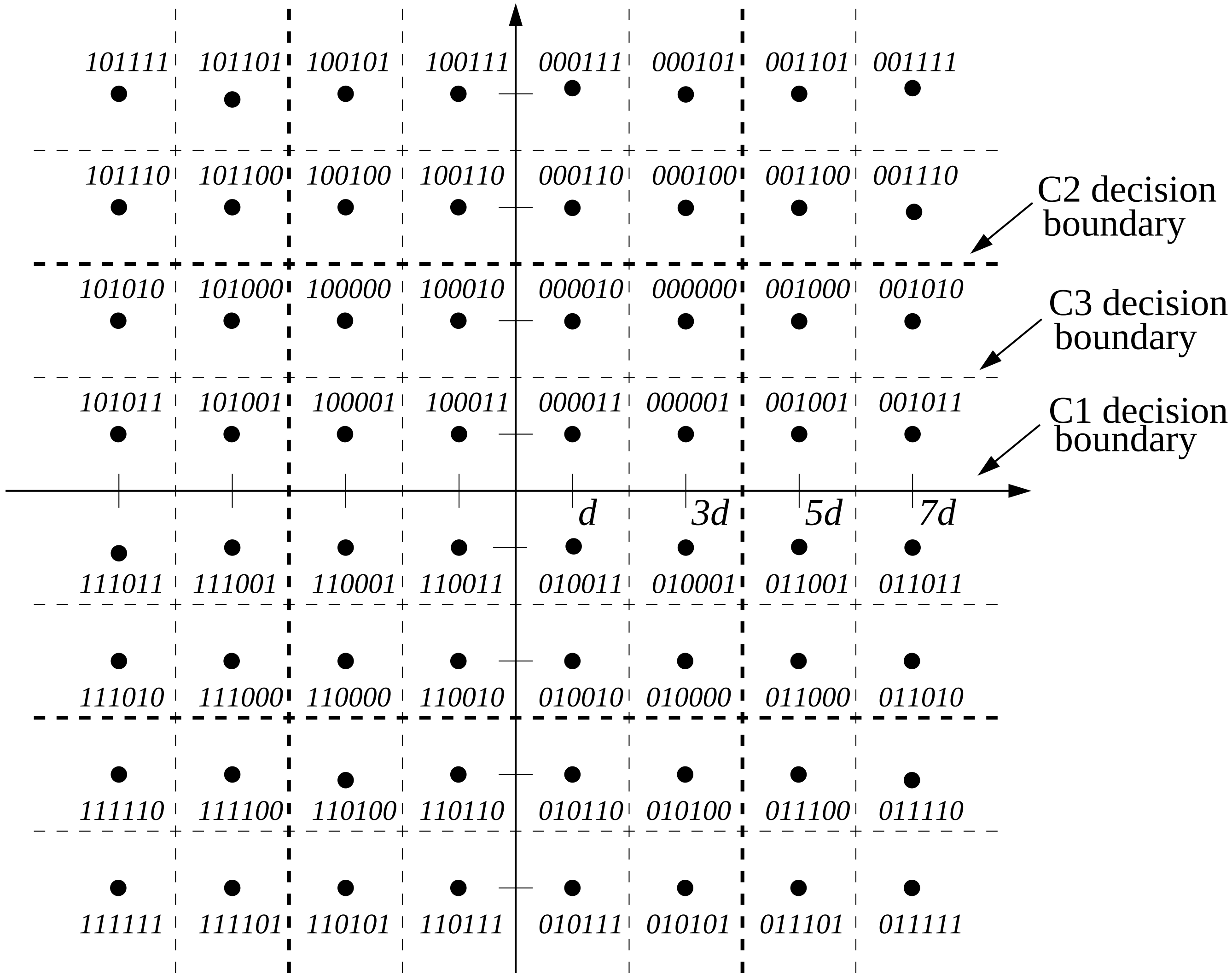
$$P_e = \tfrac{7}{12}Q\left(\sqrt{2E_s/21N_0}\right) + \tfrac{1}{2}Q\left(3\sqrt{2E_s/21N_0}\right) - \tfrac{1}{12}Q\left(5\sqrt{2E_s/21N_0}\right)$$
$$+\tfrac{1}{12}Q\left(9\sqrt{2E_s/21N_0}\right) - \tfrac{1}{12}Q\left(13\sqrt{2E_s/21N_0}\right)$$

64QAM Bit Error Rate

- 64QAM: $i_1 q_1 i_2 q_2 i_3 q_3$

  Three classes of bits

  I & Q are identical to 8-ary



64QAM BER (continue)

- C1 decision: $I,Q > 0 \rightarrow i_1, q_1 = 0,\ \ I,Q \le 0 \rightarrow i_1, q_1 = 1$
- C2 decision: $I,Q > 4d$ or $I,Q \le -4d \rightarrow i_2, q_2 = 1$

$$-4d < I,Q \le 4d \rightarrow i_2, q_2 = 0$$

- C3 decision: $I,Q > 6d$ or $I,Q \le -6d$ or $-2d < I,Q \le 2d \rightarrow i_3, q_3 = 1$

$$-6d < I,Q \le -2d \text{ or } 2d < I,Q \le 6d \rightarrow i_3, q_3 = 0$$

- Average error of 64QAM is:

$$P_e = \frac{7}{12} Q\left(d/\sigma_n\right) + \frac{1}{2} Q\left(3d/\sigma_n\right) - \frac{1}{12} Q\left(5d/\sigma_n\right) + \frac{1}{12} Q\left(9d/\sigma_n\right) - \frac{1}{12} Q\left(13d/\sigma_n\right)$$

- Noting the average symbol energy of 64QAM is $E_s = 42d^2$,

$$P_e = \frac{7}{12} Q\left(\sqrt{E_s/21N_0}\right) + \frac{1}{2} Q\left(3\sqrt{E_s/21N_0}\right) - \frac{1}{12} Q\left(5\sqrt{E_s/21N_0}\right)$$

$$+\frac{1}{12} Q\left(9\sqrt{E_s/21N_0}\right) - \frac{1}{12} Q\left(13\sqrt{E_s/21N_0}\right)$$

Summary

- Implication of optimal Tx and Rx filter design: the area under $|G_{\mathrm{Rx}}(f)|^2$ is unity
- Decision theory: error probability, Bayes theorem, a priori probability, conditional probability, Q-function
- 4QAM (I & Q are 2-ary or BPSK): decision rule, BER derivation
- 16QAM (I & Q are 4-ary): C1 and C2 bits, two virtual sub-channels and different noise immunity, decision rules, BER derivation
- 64QAM (I & Q are 8-ary): C1, C2 and C3 bits, three virtual sub-channels and different noise immunity, decision rule, BER derivation

For 256QAM or higher, simplified approximation rather than exact derivation is used for BER calculation